Dynamical Systems and Control
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Lecture - 46
Introduction to Discrete Systems - II

Hello viewers. Welcome to this lecture on discrete systems. So in this lecture, we will see some results on the state transition matrix, solution of discrete system and the controllability of discrete systems which is these results are analogous to the continuous dynamical systems and control systems.

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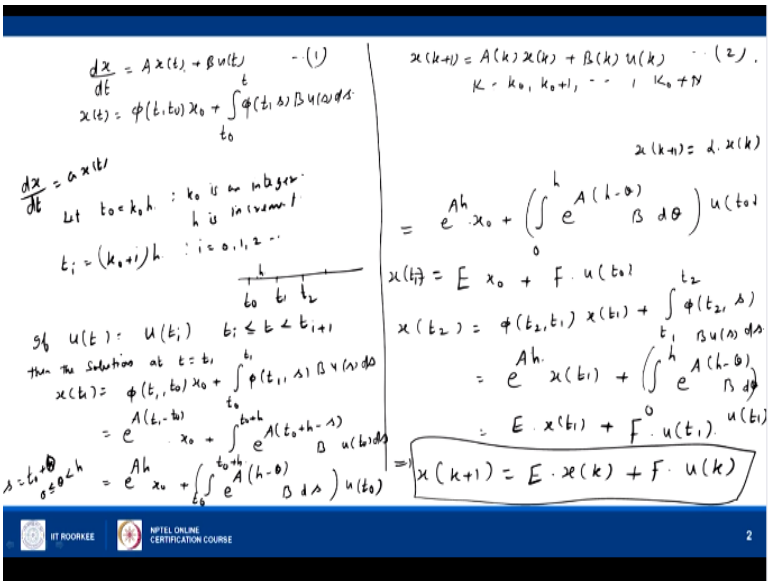

So earlier we have seen the continuous control system of the form dx/dt=Ax t and Bu t, so here A maybe a constant matrix or time variant matrix, similarly, B maybe constant or time variant. So this continuous control system has solution x of t=phi t, t0 x0+integral t0 to t phi t, s Bu s ds. So this is the solution of the control system for the initial condition x of t0=x0.

Now we have also introduced some discrete control system or discrete dynamical system which is of the form x of k+1=A of k x of k+B of k u of k is a discrete control system where k is varying from k0, k0+1, etc k0+n. So we have the discrete control system in this form. So we call it as 1 and this as 2. Now how to write the solution of the discrete system and we can also see what is the connection between a continuous control system and a discrete control system.

So we can formulate various discrete control system on its own, various problem like population dynamics can be for example written as x of k+1=some constant times x of k. So this maybe representing or a mathematical model for a population dynamics or some such similar problems as we have seen population dynamics for a simple model of population dynamics in terms of a continuous dynamical system can be written like this where A is the rate of increase of the population.

And here also alpha is the rate of increase of the population and so it gives a simple mathematical model whether it is continuous or discrete system but for certain problems discrete systems are more suitable and certain problems continuous models are more suitable. Similarly, if we have a continuous model and if you want to solve it using a digital computer, in that case we have to convert this continuous system into a discrete system.

So we can also find such relation in the following way. So if we consider let t0 that is the initial time instant which we have taken if you write it as k0 times h where h is a small time instant which we take and k0 is an integer such that k0 is an integer and h is increment in time and if you are interested in the values of x of t at various instances of time, let us say let ti it is=k0+i*h for i is=0, 1, 2, 3, etc.

So when we substitute i values differently, we will get the discrete points from t0, t1, t2, etc and the time increment is h here this. So now if u of t because while doing or while applying a control in a practical problem, it is not possible to give the control in a continuous fashion but we can give it in a discrete time instance. So u of ti we assume that it is=sorry u of t is=u of ti for the interval ti<=t<ti+1.

So if we assume this way, the control is applied in a discrete fashion for this continuous control system. Then, the solution of the system can be written in this following way. Solution for example at u of t1, the solution at t=t1 that is=x of t1 is written as phi of t1, t0*x0+integral t0 to t1 phi of t1, s Bu s ds which is same as e to the power A*t1-t0*x0+integral t0 and t0+h.

This is e to the power A*t0+h-s B. Now u s in the interval t0 to t1 is u of t0 because we are assuming that u of t is measured in a discrete fashion. So u of t=u of ti in this interval, so we substitute u of s to be like this in the interval t0 to t0+h and this is nothing but e to the power

Ah t1-t0 is h*x0+ here now we can convert this into the following way, s is between t0 to t1 so if you substitute s is=t0+h sorry t0+theta where theta is between 0<=theta<h.

Because t is or s is between t0 to t1, so we can substitute this way. So if we substitute s is=t0+theta here, t0 gets canceled so this will be=e to the power A*h-theta and this is varying from t0 to t0+h B*ds*u of t0 this expression. So this further can be written as the right hand side is e to the power Ah*x0+now if we convert this integral in terms of d theta here, so ds is=d theta and when s is=t0 theta is 0.

And here it is 0 to h e to the power A*h-theta B*d theta*u of t0. So similarly we can so we can call this as the matrix E, e to the power Ah can be called as a matrix E*x0 and integral 0 to h of this expression we can call it as F*u of t0. So in the similar way, we can calculate x of so this is nothing but x of t1 is given by this expression. Similarly, we can write x of t2 in the place of t1 if you put t2, here it will be phi of t2, t1.

Because initial position is t1, initial time is t1, final time is t2*x of t1+integral t1 to t2 of this similar expression phi of t2, s*B us ds. So here u value is u of t1 that is a constant throughout the interval t1 to t2. So that will be out of the integration. So again we can see the same thing, in the place of e to the power A*t2-t1, so it will also be=e power A*h x of t1+here also we will get the same thing e to the power A*h-theta B d theta and u of s is u of t1.

So we can see that the expression is again E matrix*x of t1+this matrix is same F*u of t1. So if you replace this t1, t2, t3, etc by some suffix itself. So this implies that in general we can write x of t suffix k+1 if you write but we can replace it by x of k+1 just as a notation x of k+1 it means x of t suffix k+1 that is=E*x of k this matrix*x at t suffix k+F*u of k. So we will get a discrete control system in this particular fashion.

And this h is the time instant at which we find the values of x as well as use the control at these time instances for this control system. So when we solve a continuous control system using a computer, we can convert that system into a discrete system and then solve in a discrete fashion. So this will give various simpler results for solving a continuous control system in the form of discrete system as follows. So here we will see a various analogous results as in the case of continuous system.

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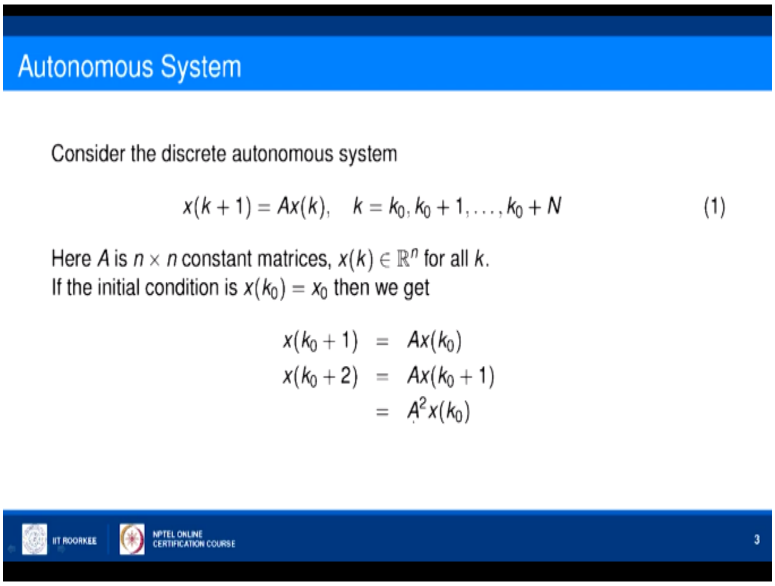

For example, the state transition matrix we have defined for a continuous system earlier, so similarly we can define it here. So consider the simple system x of k+1=Ax of k where the time instances are k=k0, k0+1, etc. Now repeatedly applying the equation 1 except k0+1 is Ax k0 and x of k0+2 is A square x of k0.

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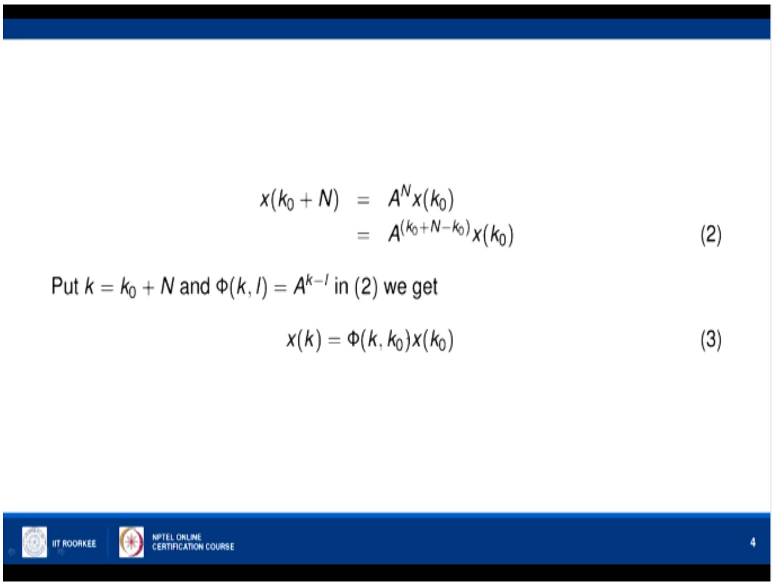

And similarly x of k0+N, it can be easily seen that it is A to the power capital N x of k0, capital N can be written as k0+N-k0. So x of k0+N=A power N x of k0. N can be written as k0+N-k0. So we get and if you substitute k=k0+N and the notation phi of k, l means A to the power k-l. If we use this notation, then we can write the solution x of k=phi of k, k0*x of k0. So it looks similar to the continuous system state transition matrix. So we will see the analogous result of that one.

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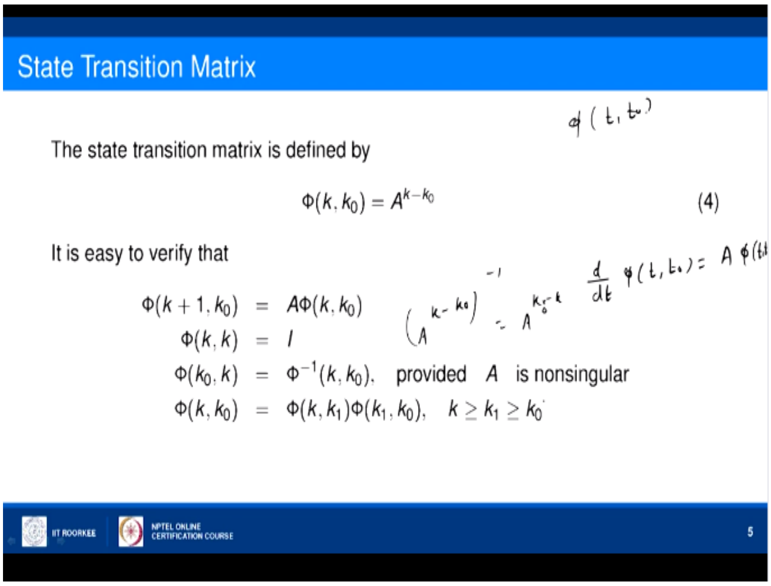

Earlier, we have seen phi of t, t0 is the state transition matrix for a continuous system and if it is a constant matrix we have seen it is equal to e to the power A*t-t0, otherwise we use the Peano-Baker series to find the state transition matrix but here in the discrete system if A is a constant matrix, the state transition matrix is phi of k, k0 is A to the power k-k0 and the property of continuous system we have seen d/dt of phi t, t0 is A*phi t, t0.

So similar result in the discrete is phi of k+1, k0 is A times phi of k, k0 that can be directly seen from here. If you put k+1 here, it will be A to the power k+1-k0 which is A times A to the power k-k0 and this first one and if you put both variables are same k=k0 we will get A to the power 0 which is the identity matrix and if you reverse this A k0-k that is nothing but A k-k0 its inverse is nothing but A to the power k0-k.

So we get this third result and fourth can be easily verified. So these 4 properties are analogous to the continuous case.

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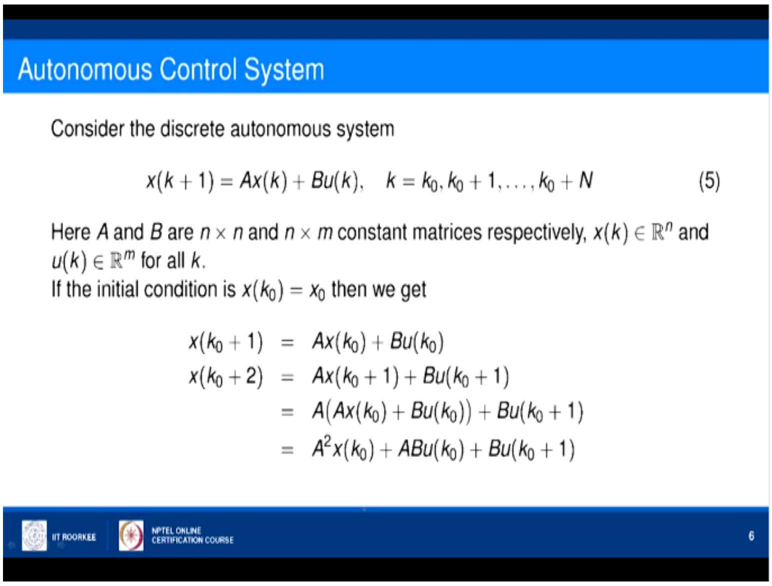

Now if you consider the control system x of k+1=Ax k+B u of k for these time instances, it can be easily verified by substituting one by one starting from k0, so x of k0+1 is Ax k0+Bu k0 and substituting k0+2 here, here it will be k0+1+Bu of k0+1, etc.

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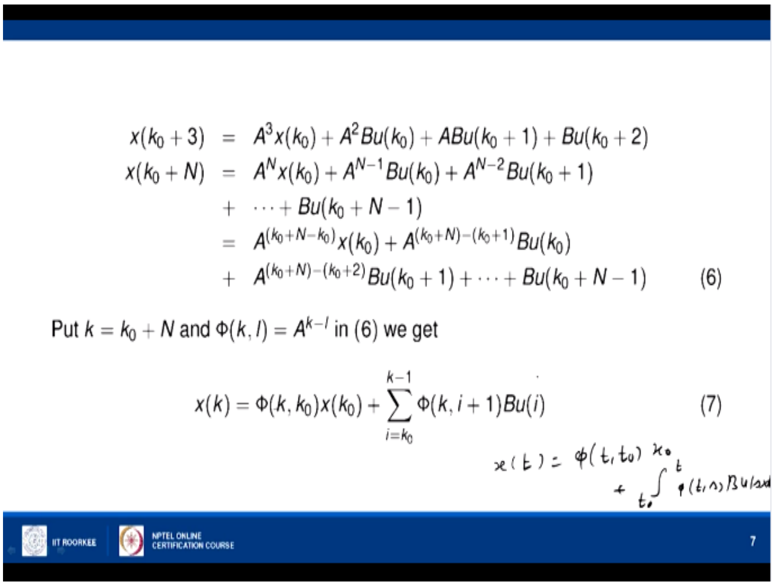

So we will get the formula easily Ax of k0+N we will get A to the power N x of k0+this quantity which can be written in the similar as we have seen in the previous case. N can be written as k0+N-k0, so by using the notation phi of k, l is A to the power k-l we get the formula x of k where k is k0+N is=this is phi of k, k0 this into x of k0+the remaining portion can be written in the summation i is=k0 to k-1 of phi of k, i+1*Bu i.

So this is the formula and it is analogous to the formula for the continuous case x of t is phi of t, t0 x0+integral t0 to t phi of t, s Bu s ds. So instead of integral, we have summation and the discrete points we are using for the formula for the solution of a discrete system.

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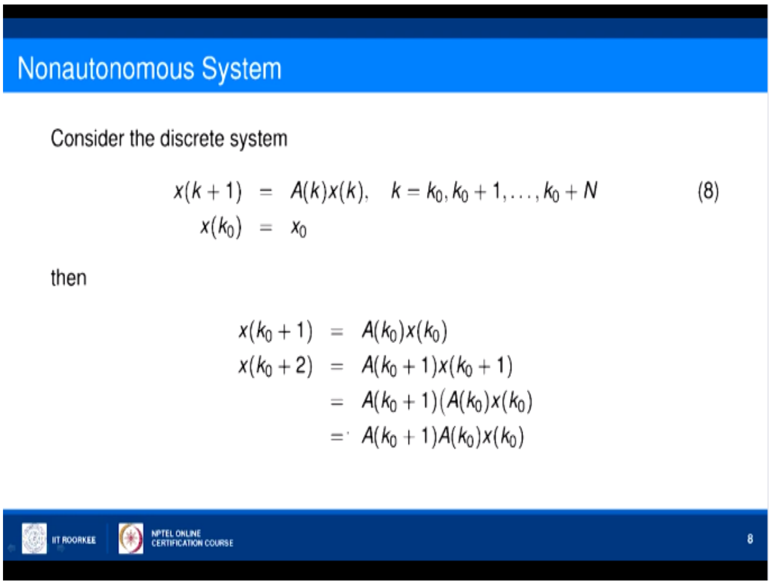

Now if you take non-autonomous system where A is not a constant, we can derive in a similar fashion, only thing is A of k should be replaced at every place. So x of k0+1 is A of k0*x of k0, x of k0+2 we can get A of k0+1 A of k0*x of k0. So for k0+N it is clear that the right hand side we will start with A of k0+N-1 and then N-2, etc up to the initial point k0.

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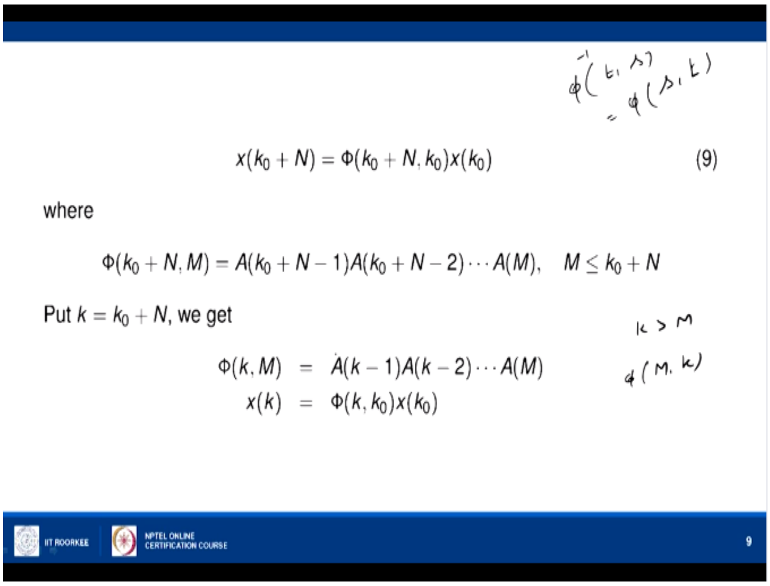

So we will get this result x of k0+capital N will be=A of k0+N-1 and k0+N-2, etc up to the initial point that is k0 and then multiplied by x of k0. So in general we can give the notation phi of k0+N, M it means we will start the A of k0+N-1 then end with the last integer M here.

So this gives the state transition matrix for the time variant matrix here and if you give the notation k=k0+N we get the formula phi of k, M is A k-1 up to A of M and the solution 9 is written as x of k=phi of k, k0*x of k0.

So now we have to note here if you write phi of M, k. Here k is>M when we write like this. So k-1, k-2 up to M it can decrease but when we write phi of M, k we cannot make use of a similar formula, we cannot start with M-1, M-2 and we cannot end at the point k here because M is<k. So we have to make how to write analogous to the continuous system, continuous system we always write phi of t, s inverse it is=phi of s, t.

So we do not consider, we do not bother about the t>s or<s. In all cases, we can write in this particular fashion but here it is not possible to use this notation as it is here.

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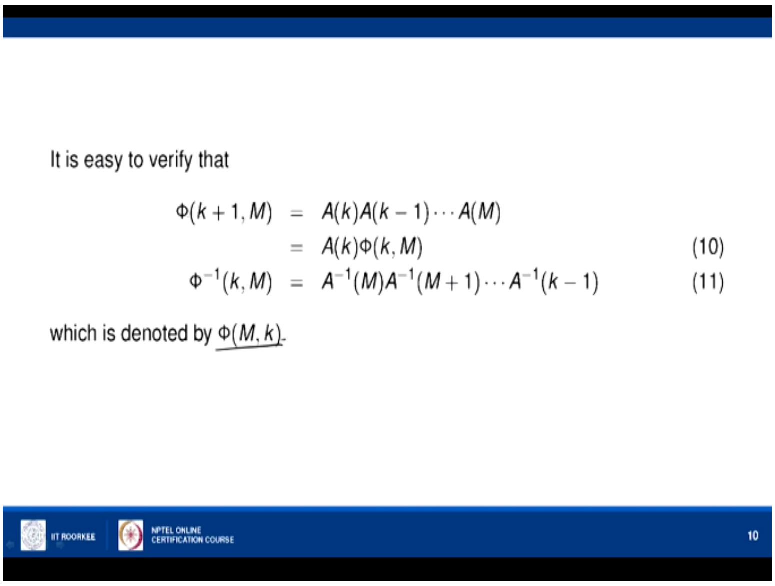

So we have to consider like this, so phi inverse of k, M is from the definition of phi of k, M if you take inverse both sides, we will A inverse M first A inverse-M-1, etc the last will be A inverse k-1. So that is the straight forward formula phi inverse of k, M is given by equation 11 but we will give a notation for this to be phi of M, k. So it is not directly using the definition of phi but it is only a notation for phi inverse k, M we will write it as phi of M, k.

And the property similar to the continuous case phi of k+1, M is given by this expression which is same as A of k*phi of k, M. So we can prove all the 4 properties of the state transition matrix using these notations. So whether it is an autonomous case or non-autonomous case before properties of the state transition matrix is proved in this way using this particular notation.

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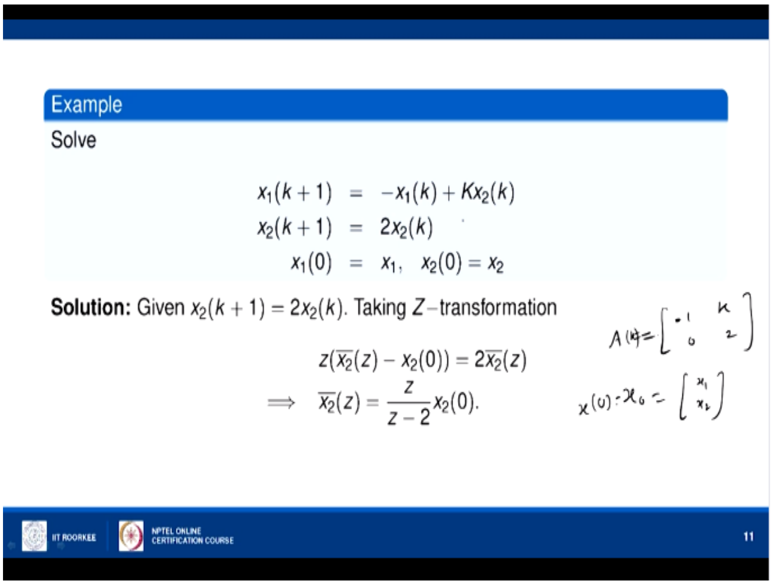

So now let us illustrate with an example how to solve a discrete dynamical system. So let us consider a second-order system x1 of k+1 is –x1 k+k times x2 of k, the x2 of k+1 is 2 times x2 of k, initial condition is x1 0=x of x1 and x2 of 0 is x of x2, two numbers. So now here the matrix A is a time variant matrix because A is -1 k and 0 2. So k is varying it is A of k is=this. So it is a dynamical system and the initial condition x0 it is nothing but x1 x2.

The vector is given x at 0 is given by this expression. Now we can solve the equation directly by using this z transform, taking z transform both sides of the first equation, sorry of the second equation, so we will get z transform of x2 of k+1 that is=2 times z transform of x2 of k and using the formulas of z transform, we will get z*the z transform of x2-x2 at 0=2 times z transform of x2 and so the z transform of x2 is given by z/z-2*x2 of 0.

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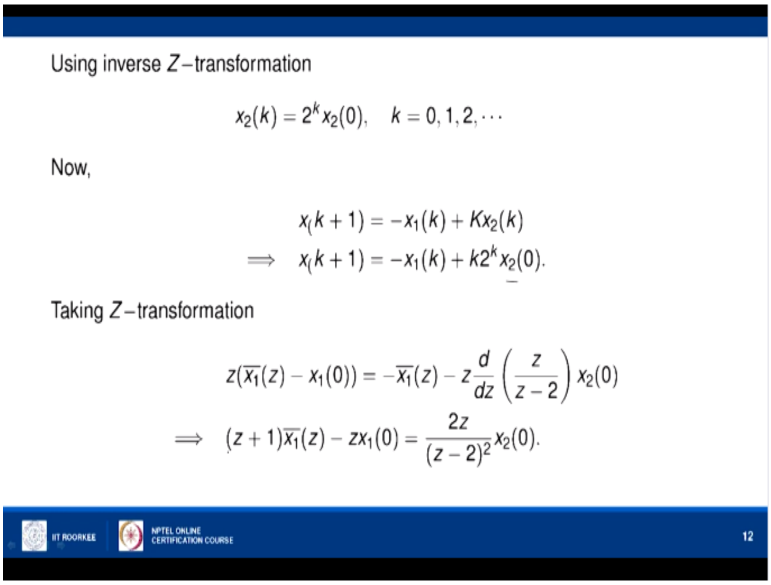

So by using the inverse z transform, we will get x2 of k to be 2 to the power k x2 at 0. So that can be easily verified. Now substituting x2 of k in the first equation which is x1 of k+1 is –x1 of k+k times x2 of k is given by this expression, so that is substituted 2 to the power k x2 of 0. Now again taking z transform both sides of this equation, we will get z*z transform of x1-x1 0 is -z transform of x1.

And z transform for this expression is -z times d/dz of the z transform of 2 power k that is z/z-2*the initial condition.

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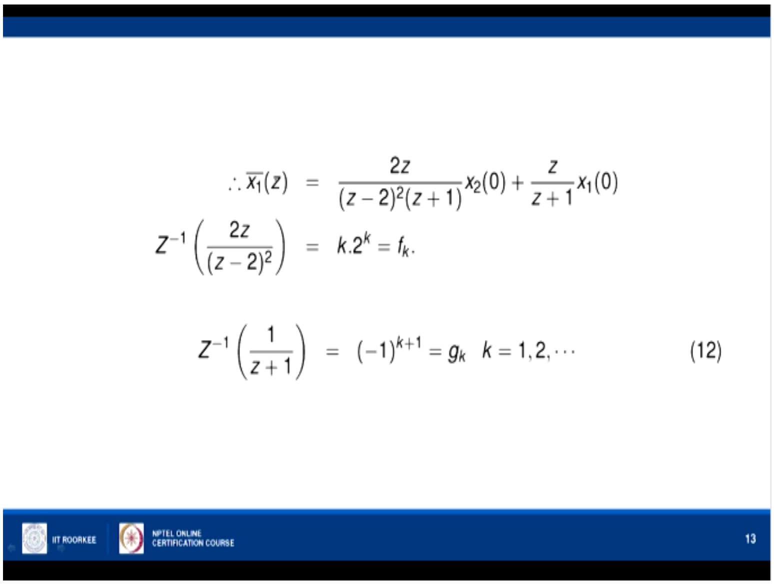

Now collecting the z transforms in one side and taking the inverse z transform, we get the z transform of 1 is given by this expression and the inverse z transform gives the equation 12. So the solution the inverse z transform of 1/z+1 is given by this. So this expression is the product of the two z transforms, f suffix k and g suffix k are the inverse z transform of this expression. So we can use the convolution theorem to find the inverse z transform of the expression 2z/z-2 whole square+z+1.

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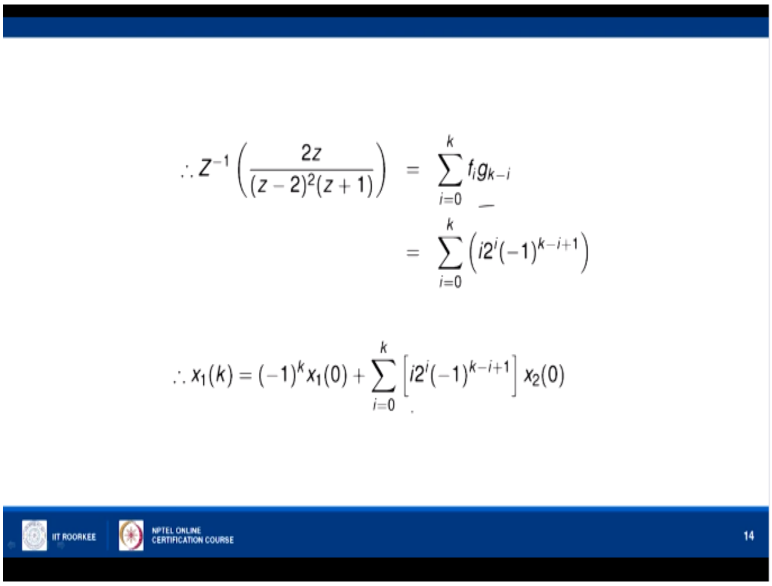

So the convolution formula is given in the right hand side here and the expression is given by this one. So the solution x1 of k is given in the last line here.

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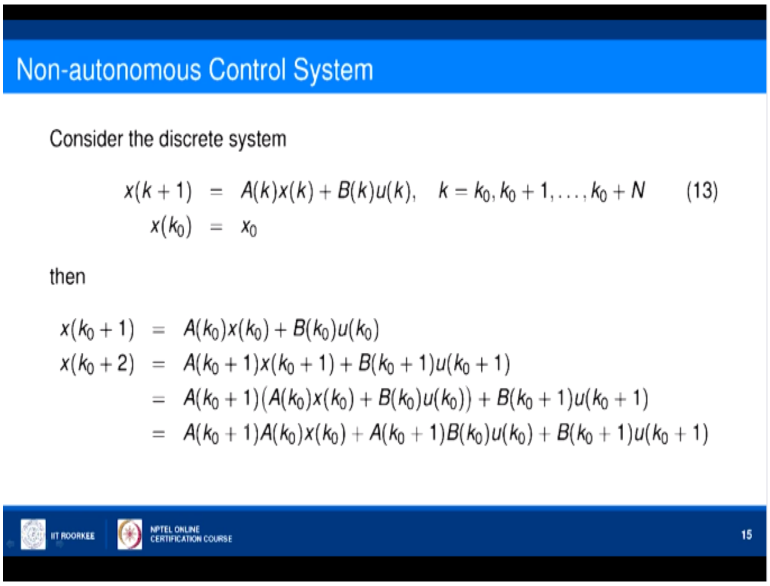

Now we come to the control systems. Consider the non-autonomous control system in which the matrix A and B are functions of k, the time instances and initial time instant is k0 and initial condition is x of x0. Now making use of a similar step-by-step method, so if you calculate x at k0+2, we will get this expression given in the last step.

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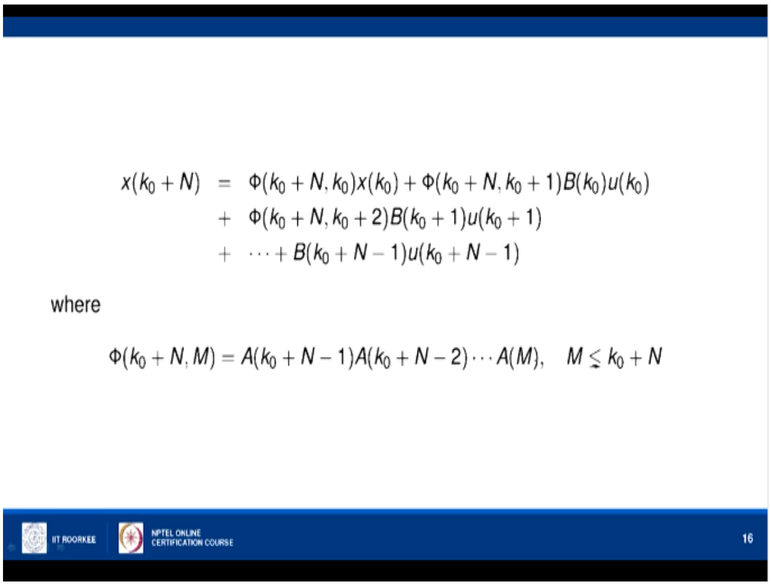

So this is only substituting step-by-step, so except k0+N that is at the Nth time instant is given by the formula as shown in the right hand side and here the state transition matrix phi of k0+N, M is given by this expression where M is<=it should be strictly<okay. M is strictly<k0+N because the right hand side it starts with k0+N-1 and decreasing up to capital M, so M cannot be equal to this expression okay.

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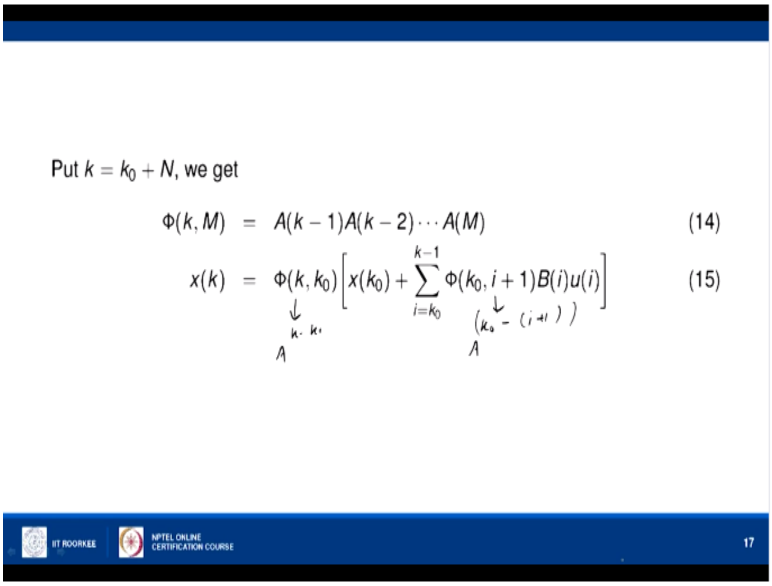

So by substituting k is=k0+N we can see that the state transition matrix phi of k, M is A k-1 up to AM the product in the right hand side and the solution x of k is given by the formula as shown in the expression 15 here, the solution in the last step x of k0+N is given in this right hand side so we can easily compare it with the compact expression which is given in the equation 15 as the solution of the problem.

So now so this expression is common to the time variant case or time invariant case. So in the time invariant case autonomous case, phi is replaced by simply A to the power this expression k0-i+1, so that is the only difference wherever phi is there we will replace it with A to the power k-k0 here and here we will replace it, so that is for the time invariant case and this is 15 is the general formula for the time variant case.

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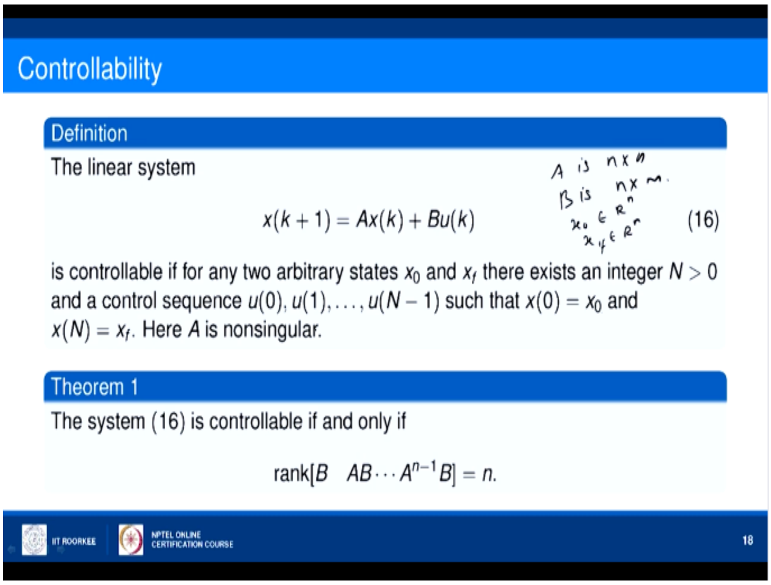

Now we define the controllability of the non-autonomous system, sorry one minute so now we will see the controllability of the autonomous system x of k+1=Ax of k+Bu of k. So this system is said to be controllable if for any arbitrary state x0 and x1, two vectors in Rn, so here we will consider A is a n x n matrix, B is a n x m matrix. So two arbitrary vectors x0 and x1 both belong to the state space Rn.

There exists an integer capital N, that is the final time capital N and a sequence of control u0, u1 up to un-1 such that the solution of the system 16 satisfies the condition x of 0=x0 and x of capital N is xf, x of f is the final. So the system is controllable for any two arbitrary vectors x0 and xf given in the state space Rn there exists an integer N as final instant and a sequence of control u1, u2 up to un-1 such that the solution satisfies the initial and the final conditions.

So here in all these problems, we note that A should be a nonsingular matrix because we have seen that the state transition matrix for satisfying the properties 4 properties it should be a nonsingular or invertible matrix. So now we prove the theorem which is analogous to the continuous case where we have defined the controllability Gramian matrix, so the system where we have proved the Kalman theorem earlier.

So the system 16 is controllable if and only if the Kalman matrix B AB A square B up to A power n-1 B has rank n. So the similar theorem we have proved it for the continuous case earlier.

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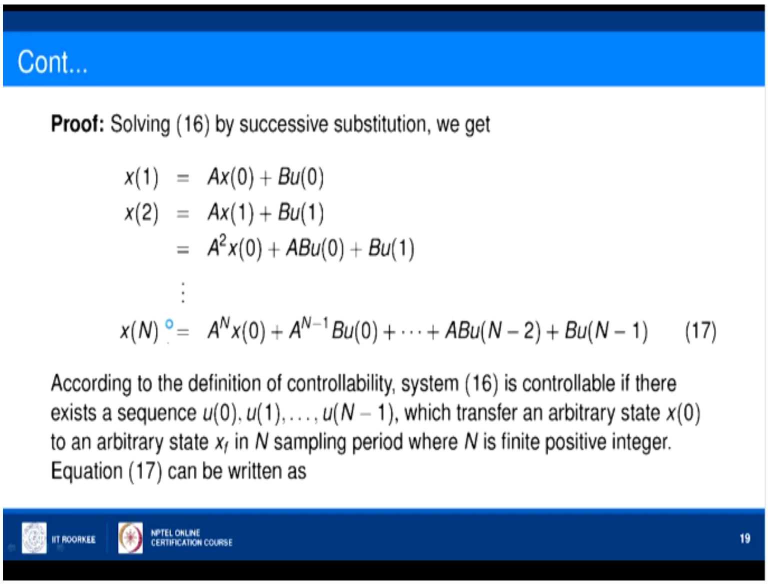

So the proof is a quite straight forward here. So now the time instances are 0, 1, 2, 3 up to capital N. If you substitute each instant one by one, so x of 1 is Ax 0+Bu 0. Similarly, x of capital N is A power N x of 0+etc and we get the equation 17 by substituting step-by-step. Now the definition of controllability is we have to find an integer capital N so that the solution satisfies.

We have to find an integer capital N as well as a sequence of control that is u0, u1 up to un-1 so that the solution satisfies the condition x of 0 is x0 and x of capital N is x suffix N. So we will see under the condition the rank condition given in the last line, we can prove that the system is controllable.

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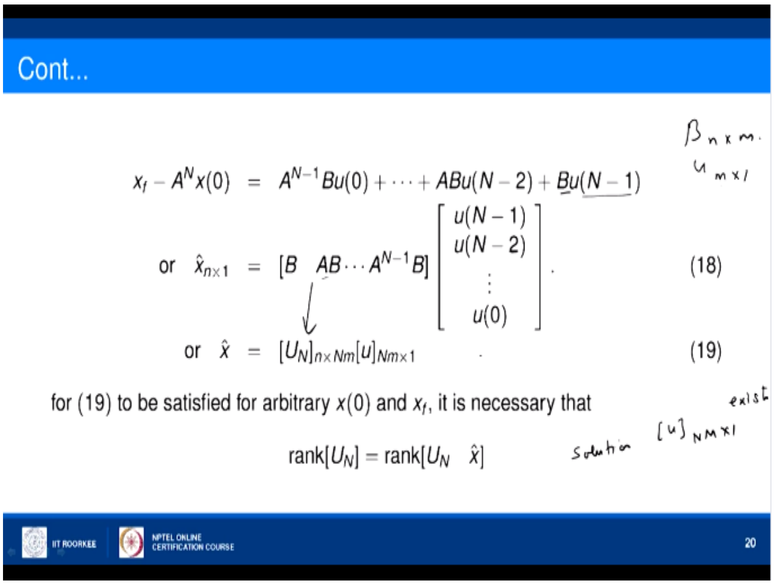

So now xf if we substitute x of capital N=xf in the equation 17 and taking this A to the power N x of 0 in the left hand side, we will get the first line here xf-A to the power N x of 0 it is equal to the remaining terms which can be denoted in this following way. So let us write the left hand side as x bar which is a vector in Rn and because x0 and xf both are arbitrary, this x bar is also an arbitrary vector in Rn.

So the remaining terms in the right hand side can be written as B AB A to the power N-1 B here*u of N-1. We can note that the B matrix is multiplied by u of N-1, so that is here. AB matrix is multiplied by u of N-2, etc. A to the power N-1 is multiplied by B matrix okay. A to the power N-1 B is multiplied by u of 0 so this expression can be written in the form of equation 18.

Now the size of the matrix if you call this as the U matrix capital U, its size is n x N*m because B is a n x m matrix and AB is a n x m matrix, etc. So the size of this matrix capital U is n x capital N*m. Now u is a vector which is m x 1 vector. So each u is m x 1 vector, so totally you have N*m x 1 vector here in the equation 18. So equation 19 it is simply a system of algebraic equation where x cap is an arbitrary vector, U is the matrix given here and small u is the vector of size capital Nm x 1.

So if the rank of this matrix U is N then it is clear that there exist a solution for this system because for existence of the solution of 19, the rank of u and rank of the augmented matrix U and x bar should be equal and if the rank of u itself is N then the system 19 will have a solution. There will be infinitely many solutions because of the size of this matrix. U is not a square matrix. So it will have a solution means it will have infinitely many solutions.

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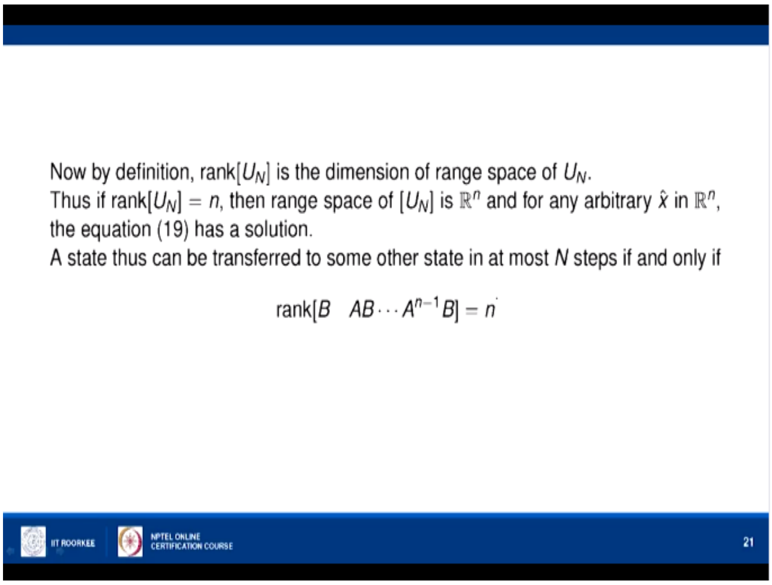

So we get if the rank of this matrix is=N then the system will have a solution that is U. So this sequence is defined from the equation 9. So we will get the solution small u Nm x 1 exist. From this, we can get the solution of the discrete control for the discrete system. So the converse part also can be proved analogous to the continuous case. That is if the system is controllable, then the rank of this matrix should be equal to n.

So it is exactly similar analogous to the continuous control system, so we need not repeat the same thing. So with this, I will complete the lecture for the discrete autonomous control systems. So in the next lecture, we will see the controllability of the discrete non-autonomous control systems. Thank you.