Dynamical Systems and Control
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Lecture – 47
Lyapunov Stability Theory - I

Dear students. Welcome to the lecture on Lyapunov stability theory 1. So in this lecture, we will consider 2 theorems on the stability of a dynamical system. One is on the stability of the system and the another is asymptotic stability of the dynamical system.

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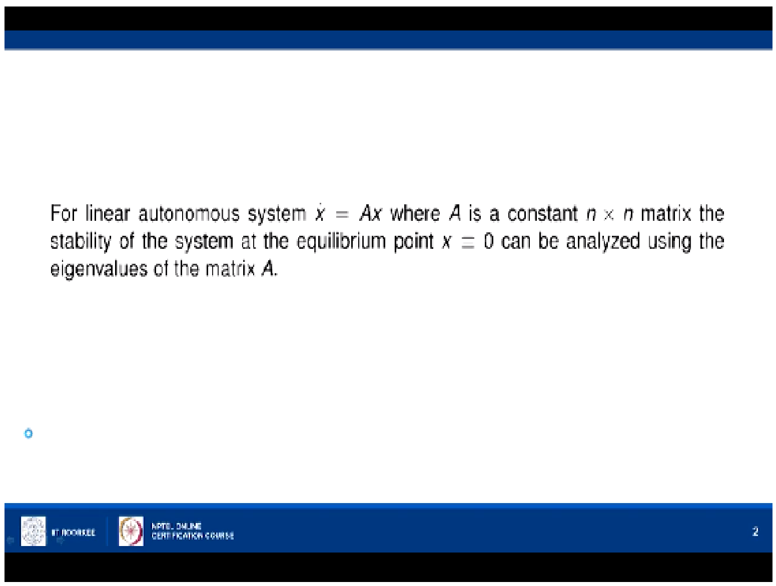

In previous lectures on stability, professor D. N. Pandey have described various theorems on the stability of dynamical systems especially on linear dynamical system. For example, if we consider the system, autonomous system dx/dt=Ax various A is n*n constant matrix. So the stability was analyzed using the eigenvalues of the matrix A. So if you recall that if all the eigenvalues have negative real part, then the system is asymptotically stable.

And even if one of the eigenvalue has positive real part, then it will be unstable. So similar such result have been analyzed earlier.

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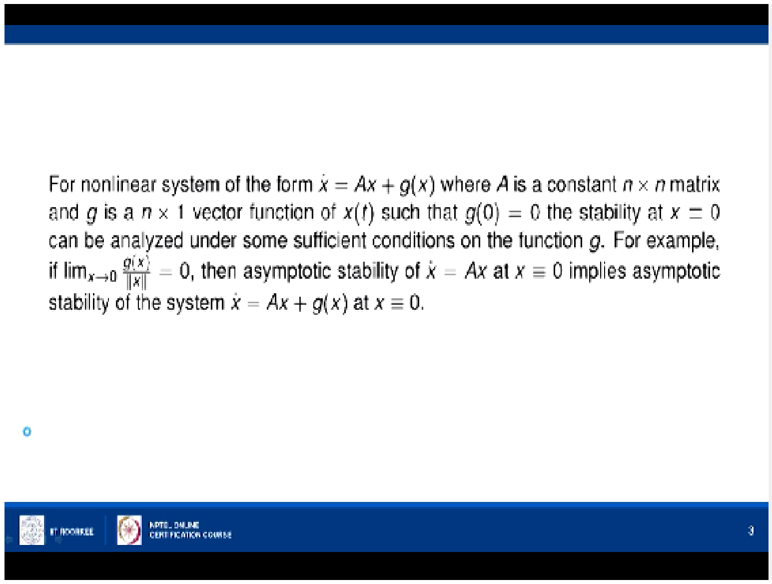

And similarly, if a system is semilinear, that is dx/dt=Ax+g of x where A is the linear part and g is the nonlinear part of the system. So if g is such that g of 0=0, then x=0 is the equilibrium point for the system. To analyze the stability of this type of systems, some sufficient conditions were imposed on the system.

For example, the condition that limit x tending to 0 of g of x/norm of x=0, if this condition is satisfied, then the system is asymptotically stable at x=0 provided the linear system x dot=Ax is asym= stable. Under the same condition, the semilinear system is unstable provided the linear system is unstable. So these 2 theorems have been already discussed.

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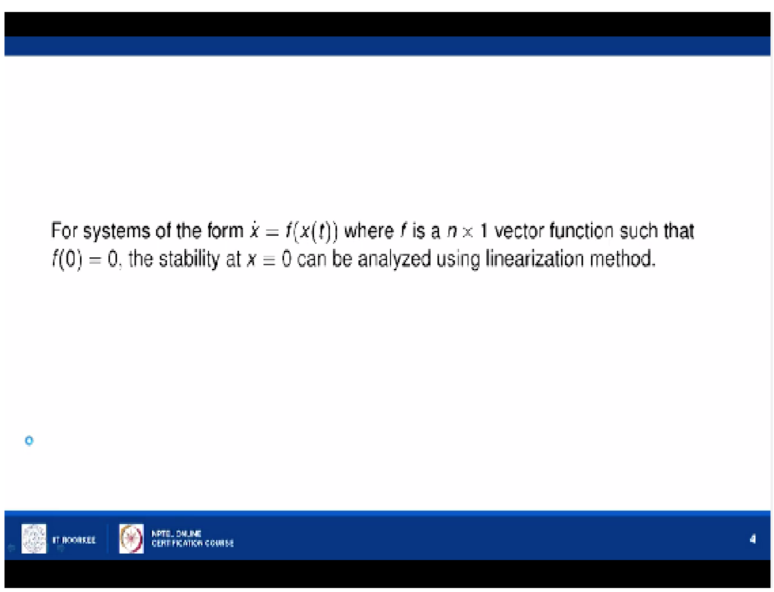

For stability of the system, also you might have studied various other types of conditions in the previous lectures.

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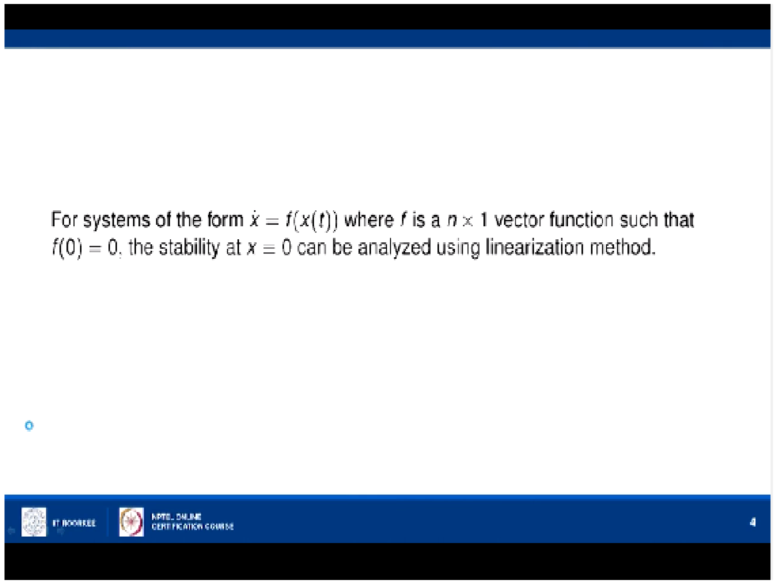

Now if the system is nonlinear, like dx/dt=f of x where f is a nonlinear vector function of n dimensional vector function and f of 0=0 is satisfied. Then x=0 is the equilibrium point of this system. To analyze the stability of this, we can linearize the nonlinear function and reduce it to the form of semilinear, one linear part + another nonlinear part using the Taylor series expansion of the function f and then again analyze the stability in a similar manner. So that is the linearization method for analyzing the stability.

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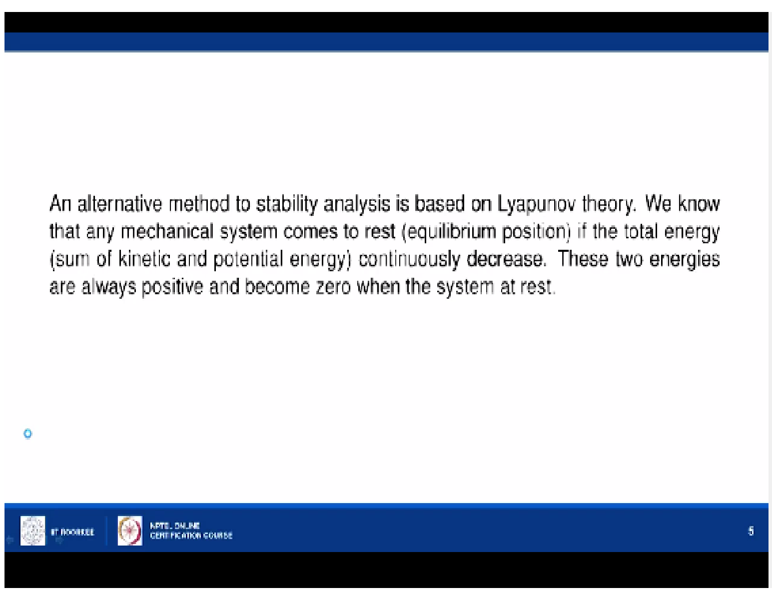

So other than this type of analysis of stability, the alternative method is the Lyapunov stability. So here may be more complicated systems can be analyzed using Lyapunov stability theory. So here, the basic idea is from the mechanical systems. Any mechanical system has 2 energies. One is the kinetic energy and potential energy. Both of them are positive values.

So if a mechanical system is in motion, then the total energy, kinetic + potential energy is positive and then when the mechanical system comes to rest, then the energy slowly decreases and comes down to 0. So this property, it indicates the stability of the mechanical system. So similarly, the Lyapunov theory defines a generalized energy function. So here, the Lyapunov function is similar to the energy function defined here.

It is always positive and then it decreases as a function of t and comes to 0 when the system becomes asymptotically stable. So it is very much similar to the energy function. So it need not resemble an exact energy function. One can define different types of Lyapunov function. But basic idea is the positivity of the given function and the decreasing nature of the function that is the derivative is negative. So these 2 properties are utilized in Lyapunov theory.

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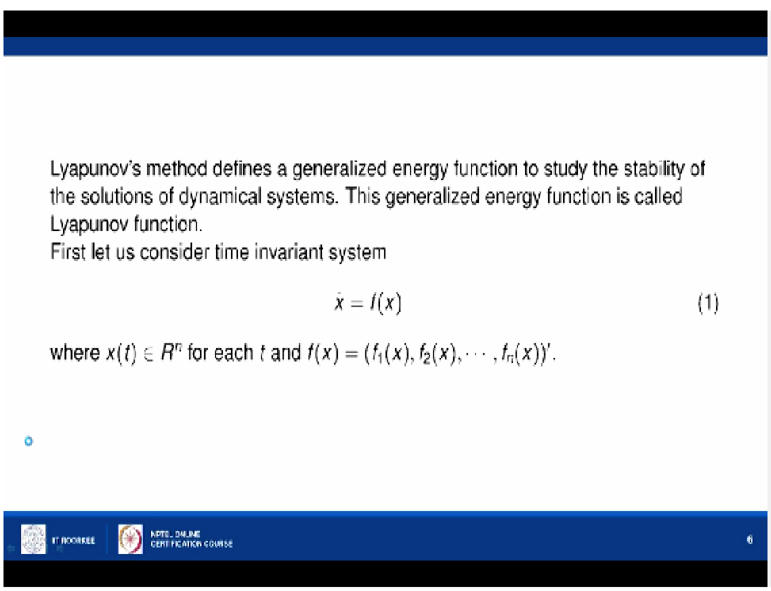

So Lyapunov method defines a generalized energy function as described in the previous slide. So let us consider the dynamical system dx/dt=f of x where f of x is given by f1x, f2x, etc. It is a column but nonlinear function fi of x is given here.

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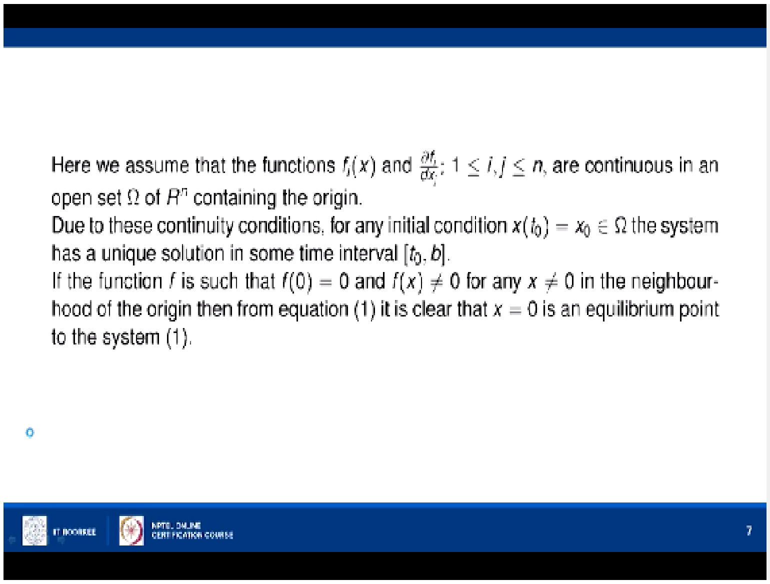

If you consider the function f of x and del fi/del xj, all these functions are continuous functions, then by the existence theory, we can conclude that the system x dot=f of x has a unique solution for any given initial condition in a suitable open set omega in Rn. So this is a standard result which we have studied in the dynamical system portion. Now if the function f satisfies f of 0=0, then x=0 is the equilibrium point of the given system.

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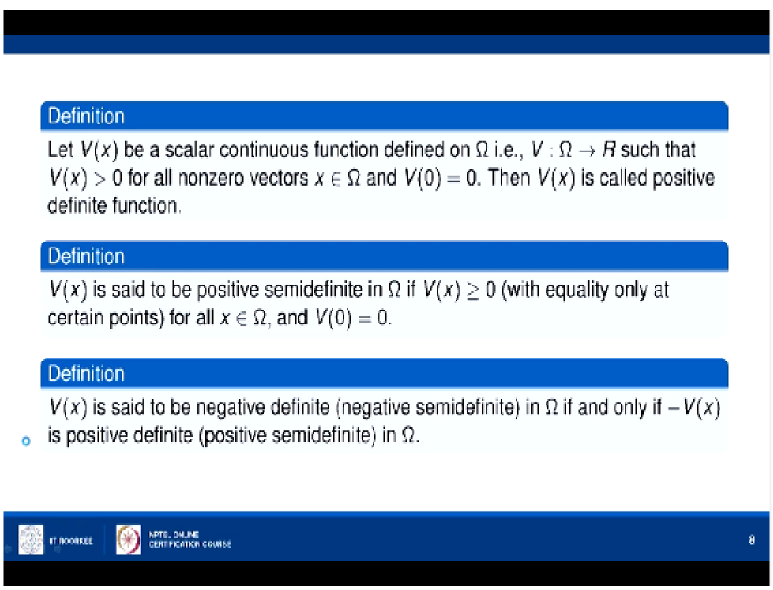

To analyze the stability, we will consider the following definitions first. And then we will define the Lyapunov function and then we prove that corresponding theorems. So first definition is continuous function. Let us consider V of x to be a continuous function and V is from omega to

R and it is such that V of x>0. It is a real valued function. V of x is positive and for all non-0 vectors, x belongs to omega and V of 0=0. Then V is called a positive definite function.

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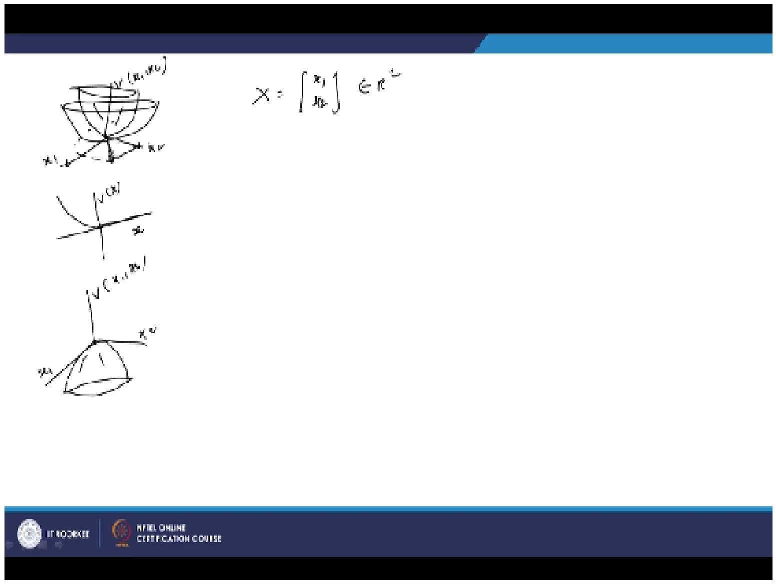

So it is like this, a positive definite function if we consider, for example in 2 dimensional setting, if we consider x=x1, x2 belongs to R2, the solution x of t has 2 components. Then if you consider this as x1, x2 and V of x1, x2 for example. So we can say that this surface represent the positive definite function. So V of x is said to be a positive definite function. If V of 0=0, that at origin, the value is 0 and for any other point x1, x2, the value of V is positive.

So this is for the 2 dimensional case. Similarly, it can be generalized for n dimensional case. And V is said to be positive semidefinite if V of x is greater than or equal to 0. And V of 0 should be equal to 0. So it means that at some places, it can also be 0. But it does not mean that V of x can be identically equal to 0. But at some points, it can be 0 and remaining points, it has to be positive.

So a surface, for example, if you have equation something like this, so this is a positive semidefinite function in 1 variable if you take x here and V of x is there, so it is positive and it also have 0, V of 0=0 and it is 0 at some non-0 places. And it is said to be negative definite or negative semidefinite if -V of x is positive definite or positive semidefinite. So if you reverse the picture, that is the bottom if we draw the picture. So this surface, so it will represent the negative definite function V of x1, x2.

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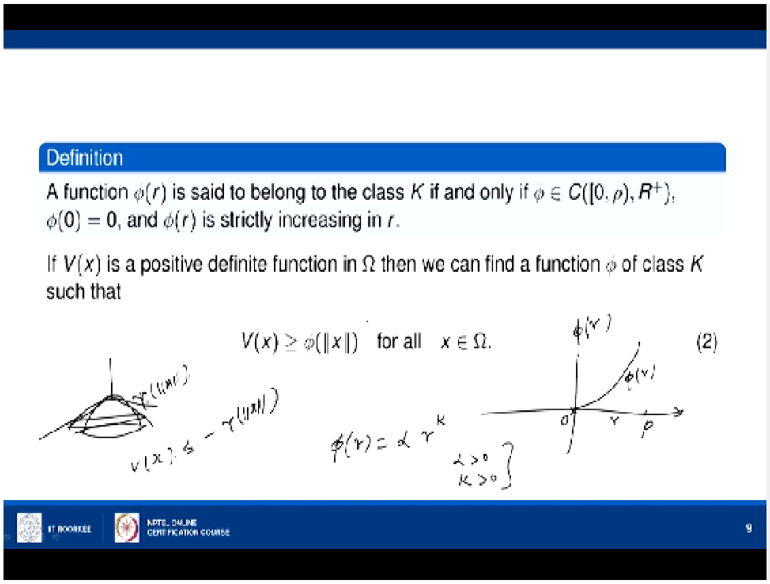

Now we define the function of class K. So a function phi of R said to be of class K if and only if phi is a continuous function in some interval. For example, the interval 0 to rho, it takes an interval in R and phi of R, this function, it is such that phi of 0=0, it is here and it is increasing, strictly increasing function. So any function of this type is phi of R is strictly increasing. It is called a function of class K.

So for example, phi of R can be any function alpha*R to the power K where alpha is positive and K is also positive. So this function, phi of R=alpha*R to the power K where alpha>0, K>0 is a function of class K because these functions are continuous functions and they are strictly increasing functions. So if V of x is a positive definite function as defined in the previous slide, then we can find a function phi of class K such that V of x>phi of norm of x.

So here, if you have a positive definite function like this as shown in this picture, then we can find a function of class K, that is phi which is operating on this norm of x. For example, if for this type of function, we can find a function which is symmetric, so here for any x if you take, the norm of x is the radius of this circle if you take this one. If you take a circle of radius norm of x, then the value of the function phi is same for all norm of x value.

So V of x is always greater than or equal to phi of norm of x where phi of norm of x is a symmetric surface which is; whether this V surface is symmetric or not but this phi of x surface is a symmetric one. And the value of the function V is greater than or equal to the value at the point norm of x that is phi of norm of x, okay. Similarly, if a function is negative definite, so it is a surface with the negative values only.

Then we can find a function, if V of x if it is negative definite, then we can find a function of class K, that is xi of norm of x, so this is less than or equal to -xi of norm of x. So for negative definite function, this is the property for positive definite function, this is the property.

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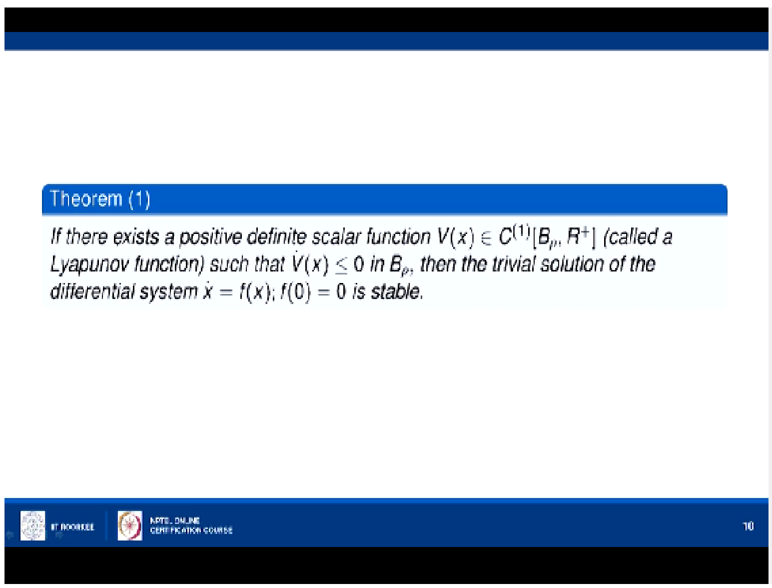

So now we can prove that Lyapunov theory if there exists a positive definite function V of x which is differentiable on the ball of radius rho. This function is called the Lyapunov function and it is such that the derivative dV/dt is negative semidefinite, less than or equal to 0 sign, it implies it is negative semidefinite function in the ball of radius rho. Then the trivial solution of the dynamical system x dot=f of x is stable.

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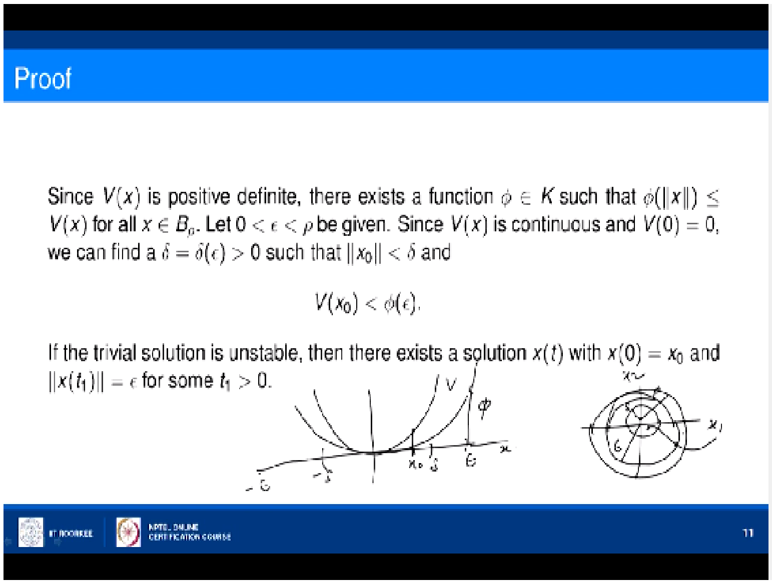

So the proof is like this. Since V of x is positive definite, there exists a function phi belonging to this satisfying this condition. Just now we have seen V of x is greater than or equal to phi of norm of x in the ball B rho. Now let us take any epsilon 0<epsilon<rho, this value. Here V of x is continuous function that is our assumption and V of 0=0, we can find a delta, positive number such that norm of x0<delta.

So we can find, for example in the case of 2 variable, x is say x1, x2, 2 variable. Then the B rho is a ball of radius rho. Then we can find a delta interval neighbourhood such that we can select an x0 within the ball of radius delta. Now and it is already selected some epsilon is already there. We can also select an epsilon radius. The smaller is the delta circle and the bigger one is the epsilon circle.

And all of them are lying within the circle of radius rho. Now according to the property of the function V, V of x0<phi of epsilon. So that can be explained from, for example in a simple picture, if V is the positive definite function, let us say with 1 variable, then we can find a phi function, a symmetric function like this which is below the surface V, that is V of x>phi of x.

Now if x0 is a point within a delta circle, let us say this is the delta region and this is epsilon, this is -delta and -epsilon, then we can find the x0 so that the V of x0, the value V of x0<phi of epsilon, so this value. So always we will be able to find the x0 so that it satisfy this condition

because of the property of the function V as well as the function phi, both of them are continuous function.

So if the trivial solution is unstable, then there exists a solution. So if you take x0 as the initial condition, we have here unique solution for the system, that solution is x of t and its initial condition x of 0=x0. And at time t1, there exists some time t1. The norm of x of t1 is epsilon because if the system is unstable, then it is from the initial condition x0, it will be going away from the trivial solution 0. So at some point of time, it will cross the epsilon circle, so we say that at time t1, the norm of x of t1=epsilon for some t1>0.

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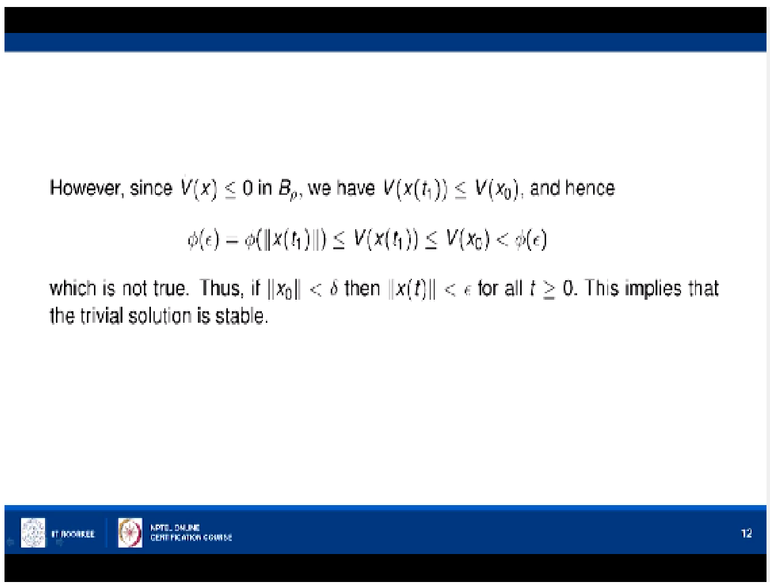

So using this condition, now V dot of x is negative semidefinite, less than or equal to 0 is given. So V is the decreasing function. So decreasing function of t because the derivative is with respect to t and it is negative, less than or equal to 0. So that implies V is a decreasing function of t. So V of x of t1 should be less than or equal to V of x of 0, that is V of x0. And hence we get phi of epsilon=phi of; because we have seen that x of t1-epsilon.

Therefore, phi of epsilon should be equal to phi of norm of xt1. So that is from this equation. And this is less than or equal to V of x of t1 because the property already we have here, phi of norm of x is always less than or equal to V of x. So we have that one and V of x of t1<V of x0 that is from the previous step and already we have shown that V of x0<phi of epsilon that has

been selected like this.

So from here, we see that phi of epsilon is strictly less than phi of epsilon which is not true. So the system, the solution cannot go out of this epsilon circle, that means always it will lie within the epsilon circle, that means the system is stable at the critical point 0.

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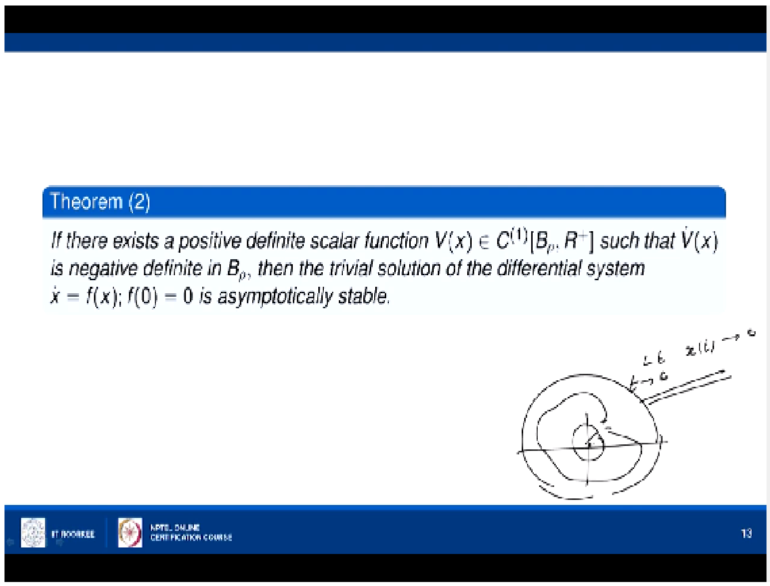

So now the second theorem is if there exists a positive definite function V such that the derivative V dot of x is negative definite in the ball of radius rho. Then the trivial solution x=0 of the equation x dot=fx is asymptotically stable. So the stability is that the solution always lies within a bounded region.

Asymptotically stability means the limit t tending to infinity of that solution, it should converge to the equilibrium point 0. So now the stability has been already proved because the condition of the previous theorem is already there. V is positive definite and V dot is negative semidefinite is there. And in addition, we have V dot is negative definite. So we have to prove that it is asymptotically stable or we have to prove this particular point.

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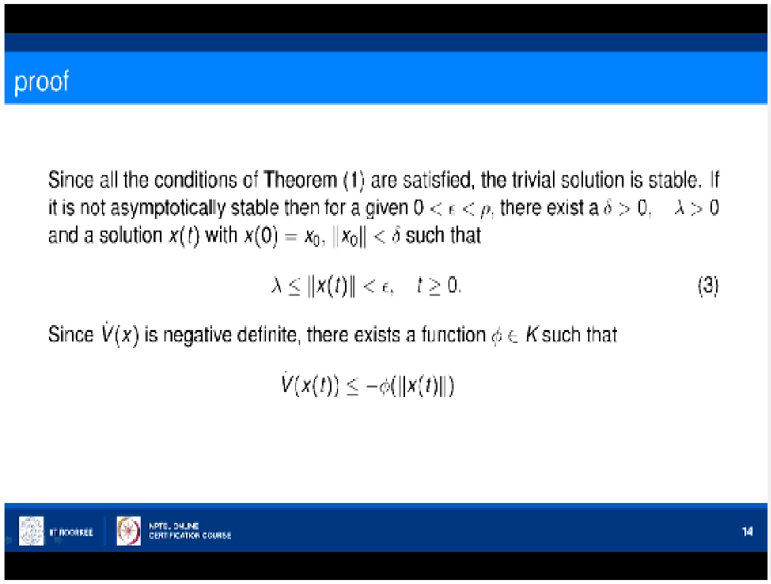

So since the condition of theorem 1 is satisfied, the solution is stable. Now we have to prove that it is asymptotically stable. If it is not asymptotically stable, then this condition will not be satisfied. Limit t tending to infinity of x of t will not tend to 0. So in other words, the solution will be bounded, that is because it is stable starting from an initial condition, the solution will be bounded but it will not reach the origin as t tends to infinity.

That means there will be a value lambda and beyond which it will not approach the 0 point. So norm of x of t will be greater than or equal to lambda and it will be less than epsilon. So this situation will occur if the system is only stable but it is not asymptotically stable. Now we will prove that this is not possible. The equation 3 is a contradiction that is to be shown. Since V dot is negative definite, there exists a phi in class K such that this condition is satisfied. V dot of x of t is less than or equal to -phi of norm of x of t. So this we have seen.

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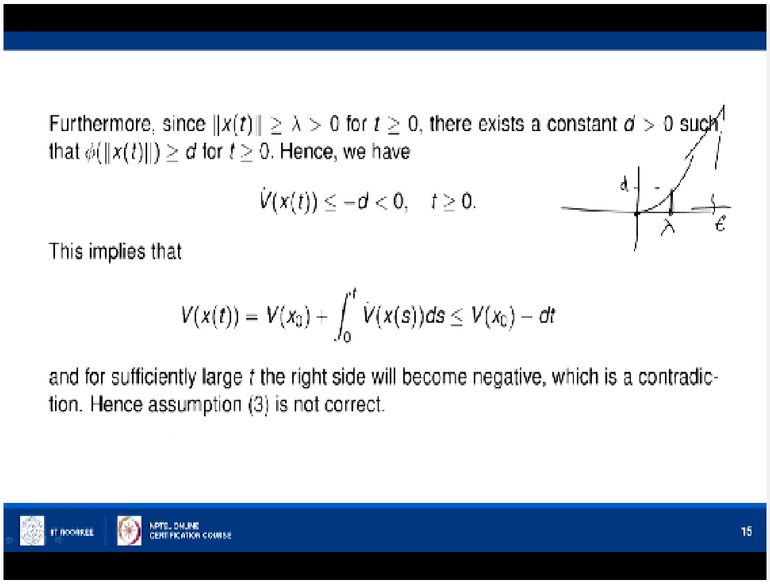

Furthermore, norm of x of t is greater than or equal to K>0 for t>0, there exist a constant d such that phi of norm of x of t is greater than or equal to d. So this is because of the property of the function phi. We have seen that phi means it is always like this. It is 0 at 0 and it is strictly increasing. So when this value norm of x of t is not approaching 0, it is between epsilon and lambda only, norm of x of t is below epsilon and above lambda always.

So phi of norm of x of t will never get the value 0. It will be always some positive value. So it will be greater than or equal to some value d, this one. This is the value d and it will lie between d and some other number. So hence, V dot will be less than or equal to -d because of this condition, V dot<-this thing and -phi of norm of x of t. Therefore, it will be less than or equal to -d for all t greater than or equal to 0.

So this implies if you integrate both sides of this expression, we will get V of x of t-V of x0, if you integrate this, that is equal to integral 0 to t V dot of xt. Now we substitute this value here. So we get V of x of t is less than or equal to V of x0-d*t. Directly by integrating both sides, we get this thing. And if you take t sufficiently large, we will get this to be a negative value. The right hand side will become negative which is a contradiction to the fact that V is a positive definite function. So this assumption 3 is not possible. All this is obtained due to the assumption 3.

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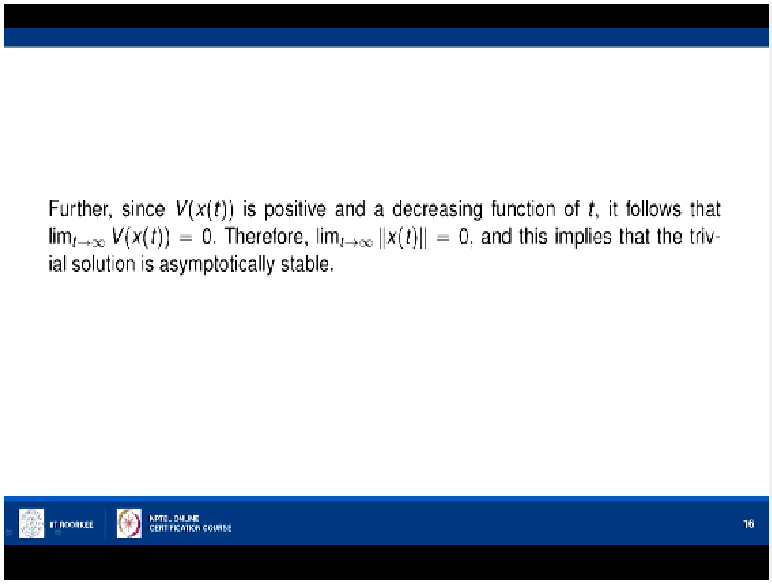

Therefore, the system is asymptotically stable because of this reason. Because since V of xt is positive and decreasing function of t, Because V dot is negative definite, it is decreasing function of t. It follows that the limit t tends to infinity of V of x of t should be equal to 0. And therefore, we get; because V of 0=0, it is not 0 for any other non-0 vector. The limit t tends to infinity of norm of xt also should be equal to 0.

So this implies the asymptotically stable of the system at the critical point 0. We have seen the 2 theorems, one is for the stability and the second one is for the asymptotically stable system. So at the next lecture, we will see various examples illustrating these theorems, 2 theorems and a theorem on the instability of the dynamical system. Thank you.