Dynamical Systems and Control
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Lecture – 50
Optimal Control - I

Dear students. Welcome to the lecture on optimal control I. In our previous lectures, we have seen various aspects of control systems namely the control ability, observability, stability aspects.

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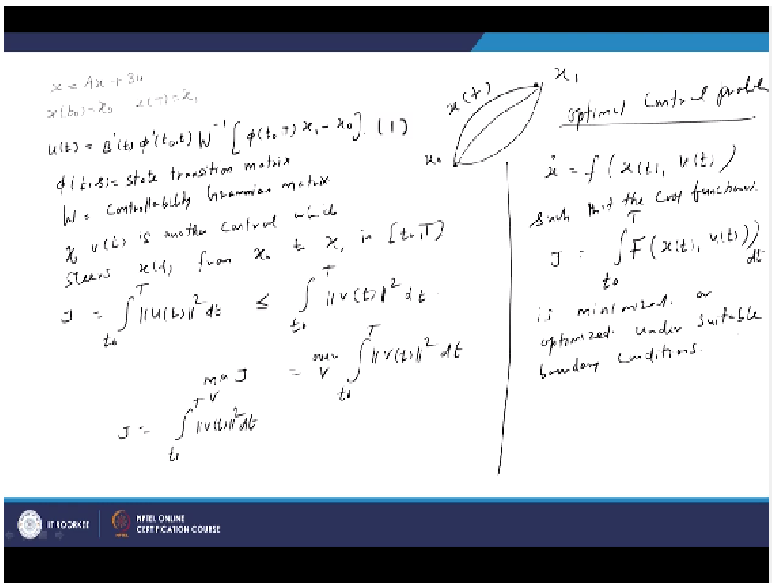

So mainly we have seen linear control system of the form dx/dt=Ax+Bu with initial condition x of t0=x0, x of T=x1. So in this case if the system is controllable, then we will be able to find a control u of t given by this expression B transpose phi transpose t0t and W inverse*phi of t0Tx1-x0. So if the system is controllable, then this control u of t will be able to steer the system or steer the solution x of t from the initial condition x0 to the final condition x1 at time T.

If here the W, notation W, it is nothing but the controllability Gramian matrix which was described in the previous lectures. And if W is non-singular, the system is controllable, that is how we will be able to find this control like this. Apart from this particular control, there may be several other control functions which can perform the same type of work that is taking the system from the initial condition to the final condition.

So there may be several such controls and there may be several solutions for this control system. So if let us say V of t is another control which steers the system, steers x of t from x0 to the x1 in the interval t0 to T, then we have also proved that integral t0 to T norm of ut whole square dt, it is less than or equal to t0 to T norm of Vt*dt. So in the sense that if you take any control which is steering that solution from x0 to x1, the control given in this particular expression has the lowest value for this integral.

So if you call this J to be this expression, so J, the minimum of J which is equal to integral t0 to T of norm of Vt square dt/all t, over all functions V all possible control functions V, if you find this minimum value, that will be equal to this J value, expression this thing. Or in another words we can say that the control given in this expression minimizes the functional J which is given in this expression.

The functional J is integral t0 to T norm of V of t square dt. The minimum value of this functional is given by the control u of t. So now this is one particular case of an optimal control problem. So in general, the optimal control problem is given by, we consider a control system x dot=x of t u of t. Let us say nonlinear control system is given by this expression which, such that the cost function between some 2 times of the expression x of t u of tdt is minimized.

So the optimal control problem is as follows. We have a control system x dot=f of x of t u of t such that the cost functional J is minimized or optimized, minimized or maximized according to the statement of the problem under suitable boundary conditions. So in this lecture, we will see how to derive conditions or necessary conditions for finding the optimal control for a system, a linear system in this expression.

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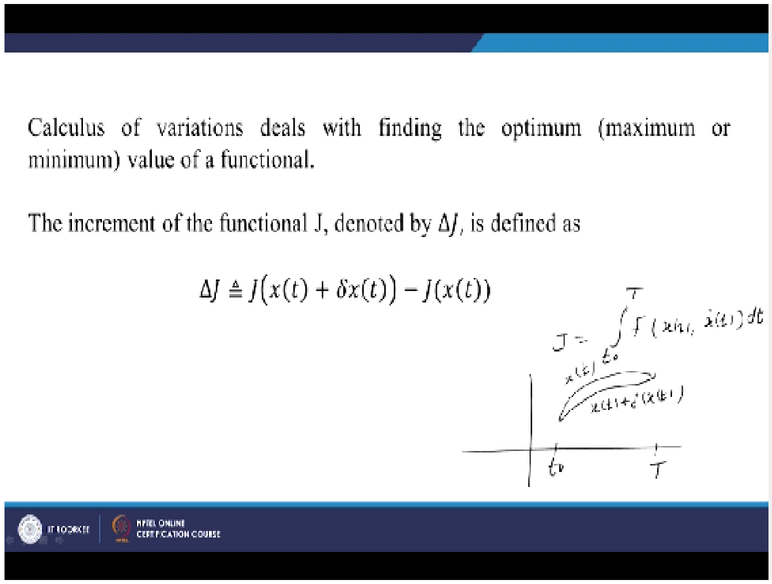

So first let us see the preliminary results which will lead to the necessary conditions. So it is based on the calculus of variation. Let J be a functional which is to be maximized or minimized and del J is the notation for the increment of the functional. So J as we have given, it may be of the form integral from the interval of a function x of t and x dot of tdt. So expression of this form if you consider, then we can find the increment J given by this expression J of x of t+del xt which is the variational in the function x of t-J of xt.

So, for example, if we consider the time t0 to T is this and x of t is a function between t0 to T, then the variation, by adding some variation, we will get a new function. So we call it as x of t+del of x of t. So if we calculate J at the function x of t, it will be of this expression and we calculate J at the incremental function x of t+del xt. We have to substitute in the place of x the expression xt+del xt and then subtract this 2, we will get the increment of this thing.

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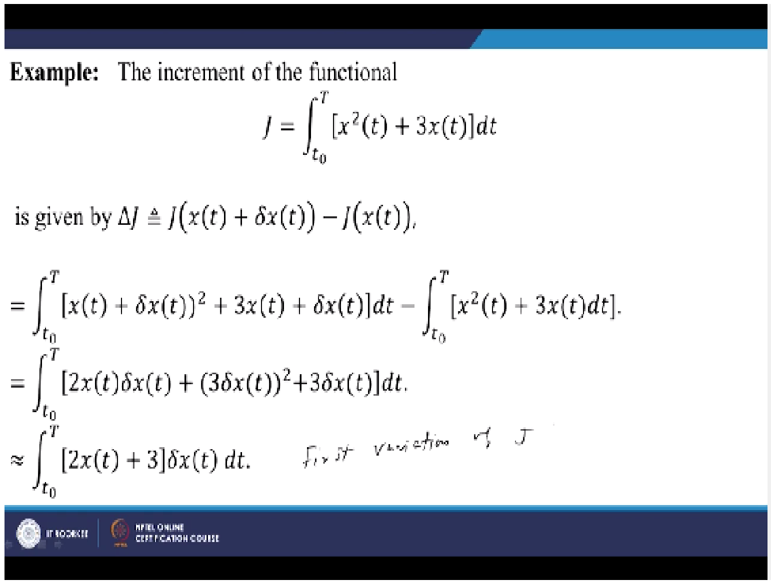

For example, if we consider an expression J as a function of x here, it is integral t0 to Tx square of t+3xt, then how to find the increment of this functional. It is given by this expression directly substituting x=del x in the place of x and subtracting J of x directly, we will get the expression. So the last line we are omitting the second variation that is del x whole square, term is omitted and remaining term gives the first variation, this is called first variation of the functional J here, okay.

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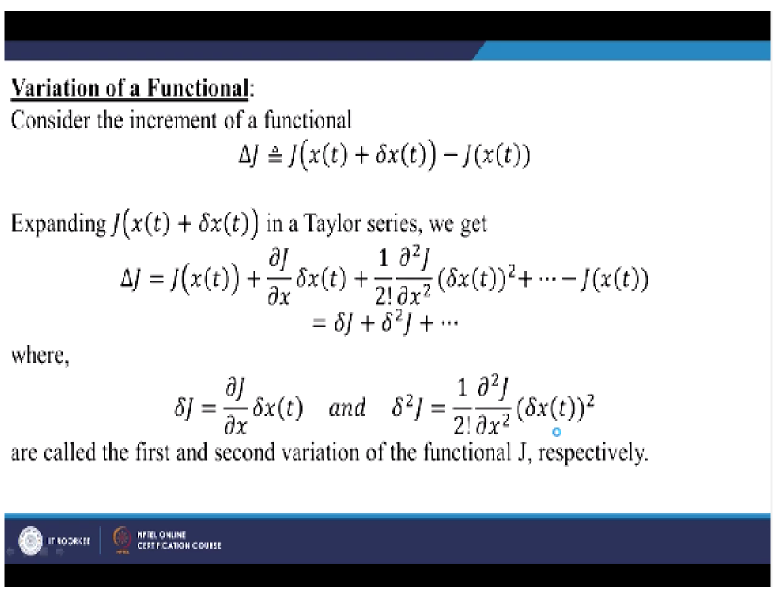

So here, again we calculate the increment of the functional del J=J of x+del x-J of x and expanding it in the Taylor series, xfJ of x+del x will give J of x + the first variation, del J/del x*the delta x of t+1/2 factorial and the second variation, etc., -J of xt. So that gives the increment and J gets cancelled, we will get del J+del square J. This is the notation for the second variation. del J is the notation for the first variation and that was shown in the previous example.

This is the first variation and now, in the previous case, the Taylor series expansion will stop up to the square term. There will not be any cube terms, etc. because it is a polynomial of degree 2 here. But in general, for any general function, we will get all the terms and then we can calculate first variation, second variation and etc.

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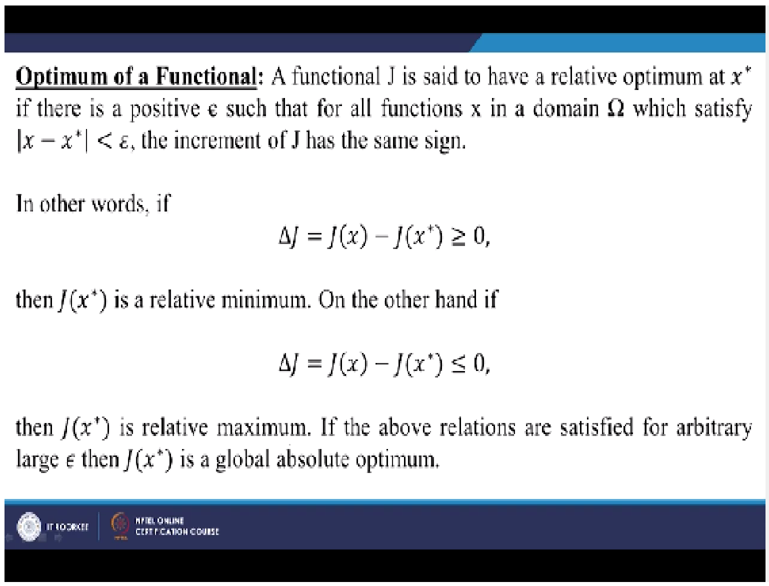

So using this first and second variation, we can get the optimal value of the functional using the following theorem. So here we define the J is said to have a relative optimum at a function x* if there exists a positive epsilon such that for all functions x in the domain omega, the neighbourhood of the function x* which satisfy this. So we take the collection of all functions which are near the function x* in the epsilon neighbourhood and then the increment has the sign, positive sign always, then it is called relative minimum.

The function x* is called the relative minimum if the increment del J is greater than or equal to 0. And if the increment del J is less than or equal to 0, then the function x* is called the relative maximum. And instead of the neighbourhood, the epsilon neighbourhood, if it is, the condition is satisfying for all values of epsilon, then we say that it is the global optimum, global minimum or global maximum depending on the increment nature.

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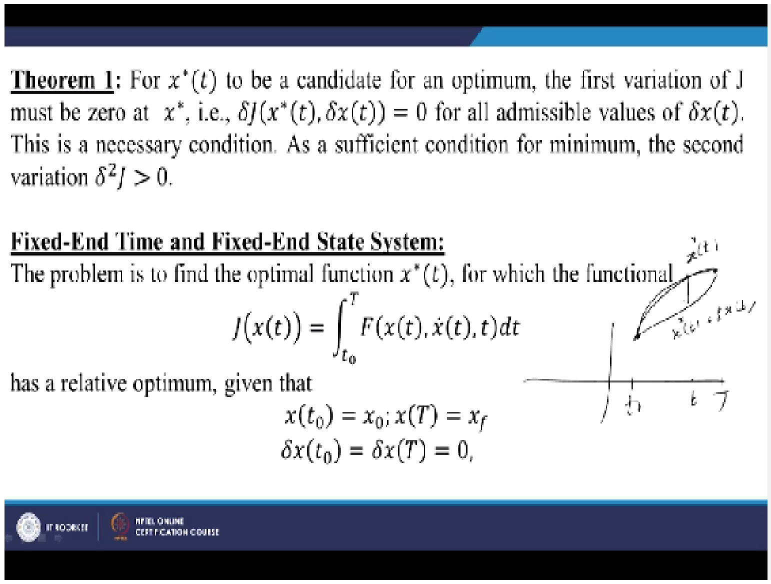

So then we can, the standard theorem on the optimum function is as follows. The function x*t is said to be an optimum function or it is a candidate for optimum function because this theorem, first statement is only the necessary condition. So it will be a candidate for optimum function if the first variation of J is equal to 0. So it is a necessary condition. Then we can confirm whether it is maximum or minimum by the second variation if del square J<0, then it is the maximum value and if del square J>0, it is minimum value.

So even though it is similar to the theorem for functions, here the similar result holds good for the functional J also, okay. Now We will come to the result on how to find the optimum function for the functional J. So let us say J of x is given by this expression integral t0 to T F of xt, x dot t, tdt with the initial condition x of t0 is x0 and x of T is xF, okay. So the function has these 2 fixed boundary conditions, that is from t0 to T.

These 2 are fixed, initial and final conditions are fixed here. Now because these 2 are fixed, the variation at the end point is 0 because if you add a delta, this is our x function and if you add a delta x function. So there is a variation at other points of t. If take general t, there is a variation between x and delta x but there is no variation at the end point. So the variation at t0 and variation at T, both are 0.

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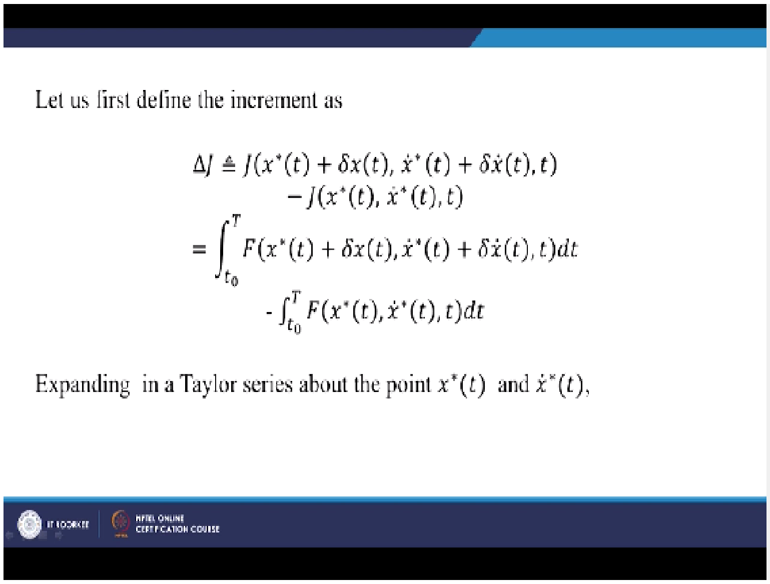

So now the increment del J is given by J of x*+del x.

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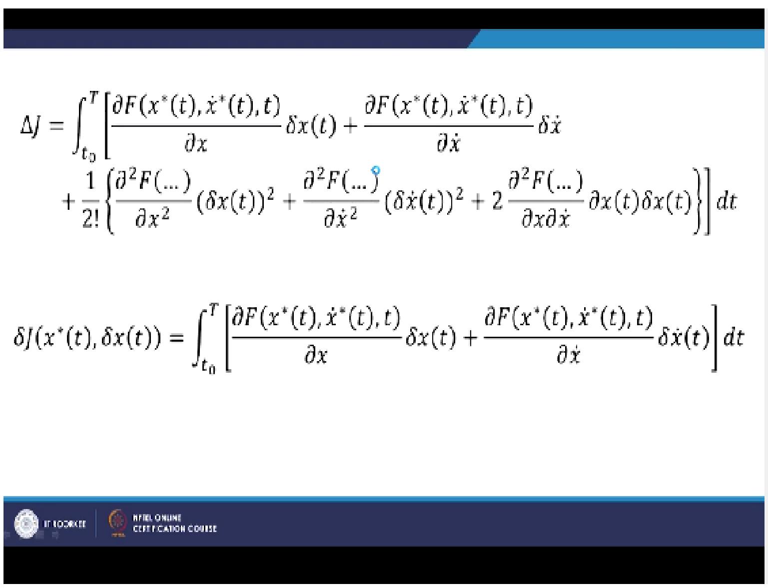

So we calculate the variation between, so let us say this is our required optimum solution. For example, if we take it as the optimum solution, then the increment is given by x*+delta x. Now calculating J value at this end, these 2 functions. So the increment is J calculated at x*+its increment, -J calculated at the function x*. So that is given by integral t0 to TF of, we substitute the x*+del x in the suitable places, -integral t0 to TF of x* and x dot*tdt.

Now we use the Taylor series expansion for the function F, term similar to the second, F of x*x dot *t is the first term of the Taylor series which will get cancelled. So the remaining terms will appear like this. So the increment J is given by del F/del x*del x+del F/del x dot*del x dot+, the second derivative terms are given by this expression, *dt. So the first variation is given by only the first derivative terms here. So it is del F/del x*del x+del F/del x dot*del x do*dt. And we omit all the second derivative terms here.

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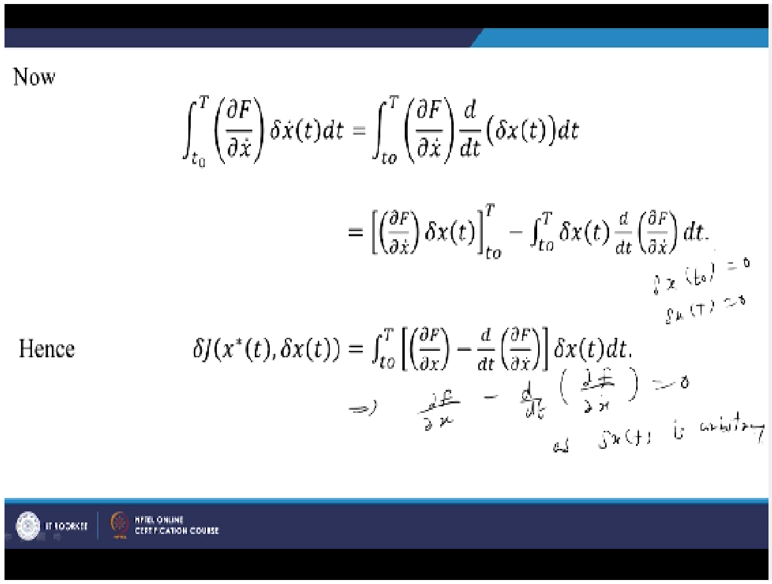

Because of the theorem statement, the first variation if we equate it to 0, it is the necessary condition for the optimum function condition. So now if you calculate this term, second term, del F/del x dot*del x dot, we take that term, del F/del x dot*del x dot is written as del F/del x dot d/dt of del x, this expression. Now we integrate by parts, this term. We get del F/del x dot*del x evaluated at t0 to T, the end points, -integral t0 to T del x of t*d/dt of the first function, del F/del x dot*dt.

Hence we get the expression of this form. The first variation in J, del J, is given by integral t0 to T of from the previous slide, this del J is given by del F/del x*del x, dt is one term and this term, second term is replaced by this expression. So ultimately we get del J=t0 to T del F/del x-d/dt del F/del x dot*del x dot. And this term is 0 because there is no variation. If you substitute t0 at T, we know that del xt0 is 0 and del xt is also 0 because there is no variation at the 2 end points as shown here the del x at t0 is 0 and del x at T is 0, so here this term will become 0.

Now this implies the variation del x is also arbitrary. From this picture, we can see that x* is a fixed, it is the function, required function. Then we can add any del x function with this, so that we get infinitely many such, this type of functions. So the variation del x is arbitrary. We get this expression del F/del x-d/dt of del F/del x dot=0, del xt is arbitrary.

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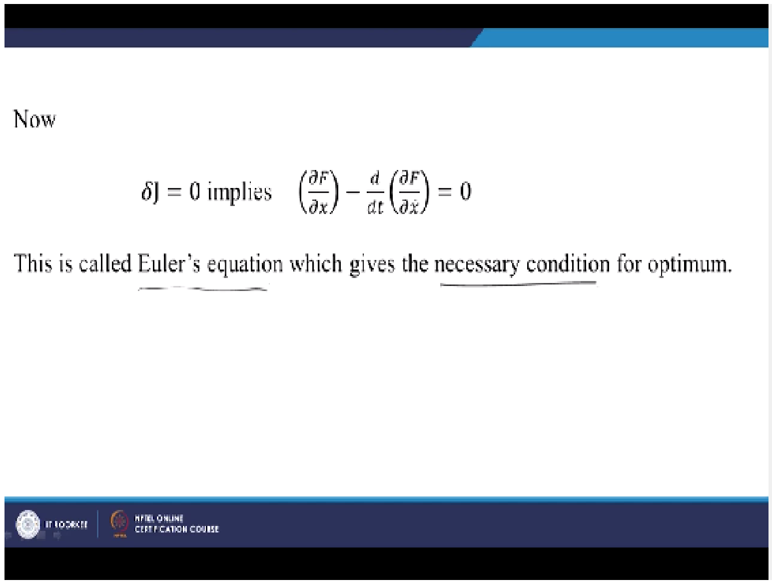

So this equation is called the Euler's equation for the minimization or maximization of the functional J here. So here there is no guarantee that this equation will give minimum or maximum because it is only the necessary condition. And if you check the second variation whether it is positive or negative at this solution of this equation, then we can conclude whether the solution is actually minimum or maximum or none of this. Euler's equation gives a necessary condition for the optimum function.

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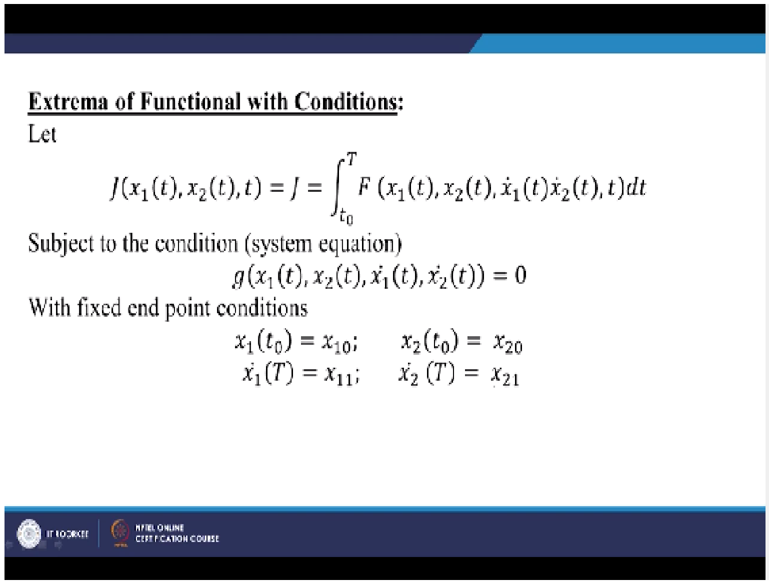

So now we quickly generalize the previous procedure for 2 variables. So J is a function of x1, x2 given by this expression integral t0 to TF of x1, x2, x1 dot, x2 dot*dt subject to the condition g of x1, x2, x1 dot, x2 dot=0 with the boundary conditions x1 of t0 and x2 of t0. Similarly, x1 dot at T and x2 dot at T, these values are given, the fixed values are given. So now we can, because it is, previously it was 1 variable, now it is 2 variable.

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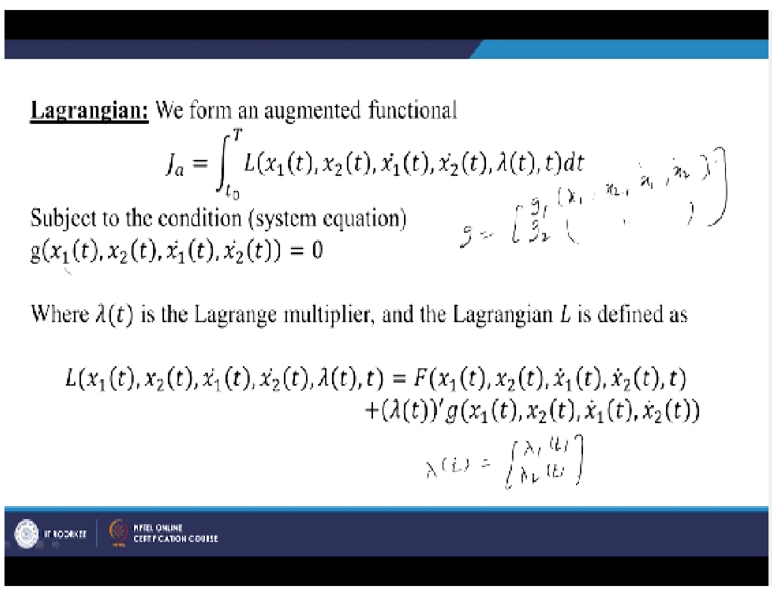

The procedure is exactly the same except that we have to write the Taylor series for the 2 variable and omit the second order terms and take the first variation=0. And the difference between the previous one and here is, there is a constraint. The minimization or maximization of J subject to the condition g of x1, x2, x1 dot, x2 dot=0. So this can be done in the usual way of the Lagrange multiplier method because this is a constraint.

So we define the Lagrange function L of x1, x2, x1 dot, x2 dot lambda, which is the Lagrange multiplier, =the function F + lambda*the function g. Here g is a vector function because there are 2 variables. It may contain 2 function g1 g2. g1 of x1, x2, x1 dot, x2 dot. Similarly, g2 also contain, so g is a vector function. And therefore, lambda is also having 2 components, lambda 1 lambda 2 and the transpose is given.

So we get a scalar functional here, L is F, which is a scalar function, +lambda dashed*g, that is also a scalar function, and now we can proceed with this Ja. Minimizing Ja will give the minimum value of the J itself, okay.

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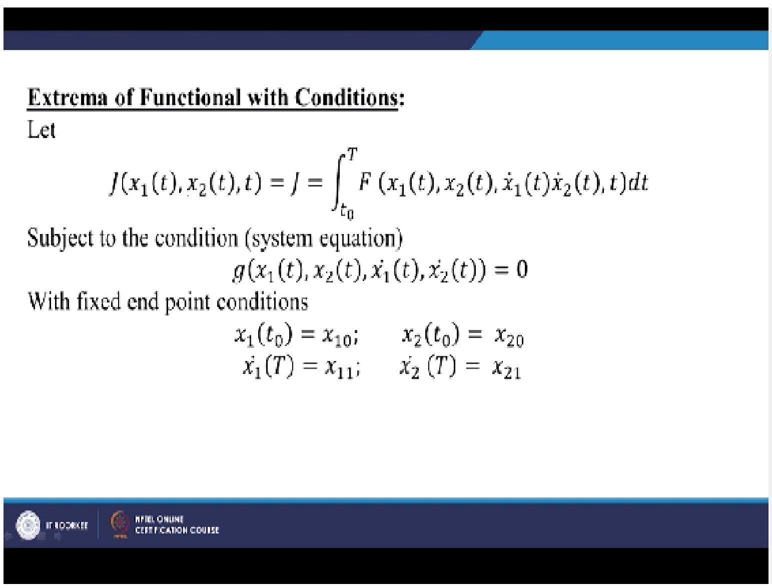

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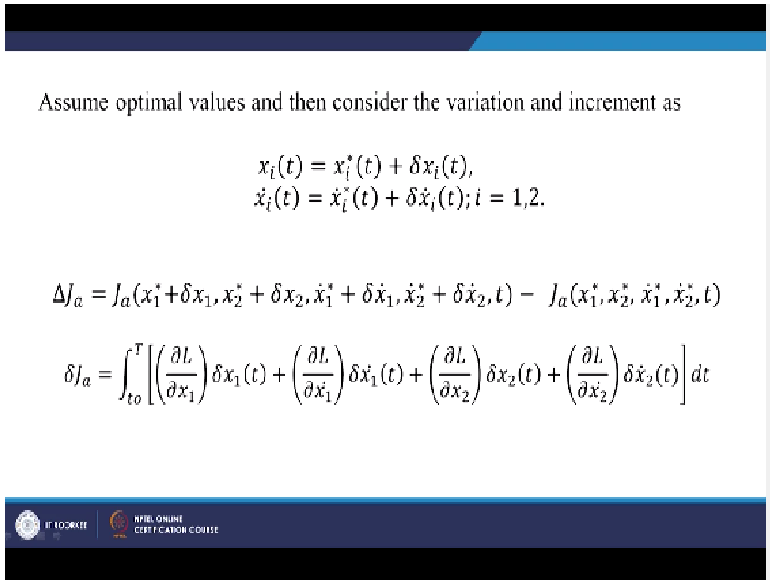

So to generalize the previous procedure, we take the incremental function, that is x1* is the required function and then if you add a variational function delta xi along with that, we get the function variation of x*. Substituting the variation and then subtracting with the original function Ja and taking only the first order terms in this way, we get the equation exactly similar to the previous one.

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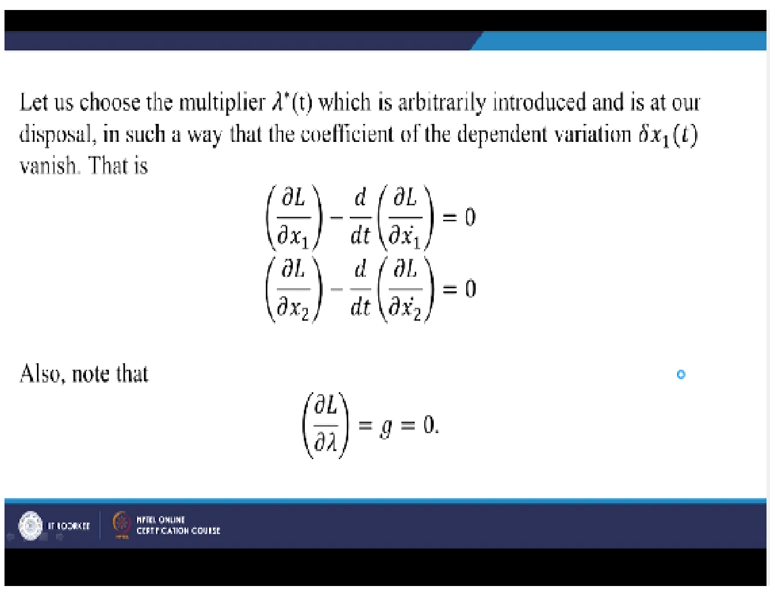

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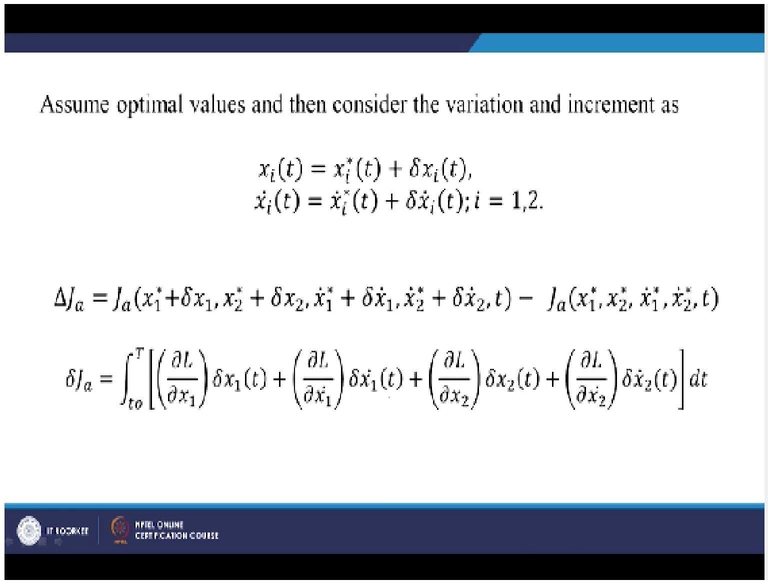

And replacing the terms containing dot, del L/del x1 dot*del x1 dot, so if you use the integration by part method as done before, we get exactly similar equation. Instead of one equation, we will get 2 equations, 2 Euler's equation and the constraint function g. g means the vector g1, g2=0 is given here.

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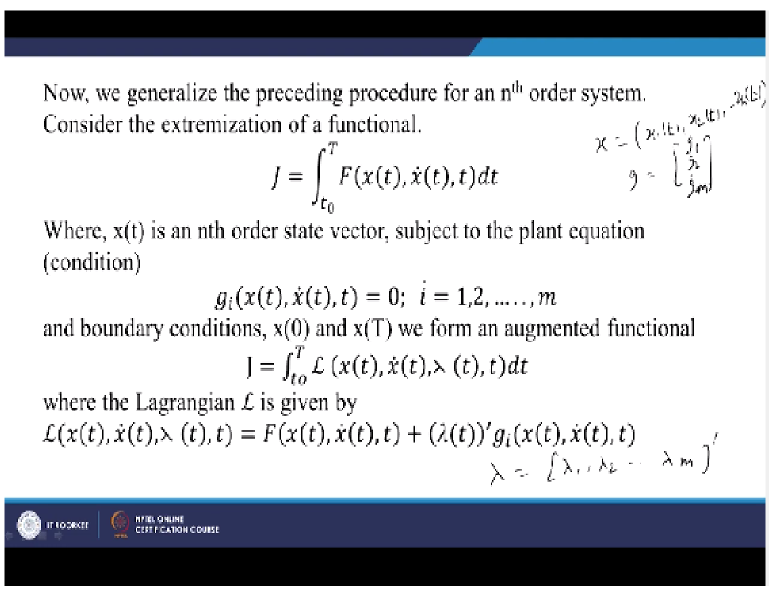

So the same thing is expand, it is extended for the general one. If your x is a vector, x1 of t, x2 of t, etc., xn of t and its derivative is given and gi, so g is g1, g2, gnm, okay. There are m constraints and n variables are given. So the procedure is exactly similar. And here, the lambda will be simply lambda 1, lambda 2, lambda m dashed which are functions of t. So we proceed in the similar manner as we did earlier.

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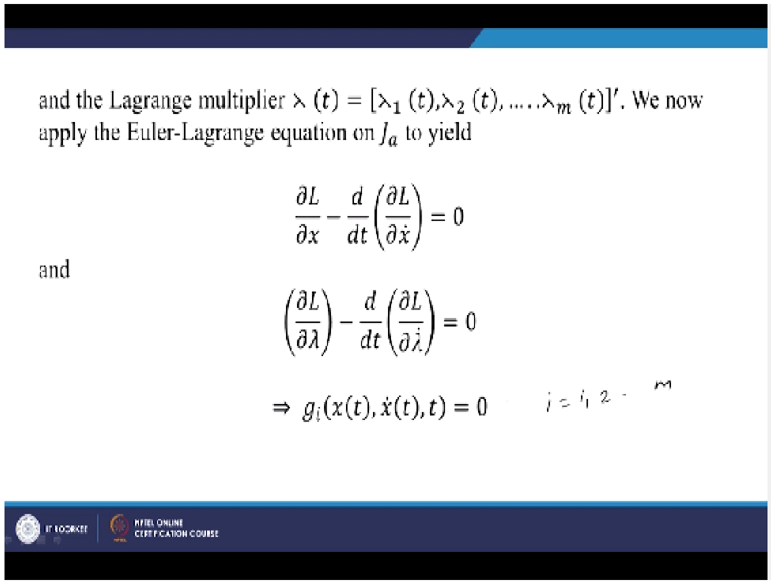

And we get the Euler's equation for the nn Euler's equation will be obtained and the m constraints are already given in the thing. And the initial conditions and final conditions are taken as in the previous case. Here we have given. so initial and final condition for 2, x1 and x2 are given. Similarly, we can take the fixed initial and final condition for n such functional. So now we will use this procedure to solve the optimal control problems as follows.

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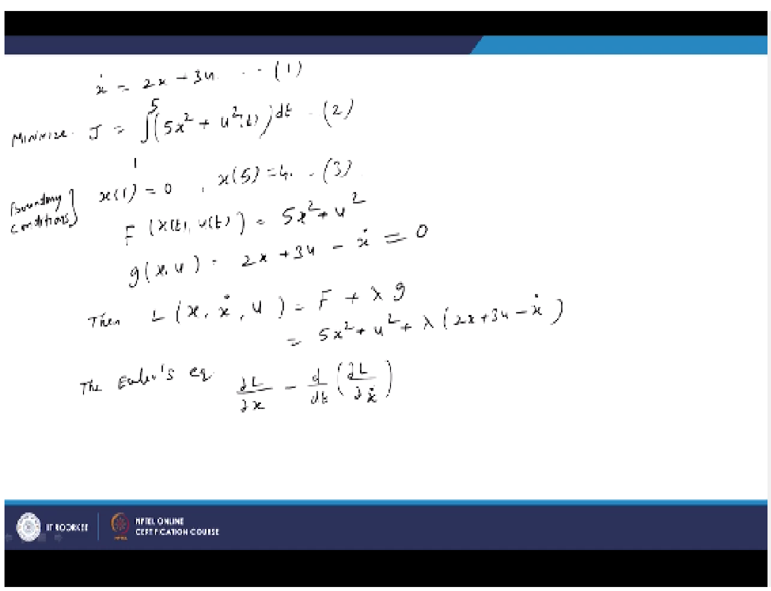

So for example if you consider the equation x dot=2x+3u and we want to minimize the functional say 1 to 5 of 5x square + u sq=, these are functions of the dt, okay. We want to minimize this expression under the condition say x of 1, these are the boundary conditions x of 1 is 0 and x of 5=4. So the equation is given. The constraint equation is given, the first one and the functional to be minimized is given and the boundary conditions are given here.

So we can construct the Euler-Lagrange equation as given here for the 1 variable case. So we have to construct L. L of xx dot lambda is given by F function, +lambda*g function. So this too we will construct here. So here, the function F of x of t and u of t is given by 5x square and u square. So this function and g of x, u; x of t u of t is given by 2x+3u-x dot. Is not it? Because g of x, u should be equal to 0 that is given from the equation 1.

Then the Lagrangian is given by, the Lagrangian is a function of x, x dot and u that is given by F+lambda*the function g. So if you take this one, we get 5x square+u square+lambda*, g function is given by 2x+3u-x dot. So the Euler's equation is, Euler-Lagrange equation is del L/del x-d/dt of del L/del x dot. See here, we note that this problem J of x1, x2 is given by integral F of x1, x2, x1 dot, x2 dot. That is to be minimized.

So now we can take in this problem x and u are there. In our statement of the problem J is 5x square+u square. So we can take this x2 function, x1 as x in the problem and x2 as the u, the control u and g of x1, x2, x1 dot, x2 dot. So x1 is replaced by x here. x2 is replaced by u. And x1 dot is replaced by x dot and u dot is not available in the equation. So we can simply write the expression like this. So g of x, u. We are writing in the place of x1, x2, we have x and u and this expression is given. x and x dot also.

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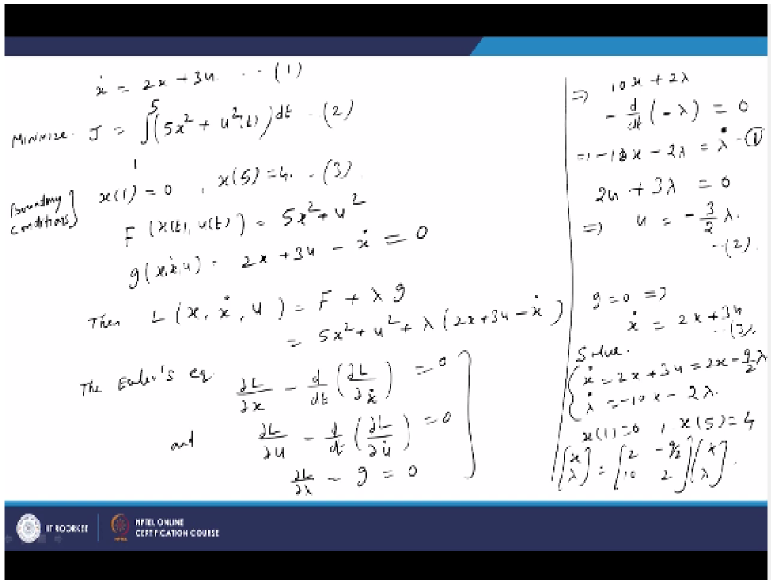

So we have this expression and we can write the Euler's equation like this with the initial and boundary condition given by 3. And del L/del x2; in the place of x2, you have u here, -d/dt of del L/del u dot. So these are the equation to be solved and we get del L/del lambda that is equal to g which is also equal to 0. So these equation to be solved. Now directly if we differentiate del L/del x, etc., we will get the equations in this form.

So this implies del L/del x from here directly if we differentiate, we will get 10x+2lambda, okay, -d/dt of del L/del x dot. There is an x dot here, so it is -lambda, =0. So from the first equation, we get this. This implies, =lambda dot, the derivative d/dt of lambda. And then del L/del u, if we differentiate with respect to u in the second equation, we get 2u and -del L/del u that is +3lambda, so that is equal to 0. So this equation implies that u=-3/2lambda.

So we get this as the first equation and this as second equation. The third equation is already given g=0 that is the given constraint. That implies x dot=2x+3u. So this is the third equation. So combining all these equation, we get; so now we have to solve the equation x dot=2x+3u from the third equation. The first equation is lambda dot=; u=-3/2lambda. If you substitute u in this expression, we will get 2x, 3/2lambda, so we will get 9/2lambda.

So we have made use of all the equation. This is the expression with the boundary condition x of 1=0 and x of 5=4. So this is the system of equation. We get x lambda. if you write in the matrix form, this equation will give 2 -9/2 and lambda dot is 10 and 2 x lambda. So we get the matrix equation in this form. By finding the exponential of this matrix, we can find the solution of this problem under the boundary condition.

Here x1 of 0=0 and x5=4. So with this we come to the conclusion of this lecture. We have demonstrated how to find the optimal control for a linear system with the fixed boundary conditions. But this procedure is not restricted to only linear system because the equation given in the procedure which we have explained earlier, the constraint is given by gi of x, x dot=0 which can be a nonlinear system also.

But in this particular example, we have taken the linear system because the solution of the linear system can be obtained analytically and the result can be obtained analytically. Otherwise, the procedure can be applied for nonlinear system also. Thank you.