Dynamical Systems and Control

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Lecture – 48
Lyapunov Stability Theory - II

Dear students. Welcome to the lecture on the Lyapunov stability theory II. So in the previous lecture, we have seen 2 theorems. One on the stability and the another on the asymptotically stability. So we have seen various conditions like positive definite function, the function of class K and decreasing function, etc. So in this lecture, initially we will see some examples of such functions and then we will proceed with the theorem on the instability of a dynamical system.

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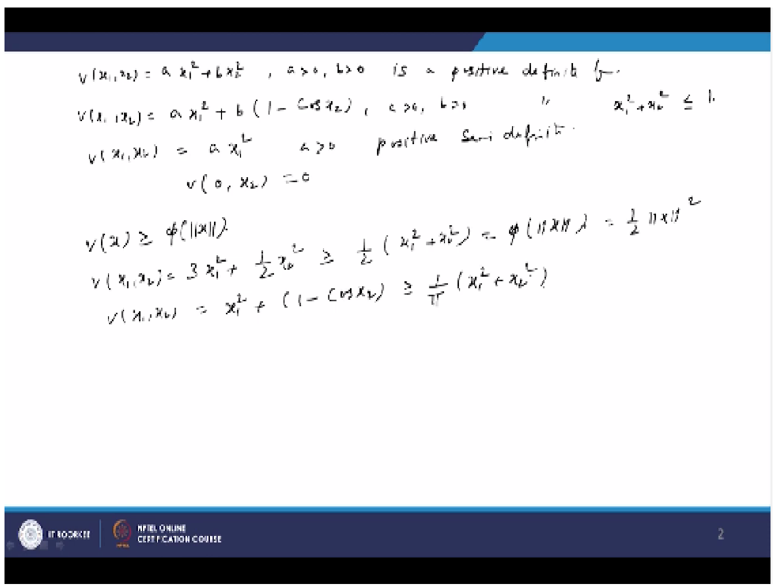

So if we consider the function V of say x1 x2=ax1 square+bx2 square where is a>0, b>0. So this is a positive definite function. So it is very clear that V of 0, 0 is 0 and it is always positive. Similarly, second example like V of x1 x2 if you say ax1 square + say some b*1-cos x2. So if you consider this example, when x1 is 0 and x2 is 0, the value is 0, b of 0, 0 is 0 and always the 2 values are positive.

If a>0, b>0, it is also positive definite function provided x2. For example, now if you consider this one, V of say x1, x2, it is equal to simply ax1 square, a is positive. So this is positive semidefinite because V of 0, any x2 is also equal to 0. For non-0 points, the function value is 0 and V of 0, 0 is also equal to 0. So it is positive semidefinite function. Now if you consider the, we have seen that if V is positive definite, then it should be greater than or equal to phi of norm of x where phi is a function of class K.

So we should show that for all these examples how to find this phi expression. So for example, if Vx1, x2 is some expression, let us say 3x1 square+1/2x2 square, so it is obviously positive definite function. And we can also see that it is greater than or equal to 1/2*x1 square+x2 square. So it is the function phi of norm of x. So which is equal to the phi of norm of x function. That is nothing but 1/2, you can say, norm x square, this x.

So this is the function. Similarly, for the function Vx1, x2 which is equal to x1 square+1-cos x2. So as we have seen, it is positive definite function in some neighbourhood. So how to find a function phi for this one. So it can be verified that it is greater than or equal to 1/pi x1 square+x2 square, which is the function phi, okay. So for any positive definite function, we can find the expression, a function of class K satisfying this condition. For the theorem 1, we consider how to find a Lyapunov function.

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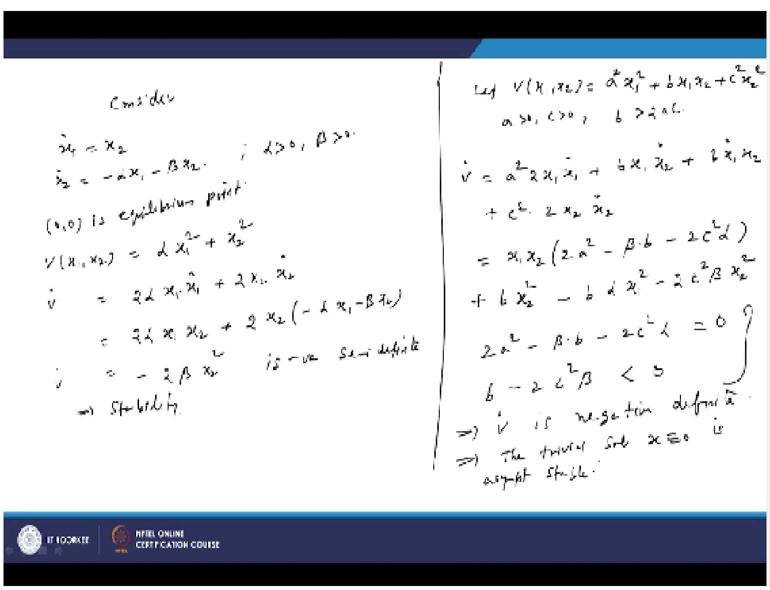

So if we consider the system x1 dot=x2 and x2 dot=-alpha*x1-beta*x2. So if we consider this system, then 0, 0 is an equilibrium point. If you equate the right hand side to 0, we get this equilibrium point. So at this equilibrium point to analyze the stability, we have to find, if possible, a Lyapunov function. So if we consider a Lyapunov function of this form, for example for this, if we consider, here alpha>0, beta>0.

So if we consider the Lyapunov function, alpha x1 square+x2 square, where alpha is given value in the system. Then it is a positive definite function and the V dot that is the derivative del V/del x1 that is 2alpha x1*x1 dot+2x2*x2 dot. So if you substitute from the equation, you will get 2alpha x1, x1 dot is x2, +2*x2, x2 dot we substitute alpha x1-beta x2. So we see that it gets cancelled, 2alpha x1 x2, remaining is -2beta x2 square.

So here it is V dot is positive semidefinite, sorry negative semidefinite. It is negative semidefinite. So this implies stability. But we can see that this system is asymptotically stable because we can find some other Lyapunov function so that it is positive definite and V dot is negative definite. So by this particular result, we should not conclude that the system is simply stable.

But if we keep trying for various types of Lyapunov function, we may conclude later that a given system is asymptotically stable also. Similarly, if you are not able to find a Lyapunov function, it does not mean that the system is unstable or we cannot conclude anything about the system just because we are not able to find a Lyapunov function. So for example, if you consider V of x1, x2 is ax1 square+, let us say a square x1 square+bx1x2+c square x2 square.

So if we take a function like this in such a way that b>2ac. So under this condition, it is clear that similarly a is positive, c is positive. So if you consider this, it is very clear that V of x1, x2 is always a positive value and V of 0, 0 is 0. So it is a positive definite function. Now by differentiating this, dV/dt, we will get the expression a square*2*x1x1 dot+bx1x2 dot+bx1 dot x2+c square 2x2x2 dot.

Now if you substitute the values of x1 dot x2 dot from the given equation, we will get the expression to be like this. If you collect the coefficient of x1, x2, we will get 2a square-beta*b-2c square alpha, then we will get +b*x2 square-b*alpha x1 square-2c square beta*x2 square. So we can see that if you select a, b, c properly, then we can make this bracket to be 0. So that is we can make -2a square b*beta-2c square alpha.

If it is 0, then x1*x2 term will become 0. And similarly, this b-2c square beta, if it is strictly less than 0, then the condition will give that V dot is negative definite. So we can easily verify that such selection is always possible. So in this case, we can conclude that the system asymptotically stable or the trivial solution x=0 is asymptotically stable according to the theorem which we have seen in the previous lecture.

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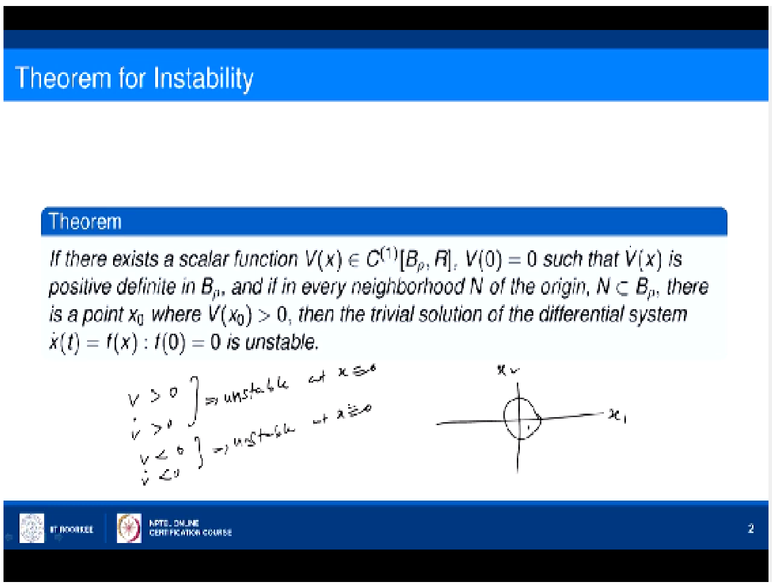

So now we will come to the following theorem on instability. So here if you can find a function V of x which is continuously differentiable in the ball of radius rho and V of 0=0 and it also satisfies dV/dt is positive definite in B rho. Moreover, if any every neighbourhood N of the origin, there exist a point x0 such that V of x0 is positive, then the trivial solution is unstable here.

So the theorem says that we consider, for example in the case of 2 variables, x is x1 x2 in R2, then in every neighbourhood, whichever neighbourhood we take however small it is, there exist at least one point x0 so that V of x0 is positive, the value is positive at the point x0. And V dot is positive definite, okay. So in that case, we conclude that the system will be unstable. So it also implies that if V is positive definite, that is for each and every point, V is positive value and V dot is also positive definite, then also it is unstable.

That is a particular case. Here more general case is if at some point in every neighbourhood, V is positive and V dot is positive definite, then the system is unstable. So the same theorem can be reversed. If V dot is negative definite and in every neighbourhood, we can find a point x0 so that V of x0 is also negative, then the system again is unstable.

So we can also say if V is positive definite, not at some point but at all the points and V dot is also positive definite, this implies the system is unstable. And if V is negative definite, V dot is also negative definite, that is also implies unstable at the critical point x, identically 0. And this is the more general statement. The theorem gives more general statement here.

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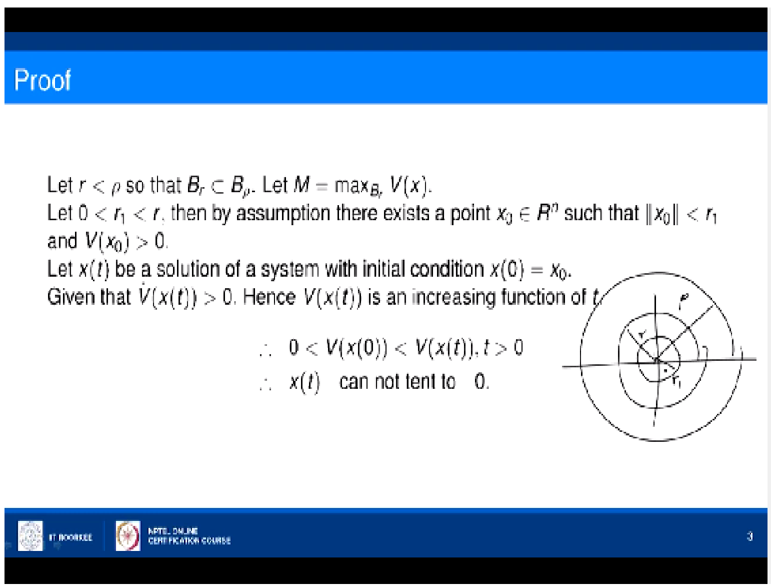

So to prove that statement, let us consider r<p and Br is subset of B rho. So if you consider a ball of radius rho and a smaller ball of radius r here. And V is we have mentioned that V is a continuously differentiable function. Therefore, it will be bounded. So there will be a maximum value for V in the ball of radius r, okay, maximum value is there. So we select another number r1 strictly less than r and take a point x0 in the ball of radius r1, smaller radius r1 as the initial condition.

So for this initial condition x0, x of t is a solution of the problem. There exists a unique solution for every initial condition. So x of t is a solution. And it is given that V dot is positive definite function, is greater than 0. Hence, V of x of t is an increasing function of t. So this dV/dt is positive. It implies that V of x of t is an increasing function of t. So if 0<t, that is for t>0, we have V of x of 0 should be less than V of x of t because of the increasing nature for all t>0.

Therefore, x of t cannot tend to 0 according to this result that V of x of 0, that is V of x0 is the value of the function V at the point x0, that is a positive value. And that is less than V of x of t, that means V of x of t is bigger than this positive value. So as t becomes larger and larger, V of x of t cannot become 0. But V is 0 only at the point 0. Therefore, x of t cannot tend to the value 0.

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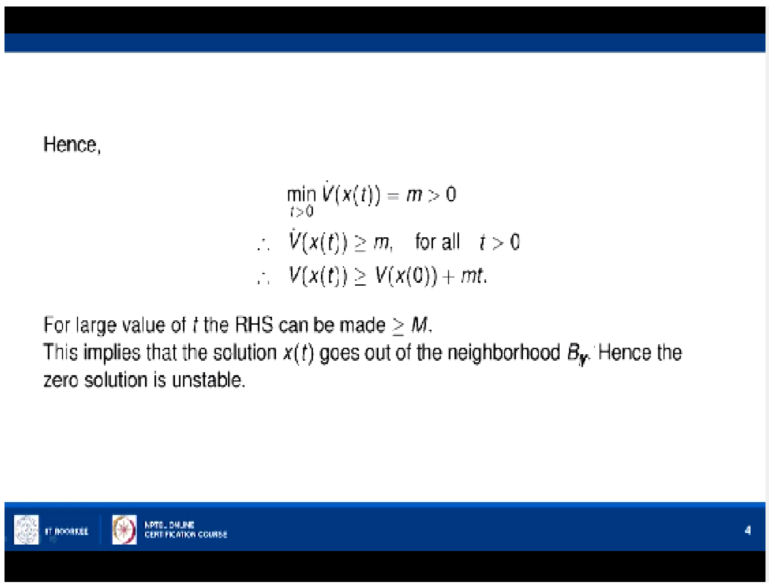

And therefore, the minimum value of the V dot x of t=m here in this case and so we can write V dot x of t is greater than or equal to m and integrating, we get V of x of t>V of x of 0+m*t. Integrating from 0 to t, we get this expression. From this for large value of t, the right hand side can be made to be greater than M where M is the maximum value which was defined here on this ball.

So it means that the, this implies that x of t goes out of the neighbourhood B rho here. The maximum is defined on Br. And because the value of V is exceeding the value M, it means that the solution x of t goes out of the neighbourhood Br. it is not rho, actually it should be corrected. It is Br. Hence the system is unstable. That is starting from initial condition, the solution goes out of the ball. Any, whatever ball we are selecting of radius r, it will be going out of that one. So the system is unstable because of the condition given here.

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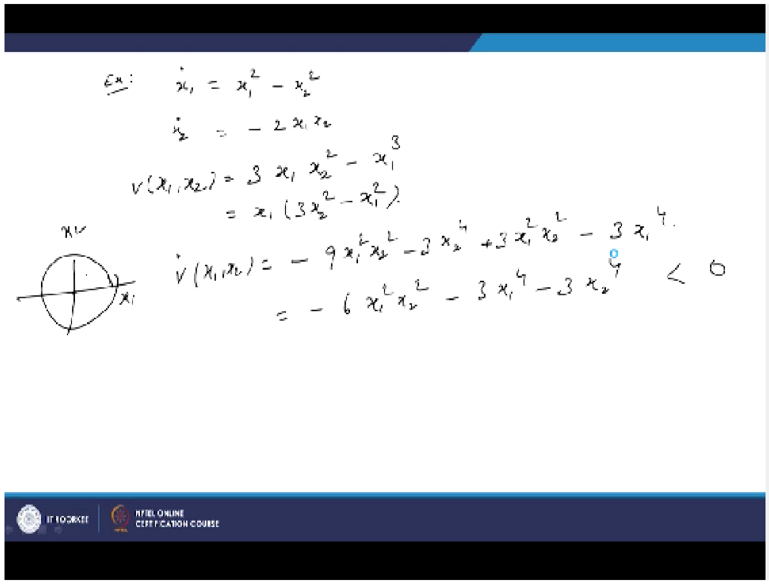

So here for example, we can consider the following. For example, let us consider x1 dot=x1 square-x2 square and x2 dot is -2x1x2. So this is a dynamical system. Now it is very clear that 0, 0 is the equilibrium point for the system. Now if you define a function Vx1x2 in this form, that is 3x1x2 square-x1 cube. So let us take this function it is given. So using this function or we can write it like this, x1*3x2 square-x1 square.

So we can see that this function has in the neighbourhood of the origin at 0, 0, if you take any neighbourhood, for points in this neighbourhood, we can see that at some point, it may be positive; at some point, it may be negative. So any neighbourhood we select, we can find some point x1, x2 for which V of x1x2 is positive or negative in anyway. Now we can easily calculate the dV/dt, V dot of x1x2. So that will give the value like this.

If you select it finally, after substituting the values of x1 dot x2 dot and after the derivative, we will get an expression like this 9x1 squarex2 square-3x2 to the power 4+3x1 square x2 square-3x1 to the power 4. So ultimately, we will get everything negative, 6x1 square x2 square-3x1 to the power 4. So this is strictly less than 0 for all the values of x1 and x2 in the neighbourhood of 0.

So this implies according to the theorem, the statement of the theorem. The statement of the theorem says that if in the neighbourhood, it is positive and V dot is also positive, the system is unstable. The reverse is if in the neighbourhood if V of x0 is negative and V dot is also negative definite, then the system is unstable. So that is the case which we are having here. In every neighbourhood, we can have a negative value for V and V dot is negative definite.

So this implies 0 is unstable, $x=0$ is unstable according to this particular theorem. So in the next lecture, we will see the stability theorem for time varying system. So far we have seen the theorems for time invariant system and now similar theorems with some modifications are available for time varying system. So that we will see in the next lecture. Thank you.