Dynamical Systems and Control

Prof. N. Sukavanam
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 49
Lyapunov Stability Theory - III

Dear students. Welcome to the third lecture on Lyapunov stability theory. In the previous 2 lectures, we have seen the sufficient condition for the stability of the nonlinear dynamical system dx/dt=f of x, f of 0=0.

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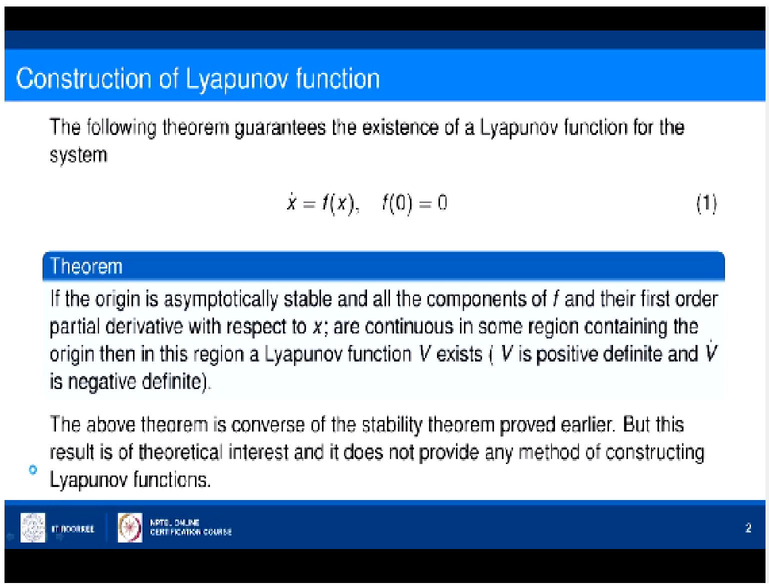

The stability at the critical point was analyzed. So in those 2 theorems, it was given that if the Lyapunov function V is positive definite and V dot is negative definite, then the system is asymptotically stable at x=0. Similarly, if V and V dot, both of them are of the same nature, positive definite or negative definite, then the system is unstable. And other theorems have been seen. The theorem which is given below, it states a converse of the theorems which we discussed earlier.

So here if the system is asymptotically stable at x=0 and if the function f is sufficiently smooth, fx is continuous and the partial derivative are continuous around the origin, in the neighbourhood of the origin, then there exist a Lyapunov function V. So this is the statement of the theorem but this theorem is of theoretical importance. We cannot, it does not propose any procedure for computing the Lyapunov function as it is. So in this lecture, we will see 2 procedures for computing the Lyapunov functions.

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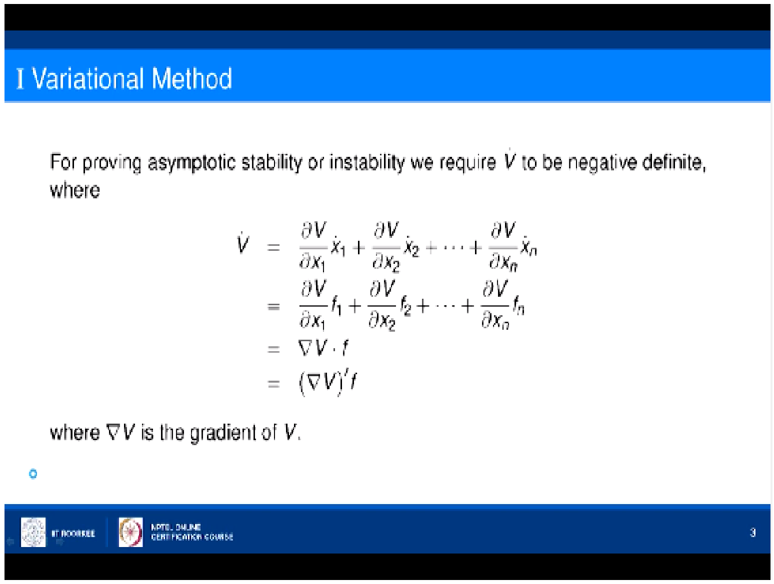

So for computing Lyapunov function, we require the derivative with respect to t which is defined like this V dot=del V/del x1*x1 dot + etc. and x1 dot x2 dot, etc. are the dynamical system equation. So x1 dot is replaced by f1, x2 dot is replaced by f2, etc. So ultimately we get dV/dt=del V dot f, the vector notation which is, where del V is the gradient of the function V.

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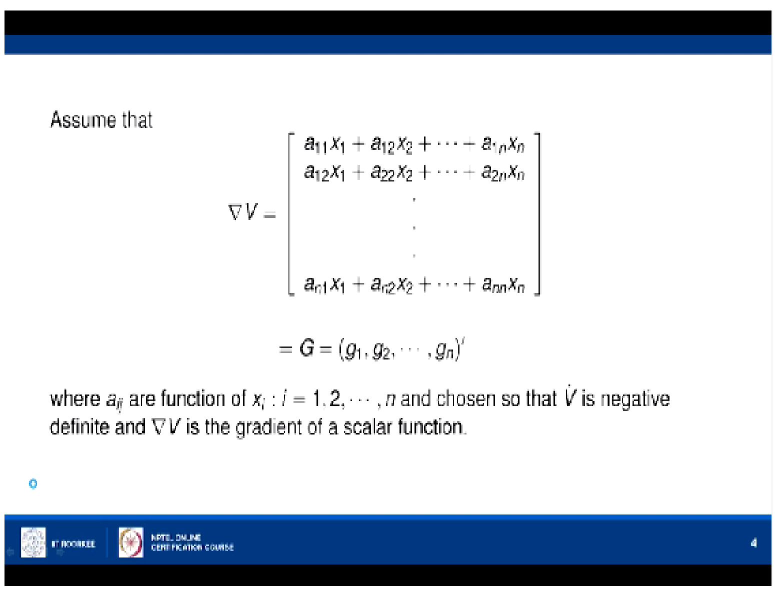

So now our aim is to find a Lyapunov function V. So for this, we just assume the following. Let us assume that del V is in this particular form because del V is a vector. We say the first component is a11x1+a12x2, etc. And the nth component is an1x1+an2x2, etc. Where aij are also functions of xi, okay. They are not constants here. We can assume them to be functions of xi, i=1, 2, 3 up to n. Now this aij must be chosen in such a way that the vector given here is a gradient vector and V dot should be negative definite.

These 2 conditions should be satisfied. Accordingly, we should select the aij functions. So if del V is the gradient vector of a scalar function, then it should satisfy this condition. It is a familiar result because del*del V should be equal to 0 and the component of the cross product should be 0 here. So it satisfies the condition del gi/del xj should be equal to del gj/del xi, and i not equal to j.

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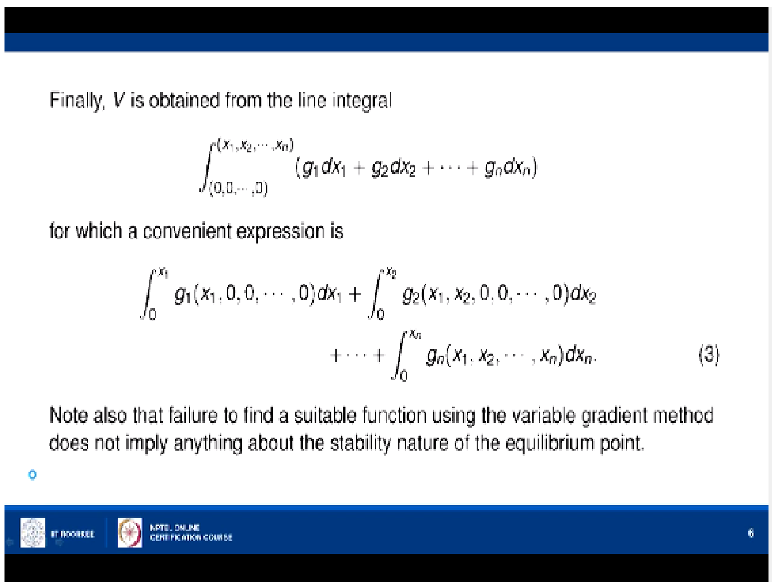

So with this condition and the requirement that V dot should be negative definite, we should select the functions aij. So that is the procedure. So to find the function V, we can follow this particular procedure because g1dx1+g2dx2, etc., it is nothing but dV/dt. And if you integrate dV/dt with respect to t, we get the function V itself. So we can just integrate the function which is given inside the bracket and get the function V of x. So a procedure is given in the equation 3. We can follow this procedure and get this one.

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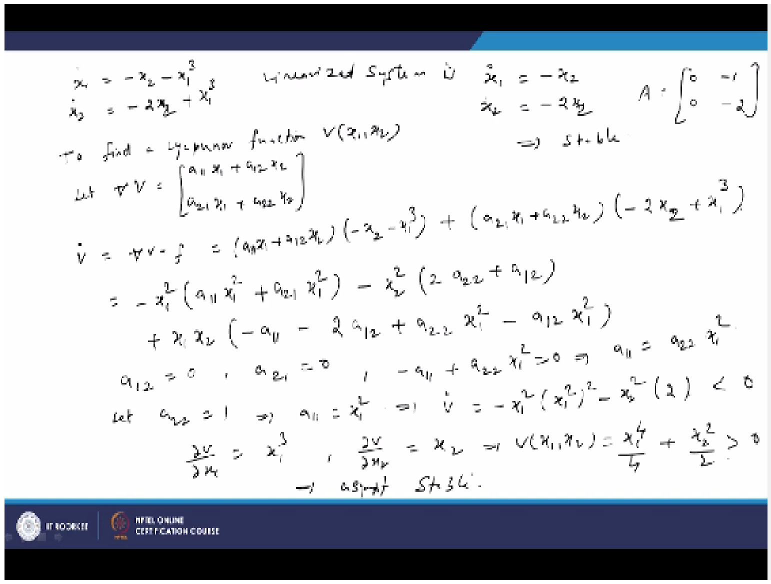

So to see an example of this procedure, let us consider the following. So we consider the equation x1 dot=-x2-x1 cube and x2 dot=-2x1+x1 cube. So here there is a linearization method in the study of stability theory if the right hand side is nonlinear, we linearize it and then analyze the stability of the nonlinear system. The linear system is asymptotically stable, then under some condition, the nonlinear system is also asymptotically stable.

Similarly, if the linear system is unstable, the nonlinear system is also unstable. But if the linear system is simply stable, we cannot say anything about the nonlinear system. So that was the theorem in the previous lectures by Professor Pandey, it was done. But here, now if you linearize this system, what we will get is? The nonlinear terms will go away. So the linear equation, linearized system is simply x1 dot=-x2, x2 dot=-2x2.

So if we convert it into the matrix form, we will get 0 -1 and 0 -2. And we can easily see that the eigenvalues of the matrix are 0 and -2. One of the eigenvalue is 0 here. So this implies the linear system is only stable. So if the linear system is stable, we cannot guarantee anything about the nonlinear system. It may be asymptotically stable or it may be unstable also. So we cannot come to any conclusion about the nonlinear system using this linear system.

So we can use this particular procedure which we have tried just now. So we want to find a Lyapunov function V of x1, x2. So for this, we assume that the gradient of this function is a11x1+a12x2 a21x1+a22x2 where aij are functions of x1 and x2. Now this has to be the gradient of a function. So now if you calculate V dot, that is dV/dt=del V dot f where f is the right hand side of the differential system.

So that is nothing but a11x1+a12x2 that is the first component multiplied by the first component of the equation-x2-x1 cube, then +, the second component is a21x1+a22x2 and this f2 is -2x2, sorry, +x1 cube. So that is V dot. We want that V dot should be negative definite as well as del V should be a gradient function. Arbitrary aij will not work here. So we have to select it suitably. If you collect the coefficient of these things, let us say x1 square we take the coefficient, it is equal to a11x1 square+a21x1 square again.

And then we take x2 square-x2 square and its coefficient, you will get 2a22+a12 and let us take the remaining x1x2, coefficient will be -a11-2a12+a22x1 square-a12x1 square. So if you take, after expanding all this, we collect the coefficient in this particular way because we want V dot to be negative definite. The bracket, the first bracket should be a positive value. Second bracket should be a positive.

And if you make the third bracket to be 0, then it will become a negative definite function. So one choice is we can write a12=0 and a21=0, so that the first equation will become a11x1 square. So we should take a11 to be a positive number. Similarly, a22 should be positive so that the first 2 terms becomes a negative definite function. Now we want to make this bracket to be 0. So for that, we should select -a11. a12 and a21, both of them are 0, so the remaining is a22x1 square, should be equal to 0.

So this implies we get a11 is nothing but a22x1 square. So if we arbitrarily select a22 some number 1, then we will get a11 from this equation it is x1 square. Now substitute all these values in the V dot equation. So V dot=-x1 square and a11 is x1 square again. So it is -x2 square and a22 is simply 1, it is 2. And x1, x2 bracket is 0, so only this number, it is strictly less than 0, negative definite function, we got the value V dot.

Now from here if you calculate V, that will be the gradient del V/del x1 that is the first value. It is a11x1. a11 is x1 square*x1, that is x1 cube. a12 is 0, so del V/del x1 is nothing but x1 cube and del V/del x2, here a21 is 0, a22 is 1, so it is simply x2. Now if you integrate with respect to x1, the first value, so this implies V of x1x2, if you integrate the first one, we will get x1 to the power 4/4.

And integrate the second with respect to x2, it is x2 square/2. So we can conclude from this, we will get del V/del x1 is x1 cube and del V/del x2 is x2 only. So this implies V is positive definite. V dot is negative definite. So this given system is asymptotically stable using this particular procedure. So this is not the only way.

There are infinitely many ways of finding this aij in this particular manner. If you are not able to find any such aij, it does not mean anything about the stability of the system. There may be some other procedure through which we can find this thing. So immediately we cannot conclude, just because this method is not working, the system is not stable or any conclusion cannot be drawn.

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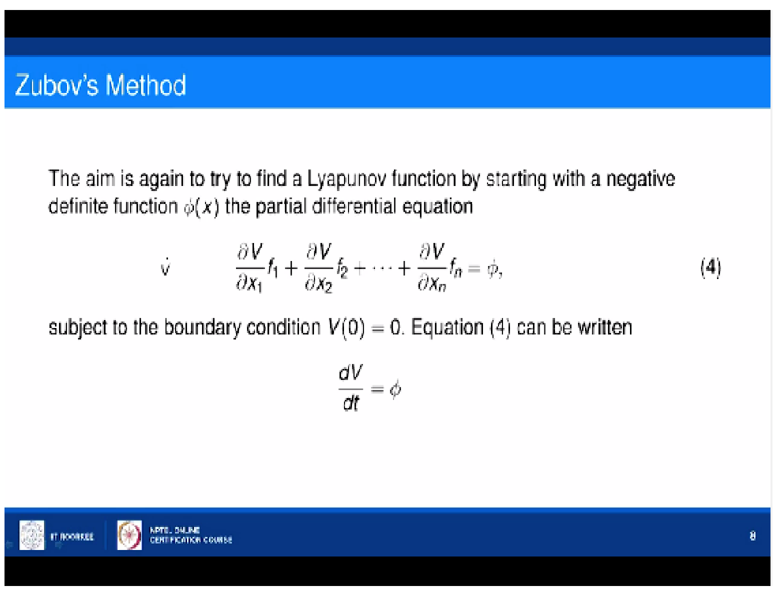

Now we will see the second method, the Zubov's method. So in this method, it is also straight forward. First thing is we want that V dot should be negative definite and V is positive definite. So that is for asymptotically stability. And similarly, if V dot is negative definite, V also becomes negative definite, then the system is unstable. So we will try to find some function V satisfying any of these conditions.

So this equation is nothing but V dot. del V/del x1*x1 dot that is equal to f1 etc. is given. We assume that V dot=phi, any function which we can select it to be less than 0, that is negative definite. We just select any negative definite function of xi and try put it in the right hand side. So dV/dt=phi is the equation which we have.

It is a partial differential equation and we can solve with using the LaGrange method in the partial differential equation and get the function V. And if V turns out to be positive; after solving this equation, if V turns out to be positive definite, then we can immediately conclude that the system is asymptotically stable. So that is the procedure here.

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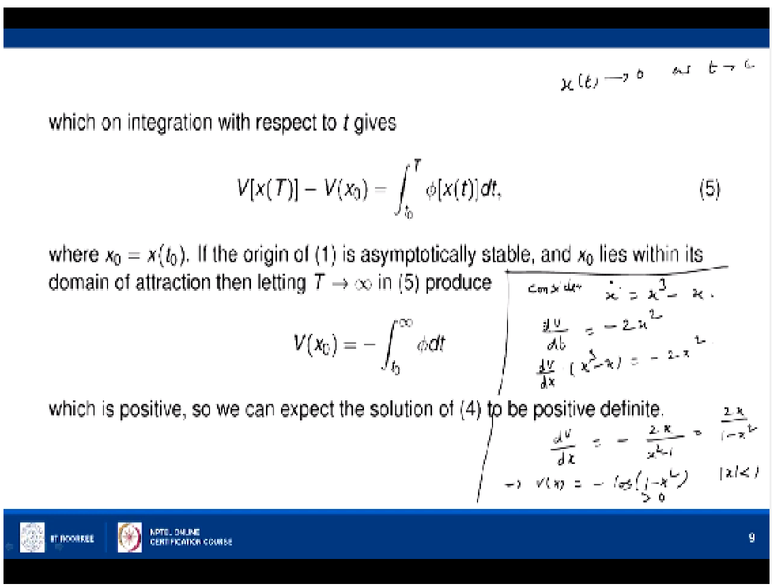

See if V is, if the system is asymptotically stable, then we know that x of t tend to 0 as t tends to infinity. So from the last step, that is dV/dt=phi, we can integrate it from 0 to some T. We will get V of x of T-V of x of 0, the initial condition=the integral t0 to T phi of t and as t tends to infinity, x of t will tend to 0 and we want that V of 0 should be equal to 0.

So we can write that as t tends to infinity, this becomes 0 to infinity phi of x of tdt and this becomes -V of x0 and this becomes 0 because V of 0 should be 0. So we get this particular expression. And from here, we can get the value of the, it is a positive definite because -phi, phi is negative definite, so -phi is positive definite and this expression is a positive definite function.

From here, we can obtain this thing, okay.

So we can make use of this particular procedure to find here Lyapunov function in the following way. So let us try to find the following equation. For example, if you consider the equation V dot, the simple equation, x dot=x cube-x. So to analyze, this is a 1 variable function. So obviously it can be easily analyzed and this can be checked to be asymptotically stable at x=0. But using this particular procedure, we will write the equation dV/dt by making use of this particular, V dot=phi.

So let us assume any negative function. Let us say -2x square. It is arbitrary choice. It is always less than 0 except at x=0. Now if you solve this equation, this can be written as dV/dx and dx/dt, that is equal to x cube-x, that is equal to -2x square. So this implies that dV/dx=-2x/x square-1 or it is 2x/1-x square. So if you integrate both sides with respect to x, we will get this to be -log of 1-x square. And in the interval, mod x<1, this is valid.

The Lyapunov function, it is always positive value because log of the number<1 is negative and there is a negative sign. It is always a positive value and V dot is negative definite, it is we have already selected. So this implies that we are able to get a V and V dot as we wish and the theorem for asymptotically stable condition. So this implies that the system is asymptotically stable. So this is our second method and if we have more than 1 variable, we will get a partial differential equation like this and it has to be solved using the LaGrange method.

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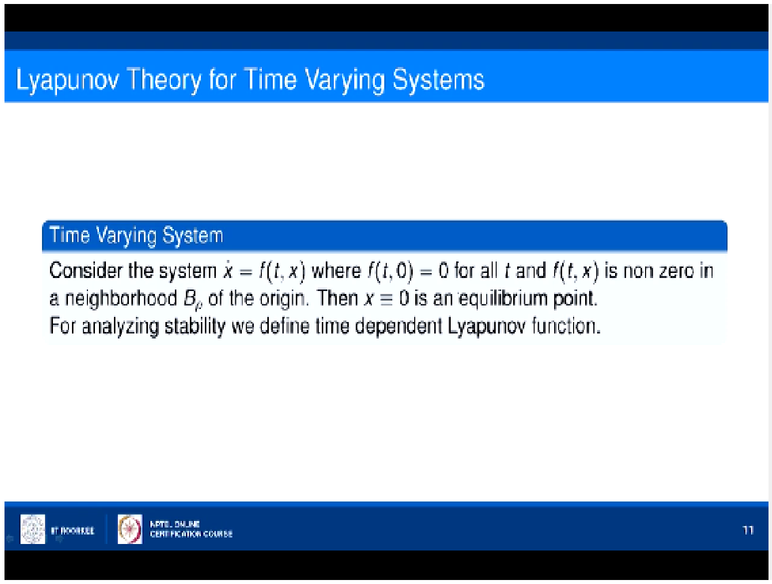

Now so far we have seen the Lyapunov theory for time invariant system, that is autonomous system. Now for the non-autonomous system, we consider the system x dot=f of t, x where t appears explicitly in the dynamical system and f is such that f of t, 0=0 for all t. So in this case, if this implies that x=0 is the equilibrium point, f should satisfy the condition f of t, 0=0 as well as f of t, x is non-0 in a neighbourhood of the origin.

So then x=0 is the isolated equilibrium point and its stability can be analyzed by Lyapunov theory. So for the time varying system, the Lyapunov function also should be a time dependent function.

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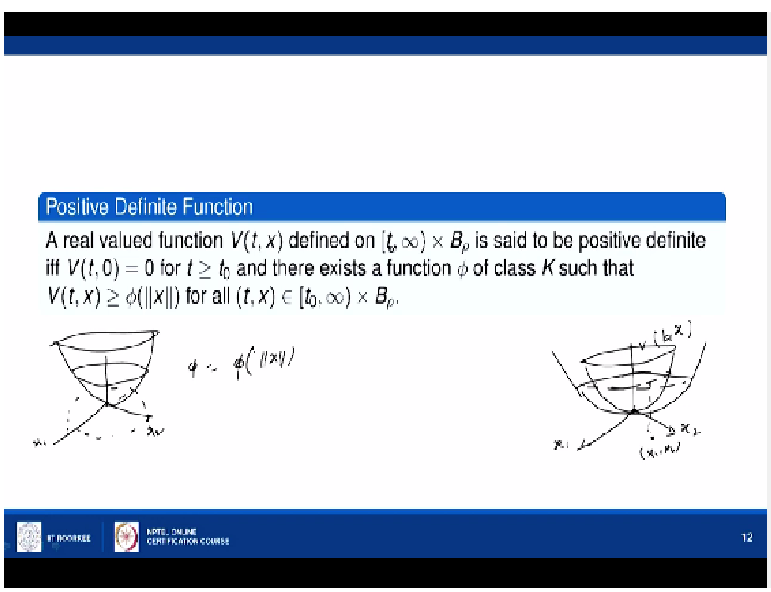

So that has to be defined like this. So for doing that, first we define what is the meaning of positive definite function for a function depending on t also. Vt, x is defined on the interval t0 t0, some initial condition, to infinity*the B rho, the neighbourhood of the origin. So for example in 2 variable, let us say x means x1 x2 in R2. The value of the function V of t, x, x1 x2. So in that case, for each t, if you fix one particular t, then we will get the surface something like this.

If V is positive definite function, it will be 0 at the point 0, 0 and for any x1 x2, the value of the function V is a positive value. So we will get a surface. Now if you change the value of t, then we will get another surface. So in the case of time invariant system, we got V as a single surface. But here, the surfaces keep on changing as t changes. So it is not only 1 surface. A continuous change of surfaces we will get.

Now if it is positive definite function, then there will be a function of class K. We have seen function of class K in the previous lecture. So we will find, so a function of class K if we plot the graph of this in 2 variable, phi of norm of x, so this is the value of the function. That means if norm of x is constant, if you take a circle of radius r, its norm of x=r, then the phi value for all this value will be the same.

So it means it will be a symmetric type of surface. So the function of class K, if you draw the surface, we will get in this particular form where the level curves or simply circles always. Here if you take level curves, they may not be circle. It may be ellipse or any kind of closed curve available for the surface V. But here, it will be always a circle.

So if V is a positive definite function, then we will be able to find a surface of class K, a function of class K like this and which will be like below that one, okay. Always for any x1 x2, Vt, x1, x2 will be greater than or equal to this phi of this x1 x2 at the same point. So we will be able to find a surface of class K in this particular way. Then only it is called a positive definite function. So let us see some examples of this thing.

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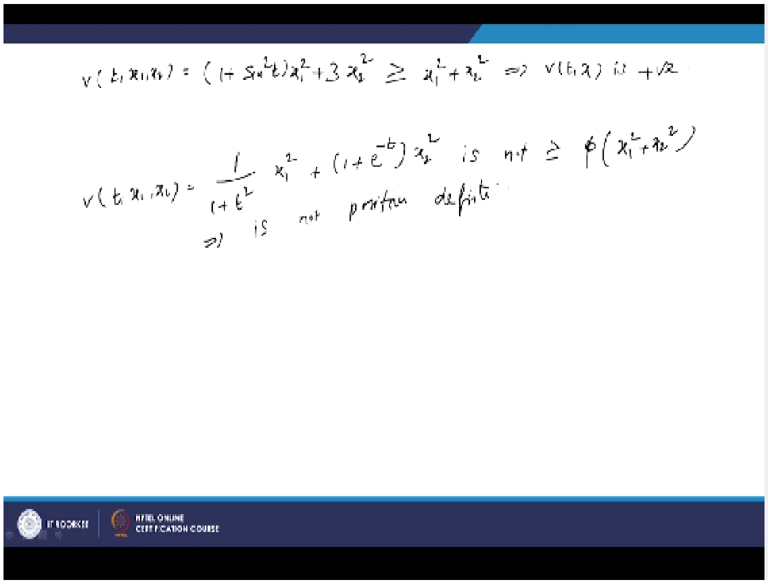

So if you take Vt, x1, x2, it is equal to 1+sin square tx1 square + let us say 3x2 square. So it is always positive, both the terms are positive, and it is 0 only at x1, x2 are 0, okay. So it represent a positive definite function because we will be able to find a surface like this. It is greater than or equal to, the first term is always greater than or equal to 1 and the second coefficient is also greater than or equal to 1.

We can say that it is greater than or equal to x1 square+x2 square, this particular expression. So this implies Vt, x is positive definite function. Similarly, we can find, let us say Vt, x1, x2, if we take this to be; in the second example, we take V of t, x1, x2 is 1/1+t square x1 square+, say, 1+e to the power -tx2 square. So if you observe here, it is also positive, always for any value of t and any value of x1, x2, both of them are positive terms and it is 0 at x1 x2=0.

But here we cannot find any surface, there exist no surface so that it is greater than or equal to some phi of x1 square+x2 square. Is not it? It is not greater than or equal to phi of x1 square+x2 square. This type of expression, we will not be able to get because whatever surface we take, as t tends to infinity, this will become smaller and smaller. The first term becomes smaller and smaller.

So it cannot be greater than or equal to any particular quantity, fixed quantity. So this implies it is not positive definite function. Even though the expression shows that it is always positive, it is not a positive definite function.

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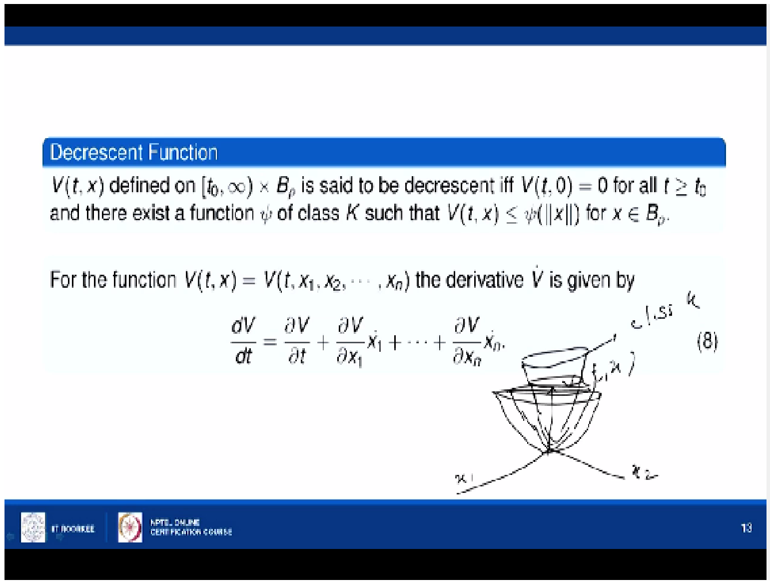

So now we define the next one, the decrescent function. The function is called a decrescent if Vt, 0 is 0. So at the origin, the value is 0 for all t, that is similar to the positive definite thing but it should satisfy the condition like this. So here we have seen that there should be a surface of class K below the surface defined by V. Now for decrescent, we should find a surface of class K above the surface V.

So that is the difference between the positive definite and decrescent. So decrescent means, if you have 2 variables, x1, x2 and this is V of t, x1, x2. So if you have a surface V, something like this, then it is keep on changing with respect to t. For 1 particular t, this is the surface. For some other t, it will be some other surface. You may get a band of surfaces, okay.

So for different t, you may get different surfaces, but all the surface from 0 to infinity should be within a band and you should be able to get another surface of class K, this is, that means the level surfaces should be circle type of thing. So that should lie above all the surfaces Vt, x. So Vt, x should be less than or equal to xi of norm of x where xi denotes a function of class K. So it should be like this. In that case, you can say that the function is a decrescent function. So for example, let us take this example.

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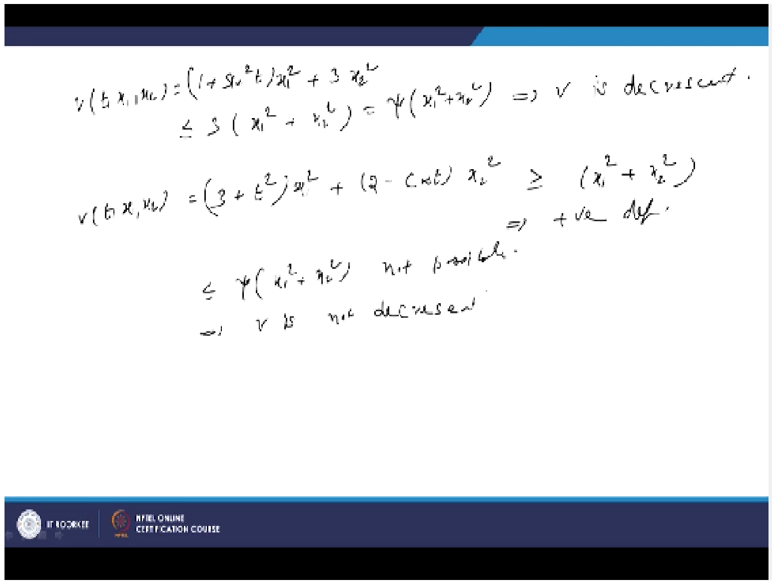

Vt, x1, x2, so first we have seen that it is 1+sin square t*x1 square+3x2 square, as we have seen it is a positive definite function. Now it is also decrescent because it is less than or equal to, first we have written that it is greater than or equal to x1 square+x2 square. Now it is always less than or equal to, we can say, 3*x1 square+x2 square because 1+sin square t is always less than or equal to 2.

So definitely less than or equal to 3 and this is also 3 here. So it is the surface xi of x1 square+x2 square. So this is what we get. So this implies decrescent. Now if we consider this surface, if it is equal to, let us say 3+t square, so both of them are positive expressions, always they are all positive. So this is positive definite because you can write it as greater than or equal to x1 square+x2 square. Because here cos t can be 0 and it can be -1, etc.

So this will be always greater than 1, this expression, greater than or equal to 2, okay. And this is always greater than or equal to 3 for t>0. So this and this, both of them are greater than or equal to x1 square+x2 square, this is the function phi. So this implies positive definite, okay. But we cannot find any surface such that it is less than or equal to xi of x1 square+x2 square. It is not possible. Because as t tends to infinity, this coefficient keep on growing towards infinity. So it cannot be bounded by any such function of class K, okay. So it is not decrescent.

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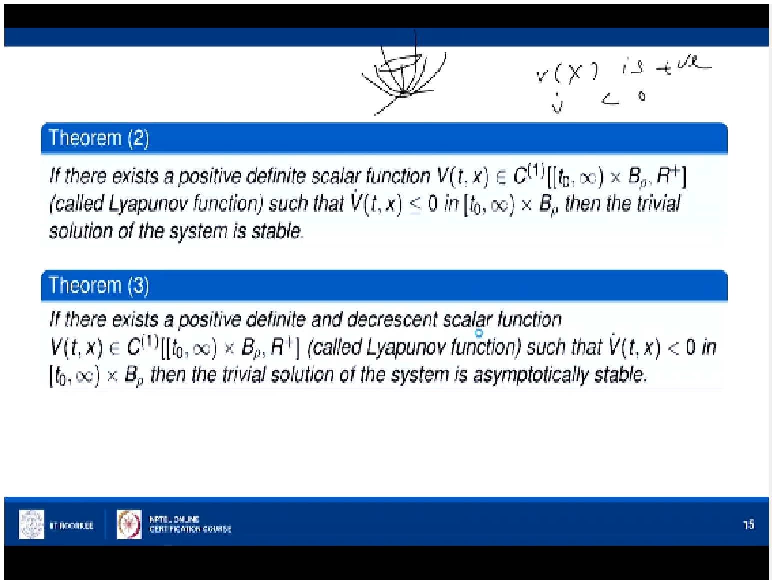

Now the theorem on the stability of the time varying system is like this. If you are able to find a positive definite function V and V dot is negative semidefinite, then the system is stable. So the theorem and proof, both of them are exactly similar to that of the time invariant case. All the steps which we have seen in the previous lecture will hold good for this theorem.

Similarly, if the function Vt, x is positive definite as well as decrescent, okay, these 2 conditions are to be satisfied and if V dot is negative definite, then the system is asymptotically stable. Here we are saying that positive definite as well as decrescent because the same condition is true in the case of time invariant also. Earlier we have seen that V of x is positive definite, that is what we have seen for, and V dot is negative definite, then it is asymptotically stable.

But for the time invariant case, if a function V of x is positive definite, automatically it is decrescent also because we can see that if a surface, because it is a single surface. We can find a surface below that as well as above that, both can be easily found out. Therefore, it is positive definite as well as decrescent in the time invariant case. But in the time variant case, we have seen example. It may be positive definite but it may not be decrescent.

So we need these 2 conditions to be specified for the asymptotically stable of the system and the proof of the same, without t also holds good here. All the steps which we have shown in the previous lecture can be applied to this also. Because in the time invariant case as we have seen, it is automatically decrescent. So whereever the decrescent step is used for that purpose, we have to mention here separately that the system is decrescent, okay.

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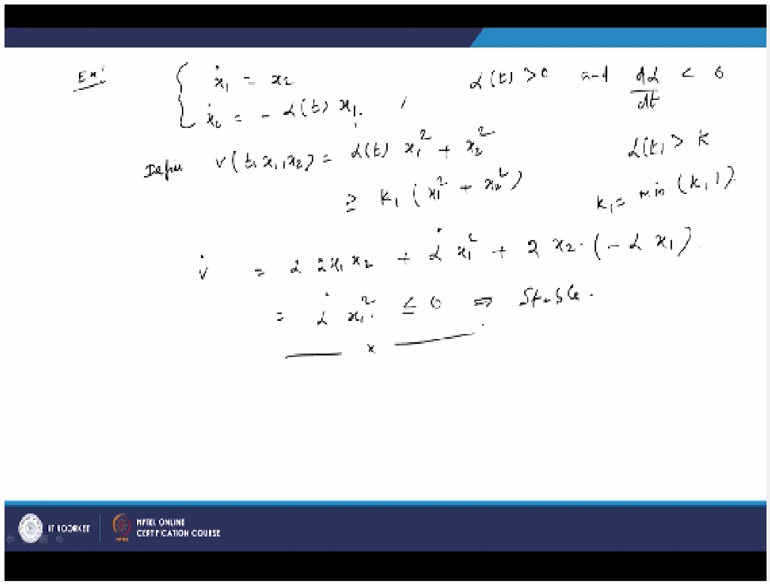

So we will see here, for example, if we take x1 dot=x2 and x2 dot is -alpha of t and x1. So if you take this system, it is a time varying system because the coefficient is a function of time. And 0, 0 is the critical point or the equilibrium point of the system. So in this case, define the Lyapunov function as this one, alpha of t*x1 square+x2 square. So if you define this Lyapunov function, it is given that alpha of t is positive definite, greater than 0 and d alpha/dt<0.

So under these 2 conditions, because this type of system occur in practical situation under this coefficient alpha to be, because if you take this, it is nothing but the harmonic oscillator in the physical situation. If you write it as a second order differential equation, this equation is nothing but the harmonic oscillator. So it is a practical problem in which this alpha should satisfy these conditions, okay.

So now we can take the Lyapunov function V to be like this. So it is positive definite function and because alpha>0, there should be a value greater than some constant K, positive constant K because it is strictly greater than 0. And therefore, we can find the function, it is greater than or equal to this one. So if you take the minimum of K and 1, let us call it as some other number K1, then it is greater than or equal to K1*x1 square+x2 square.

So we are able to find a surface like this. So it is positive definite function. And we can easily prove that V dot is negative definite, del V/del x1 that is alpha*2x1*x1 dot is x2, +alpha dot, d alpha/dt*x1 square, +2x2*x2 dot is -alpha*x1. So we get here alpha dot*x1 square. This is less than or equal to 0 because x2 term is missing here. Only positive term alpha dot*, alpha dot is negative and x1 square is positive.

So it is less than or equal to 0 because the x2 is not mentioned here. So this implies the system is stable, okay, according to the result which we have seen. So we have seen some theorems and examples related the Lyapunov stability theory. So for we have seen the theorems on the stability using the Lyapunov function for the time varying and time invariant system and some examples have been seen. So in the next lecture, we will see another type of stability theory, okay. Thank you.