**Dynamical Systems and Control**
**Prof. N. Sukavanam**
**Department of Mathematics**
**Indian Institute of Technology – Roorkee**

**Lecture - 45**
**Introduction to Discrete Systems - I**

Hello viewers. Welcome to the first lecture on discrete dynamical systems. In this lecture, we will define the state transition matrix of the discrete system and how to write the solution of the discrete system in terms of state transition matrix. Then, we will see the definition and conditions on the controllability of the discrete systems.

So in previous lectures, we have seen several examples of continuous dynamical systems in terms of the differential equations or the control systems in the form of a differential equations. So in this now we will see some examples of discrete dynamical systems. So for example if you consider prey-predator model, earlier we have considered as a continuous system, now we can also consider it as the following system.

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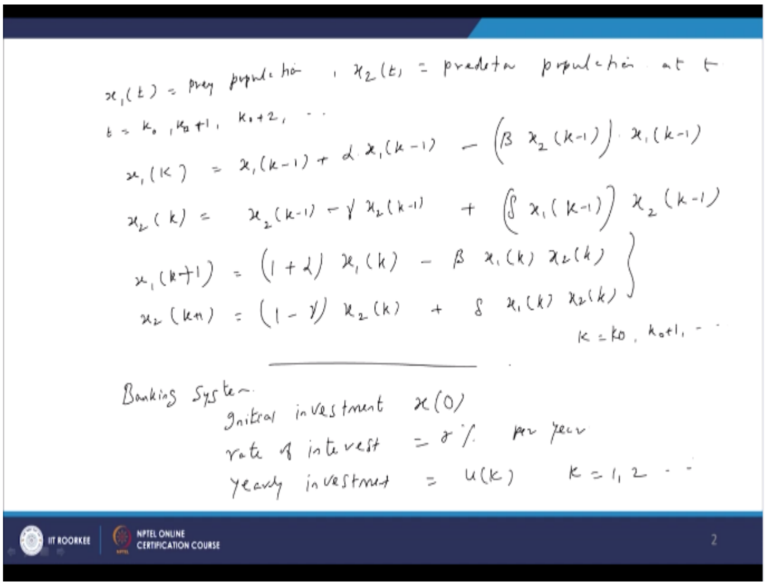


So let x1 of t it represent the prey population and x2 of t is the predator population at time t. So if the time is measured in discrete intervals, so if T is considered as say k0, k0+1, k0+2, etc in discrete time intervals. Then, the prey predator model can be written in this particular form that is the prey population at the time instant k that will be depending on the population of the previous time instant+some increase in the previous time instant population where alpha is the ratio of increment.

So this is our original population of the previous instant x1 of k-1+now alpha times x1 of k-1 population increases. So this is about the increase and then it will decrease with respect to the predator population. So beta times x2 of k-1, so in the previous instant how much predator was available that is the ratio and then we have to multiply by the previous year population of the prey, so this amount will be decreasing.

So we can say that at time instant k, it is all depending on the population of prey and predator the previous time instant. Similarly, the population of the predator, it is the previous instance population but it will decrease by some amount let us say gamma. So gamma times its population in the previous time instant, so it will decrease but it will increase with a population delta times x1 of k-1 times its own population at time k-1.

So this gives a prey predator model in the discrete case or we can write in another form. If you write k+1 as the current instant and k as the previous instant, we will write this as 1+alpha times x1 of k-beta times x1 k x2 of k. Similarly, x2 k+1 of 1-gamma times x2 of k+delta times x1 of k*x2 of k. So this gives a prey-predator model for k starting from k0 and k0+1, etc. We can give different values for this k value.

Another model which is in the banking sector, we can see the following equation. So for example let us consider in the banking system, so if we consider the initial investment in a bank as x0 x of 0 at the time instant 0 and let say the rate of interest is r% per year and the yearly investment it is let say u of k for various k values, let us say 1, 2, 3 etc. Every year the investment is given like this.

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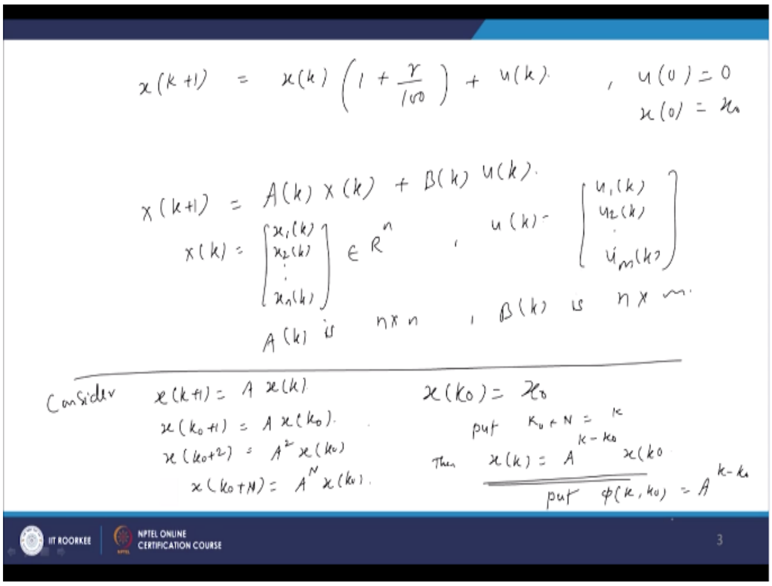


So then the amount which will be available at a particular instant of time that is k+1th year how much amount will be available that is written as using a simple interest, the principal interest and r/100. So the previous year the amount was x of k and +x of k r/100 will be the interest earned by x of k+the investment of that year is u of k and we assume that u of 0=0 because the initial investment was already x0.

So this expression gives it is a discrete dynamical system and if this u of k can be considered as a control for achieving certain particular goal that is if you want to achieve that after certain years what should be the amount available in the bank. So for that purpose, we can adjust this u of k suitably so that we can achieve the goal. So it is a discrete control system in this form.

So in general a discrete control system is written in the form x of k+1=A of k*x of k+B of k*u of k where x of k is a vector x1 k, x2 k, etc, xn k belongs to Rn. So it is the state variable and the control variable u of k will be u1 of k, u2 of k, un of k. So it is analogous to the continuous control system and A of k is a n x n matrix, B of k means n x m matrix, so almost similar to that one except that the time is measured in a discrete manner.

So if we consider only the dynamical system without the control term, so we can say x of k+1=A of k*x of k. If you consider this equation for starting from let us say x of k0=x0 initial time instant, we take it as k0 and then at every increment by 1 we will measure the state variable x of k+1. First, let us consider the case in which A is a constant matrix instead of A of k let us consider A itself and then x of k+1=A*x of k we consider.

So x of k0+1 is=A*x of k0 and x of k0+2 is A times A of x of k0. So we will get this one etc. So we will get the expression k0+n it is=A to the power N*x of k0. So if you substitute k0+N as the notation k then we will get the expression x of k is=A to the power k-k0*x of k0. So this is a very simple expression we can easily see from step-by-step. So this A of k-k0 we will call it as the state transition matrix, phi k, k0=A k-k0.

So if you give this notation phi k, k0=A to the power k-k0, we call it as the state transition matrix and this satisfy the various properties.

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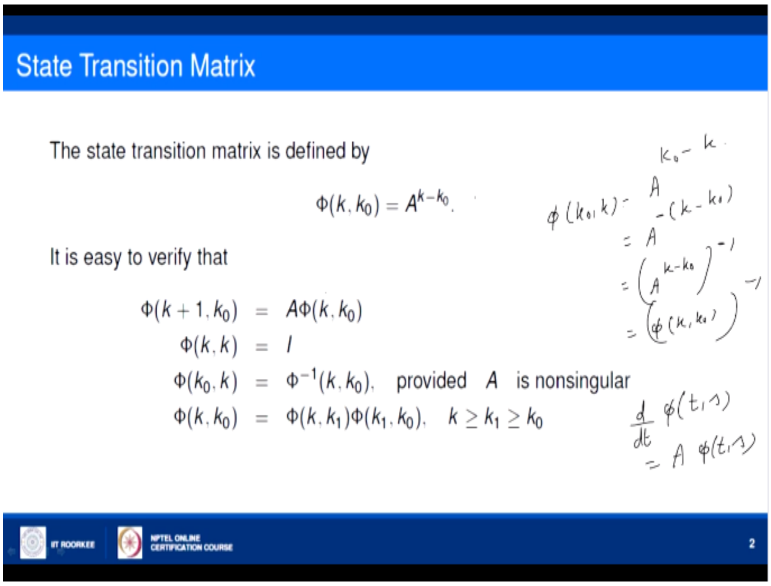


So now from here we will see that phi k, k0=A power k-k0, it is easy to verify that these properties are satisfied, phi k+1, k0 if you substitute here A to the power k+1-k0, so we can take one A out, so we will get A*A power k-k0 so this is satisfied. Now phi of k, k if you put the same thing you will get identity from this expression and phi k0, k will be it is=A to the power k0-k which can be written as A to the power –k-k0.

Or it is nothing but A to the power k-k0 inverse okay because first we can write it as A inverse whole power k-k0 which is same as this expression. So this is phi k, k0 inverse so that can be easily verified from this expression and the last one by directly substituting this we can verify. So these 4 conditions they are much similar to the conditions of the state transition matrix of continuous dynamical systems.

So there the first condition is replaced by d/dt of phi t, s is A times phi t, s. So this condition is replaced by the first one here.

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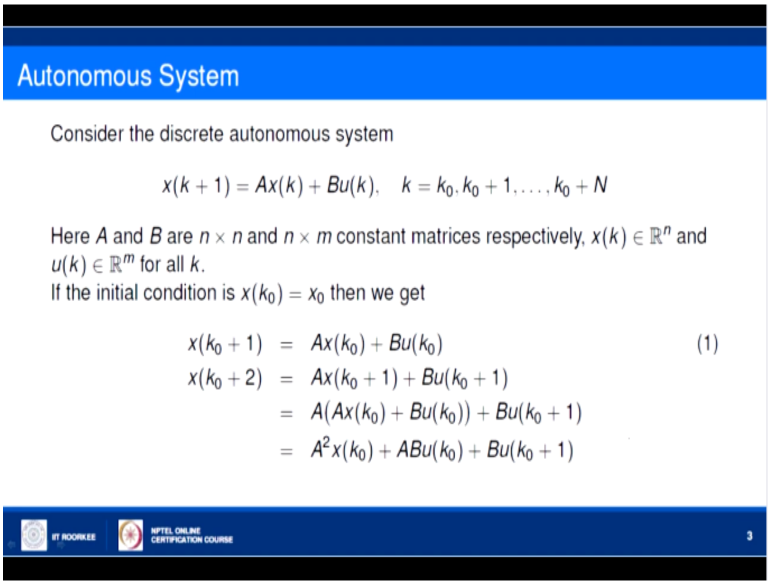


Now using this state transition matrix, we can write the solution of the equation in the following way. So x of k+1 will consider the discrete control system. So phi of k+1=A*x of k+B u of k and initial time we consider as k0. So by substituting the values one by one, x of k0+1 is A x of k0+B u of k0 and x of k0+2 is A x of k0+1. In that place, we substitute in the next step the previous equation that is A of x k0+B u x k0.

And then the sum last term is there and simplifying we will get A square x of k0+etc. The last line is there.

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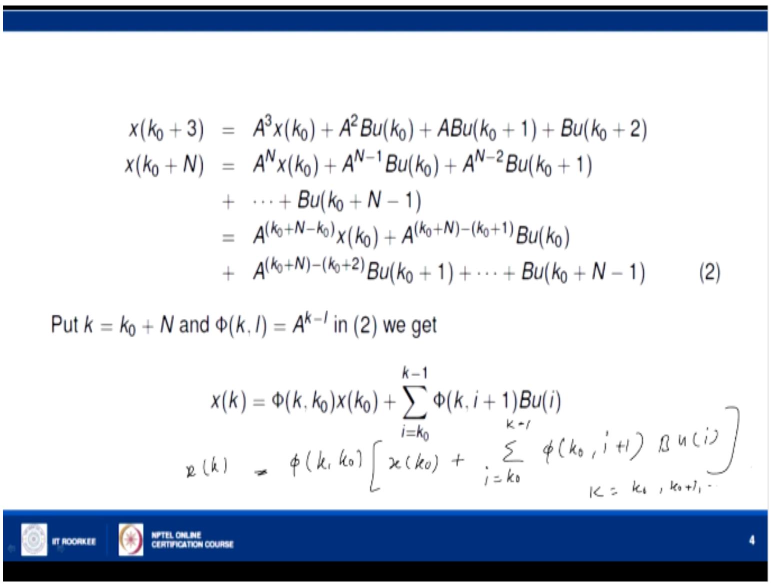

Now using the similar step, continuing in a similar manner we will get x of k0+capital N, we get the last expression as A to the power k0+N-k0 because N is replaced by this and A to the power k0+N-of k0+1 etc. So this expression by replacing this as the state transition matrix, we will get the expression as x of k=phi of k, k0 x of k0+summation i=k0 to k-1 okay.

Substituting k=k0+N and calling phi of k, l as A to the power k-l as defined earlier we get the expression which can be verified in the previous step directly by substitution. So this expression can further be simplified as this following. This can be written as phi of k, k0*x of k0+summation i=k0 to k-1. So this phi of ki+1 can be written as phi of k, k0*phi of k0, i+1. So phi k, k0 we have taken out of the bracket.

And the remaining we write it this way B of u of i. So this we call it as the solution of the system for various values of k, k0, k0+1, etc for the time invariant system.

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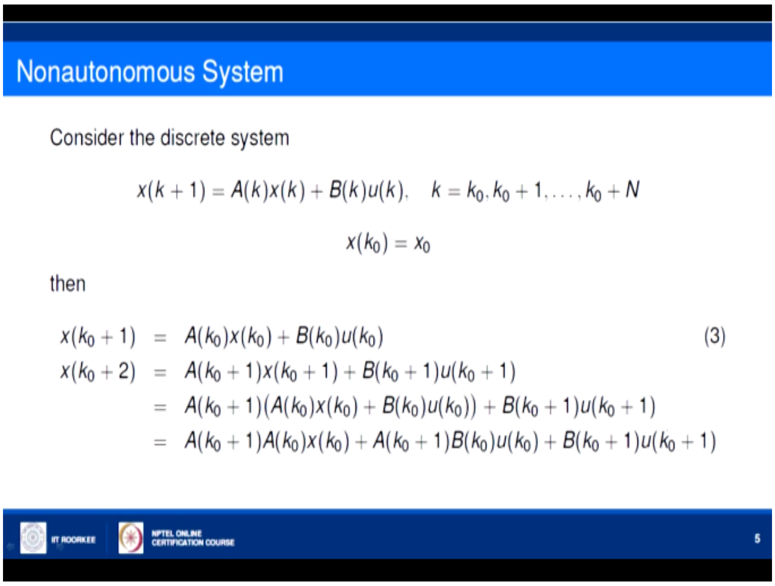


Now if you consider the system which is time variant system nonautonomous system, x of k+1 is A of k x of k+B of k u of k for the various values of k initial condition is x of k0 is x0. So proceeding in the similar manner substituting these values step-by-step, we will arrive at this one, x of k0+1 is A of k0 x k0+B k0*u of k0 that is the initial time and the second step we will get we will make use of the previous step and we arrive at the relation A of k0+1*A of k0 x k0 that is first term+A k0+1 B k0 u k0 second term etc.

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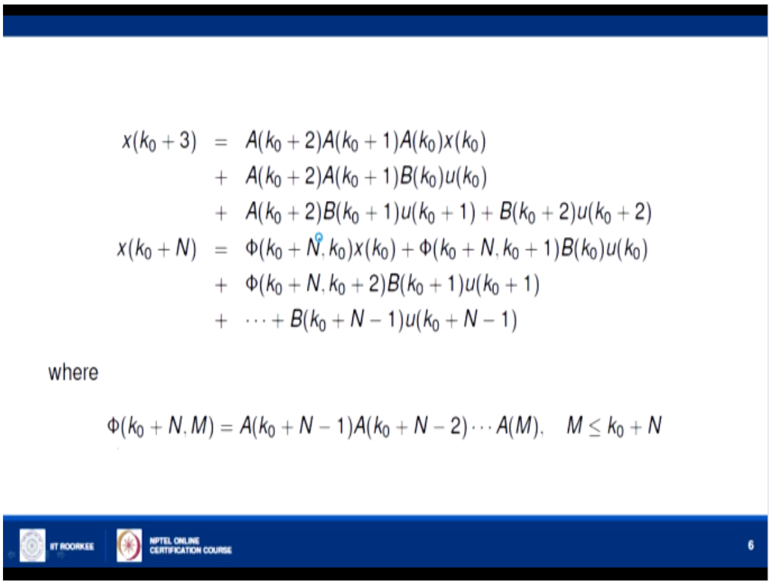


So we can easily generalize it for k0+N, we will get the expression x of k0+N is written in the form of phi k0+N, k0. So we can notice that we define the state transition matrix in this case as phi of k0+N, M is written as A of k0+N-1. So we have to notice here, for example in the third step when x of k0+3 is there, in the right hand side we have A of k0+2 and k0+1 k0. So it is in the decreasing order.

Similarly, when we have k0+N, it will start with k0+N-1 and k0+N-2 etc. So that is what we write here. We denote phi of k0+N, M as the starting will be k0+N-1 and -2, etc. It will stop at the stage capital M. So we can write the expression of the solution in this particular form. Now if you substitute k0+N as k we get phi of k, M as A of k-1.

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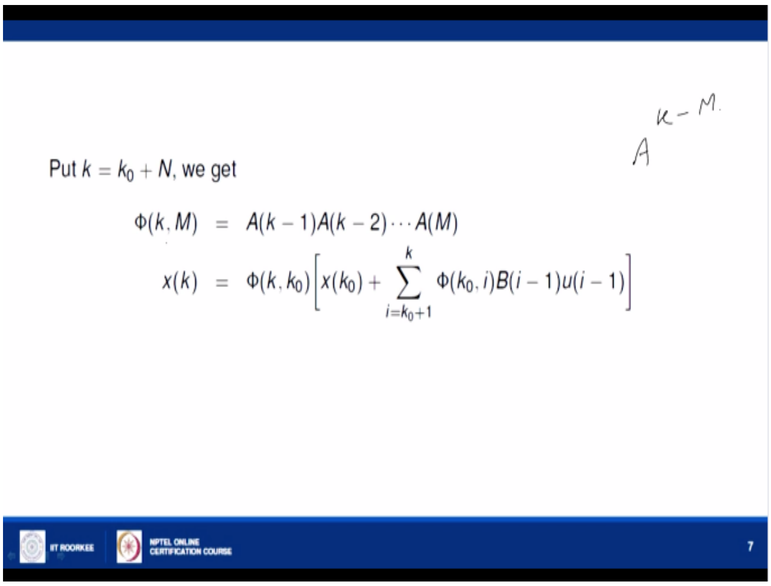

From the last step k0+N is k, so we get here k-1 k-2 up to A of M. So this is called the state transition matrix for the time variant system and if A is a constant matrix we can easily see that this turns out to be the same phi of k-M will be=A to the power k-M as in the previous case. So the solution of the time variant system can be written in this form x of k=phi of k, k0*this expression.

So here we can easily verify the 4 conditions of the state transition matrix. So in this lecture we have seen for the time invariant system how to write the state transition matrix etc. So in the next lecture, we will see about the controllability of the time varying system, similarly the conditions for the observability and stability of all these systems. Thank you.