Dynamical Systems and Control
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Lecture - 43
State Observer

Hello viewers. Welcome to the lecture on state observer. So in this lecture, we will see how to estimate the state of a control system using the observation or the measurement of the system.

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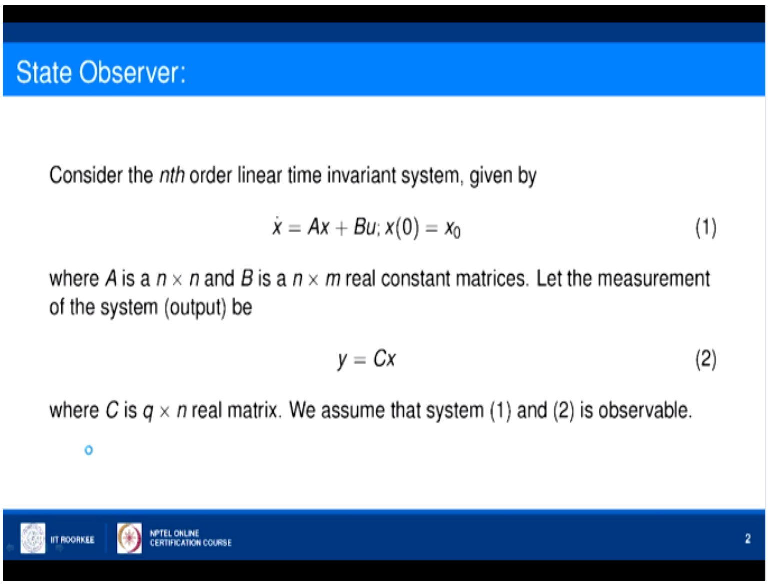

First, let us consider the given control system x dot=Ax+Bu and the initial condition is x0. So here x is the state variable and u is the control variable, A is a n x n matrix and B is n x m constant matrices and the entries are real numbers and the measurement or the observation related to this control system is y is=Cx where C is a q x n real matrix. So first we assume that the system 1 and 2 is observable.

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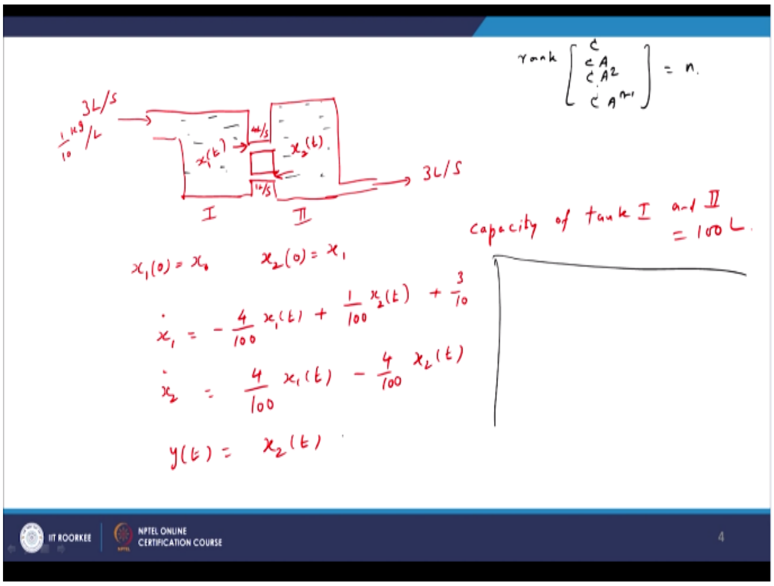

So we have seen the observability of a system, it means estimating the initial condition uniquely that is given the measurement y=Cx if you are able to estimate, if you are able to find the initial condition x0 uniquely, then the system is called observable system. So the meaning is it should be satisfying the given condition that is the rank of the matrix C CA CA square CA power n-1.

So if the rank of this matrix is n, then it implies that the system is observable or in the other way also, the system is observable then the rank of this matrix=n. So if the system is observable, then we should be able to find the value of the state x of t at each instant of time that is theoretical but in practice in many systems, the observation may be available but we may not be able to measure the system state x of t say a function of t accurately.

So it is required that we should estimate this state variable provided y of t that is the observation. So for example if you consider the mixing problem, consider two tanks, tank 1 and tank 2 with capacity let say 100 liters and the x of t denotes the amount of some substance for example salt mixed in this tank 1 and the liquid is let us say water. So the amount of salt in tank 1 is x1 of t and amount of salt in tank 2 is x2 of t at each instant of time.

And there is an inflow of 3 liter per second and in this inflow 1/10 kg of salt per liter is present. So it means 3/10 kg of salt will enter into the system continuously that is a constant rate. Similarly, 3 liters per second will be outflow from the tank 2 and there is a circulation between tank 1 and 2 that is 4 liters from tank 1 enters into tank 2 and 1 liter from tank 2 enters into tank 1, so it is circulating.

Now as time changes the amount of salt changes in both the tanks. Initially, we assume that x0 is the amount of salt in tank 1 and x1 is the salt in tank 2 and the rate of change of the amount of salt in 1 and 2 is given by this equation. It is depending on whatever how much inflow and how much outflow from each tank, the - sign and + sign denotes that one. So it is a dynamical control system and the control is a constant one here.

This 3/10 is entering into the system and the observation we are assuming that we are measuring the amount of salt coming out of the tank 2, y of t=x2 of t can be measured because it is coming out. Now we can see that there is no way of measuring x1 of t because it is completely closed, there is no outlet here. So it is to be estimated only using the information y of t.

So similarly in many problems directly we may not be able to measure the state of this, all the states of the system but we can estimate them using the output or the measurement y of t.

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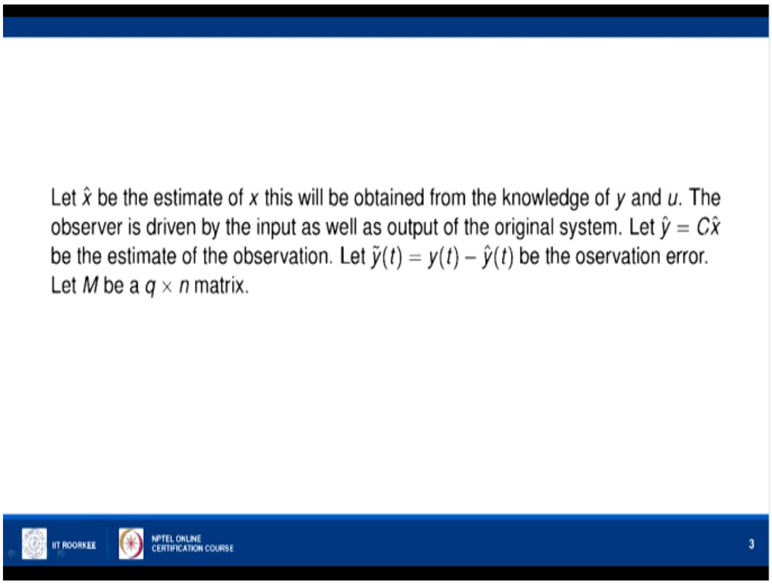

So let x tilde denote the estimate of the state variable x here and from the information of y and u. Now let us assume that y bar or y cap=C times x cap because x cap is the estimate and C times x cap is the estimate of y cap but in fact we have y of t that is the actual value of the output, y of t is available with us and actual value of x of t is not available.

We are estimating x of t, so we are giving this notation y cap as C times x cap and the difference between these two is called the y bar of t. Now if y bar of t is=0, it means that y=y cap and the estimate is equal to the exact value. So let y cap=C*x cap be the estimate of the observation and y bar is the difference between the actual value of y and the estimated value of y here. So we have these values with us, y of t and y bar of t the error. So using this we want to estimate the state value.

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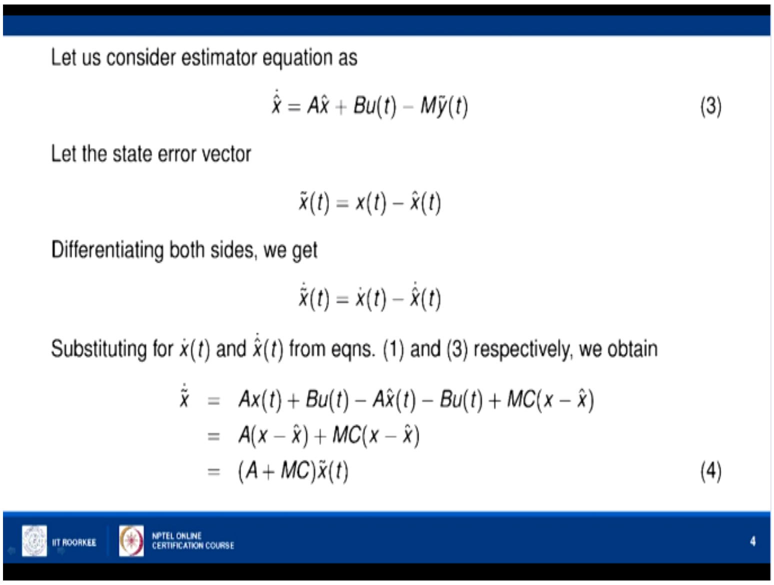

So let M be a q x n matrix a suitable matrix which will be used in the estimator equation. So let us consider this particular equation 3 that is x cap dot=Ax cap+Bu-M times y bar, y bar is the error in the observation. So this M matrix we have to find suitably so that this equation 3 gives a proper estimate of the state variable. So how to find this M matrix in a suitable manner?

So the error in the original value of x and the estimated value of x is denoted by x bar here. Now differentiating x bar that is x bar dot is=x dot-x cap dot and substituting x dot=Ax+Bu and x bar dot we are assuming from the equation 3, so we substitute it here, Ax bar-Bu and +M*y where y=C times x here. So if you substitute this expression and substituting x-x cap as x bar, we will get the equation x bar dot=A+MC*x bar of t. So it is a very simple first-order differential equation in the error of the estimation x bar.

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Thus, the error in state vector is given by

$$\tilde{x}(t) = \exp[(A + MC)t]\tilde{x}(0) \qquad (5)$$

and is independent of the applied control.
The error $\tilde{x}$ will decay to 0 if M is chosen such that (4) is asymptotically stable, i.e., all the eigenvalues of the matrix $(A + MC)$ have $-ve$ real parts. It can be easily proved that if (1) and (2) is completely observable, the matrix M may be chosen so as to place the eigenvalues of $(A + MC)$ in any desired configuration (subject to conjugate pairing).

$A' + c'M'$ has eigenvalues $\{\mu_1, \mu_2 \ldots \mu_n\}$ if rank $[c' \; A'c' \; (A')^2c' \ldots (A')^{n-1}c'] = n$ — observability of the system.

$A + BK$ has eigenvalues $\{\mu_1 \ldots \mu_n\}$ if rank $[B \; AB \ldots A^{n-1}B] = n$

So if you solve this equation, we will get x bar=the exponential of the matrix A+MC*any initial condition, let us say at the initial time t=0 we have made some estimate of the state variable x. So we denote it by x cap at time t=0. So the error in the estimate is given by x bar of t given by this expression. Now we can observe that if this matrix A+MC has all the eigenvalues having the real part to be negative, then the right hand side will tend to 0 as t tends to infinity.

So what we emphasize here is if you are able to select the eigenvalues of the matrix A+MC to be in the left half of the complex plane, then the error will tend to 0 or in other words we will say that x cap will tend to x as t tends to infinity. So theoretically, we say that t tending to infinity it will converge but in practice because it is the exponential decay, it will converge to x cap will converge to x of t as quickly as possible.

So it will reach the exact value of the state variable. So this x of t the solution of the equation 3 if you are able to find a suitable matrix M is called the estimate of the state. So this is how to find the matrix M that is the question here. So for finding the suitable matrix M in such a way that A+MC has eigenvalues with negative real part, we make use of the previous results on feedback control.

So what we have seen earlier is if A is a matrix, the state matrix and B is the control matrix, we have seen that we can find a matrix K such that A+BK has say eigenvalues with negative real parts, eigenvalues u1, u2, un. Any eigenvalue can be assigned, in particular we are interested in eigenvalues with negative real part, so we can assign any for example negative numbers can be assigned.

So this is possible provided the rank of the matrix B, AB, etc A power n-1 B if the rank of this matrix=n, in other words, the pair A and B are controllable matrices. If it is so, then we will be able to find a matrix K in a required manner. So here it is different, it is A+MC. The matrix which we want to found is in this position. So instead of finding the eigenvalues of A+MC, we will find the eigenvalue of A transpose+C transpose M transpose.

So if you are able to find the matrix M transpose in such a way that this A transpose+C transpose M transpose has eigenvalues, any assigned eigenvalues arbitrarily we will give something and eigenvalues for that we will be able to find the matrix M provided this pair A dashed C dashed is controllable. In this case, in the place of B we have C dashed, in the place of A we have A dashed.

So if the rank of C dashed and A dashed C dashed A dashed square C dashed, etc A dashed n-1 C dashed. So if this rank equal to n, then we will be able to find such a matrix M dashed. In other words, this implies the system is observable because this condition is same as if you take the transpose of this matrix, it is same as this matrix which we mentioned earlier rank of C, CA, CA square, etc=n that implies the observability.

So this implies the observability of the system. So if the system is observable, we will be able to find a matrix such that A dashed+C dashed M dashed has any arbitrarily assigned eigenvalues and the transpose also will have the same eigenvalue. So we will conclude that, we can find the matrix M so that A+MC also has arbitrarily assigned eigenvalues for this one. So we can find the estimator of the state using the equation 3. So this is the required equation for the estimation of this thing.

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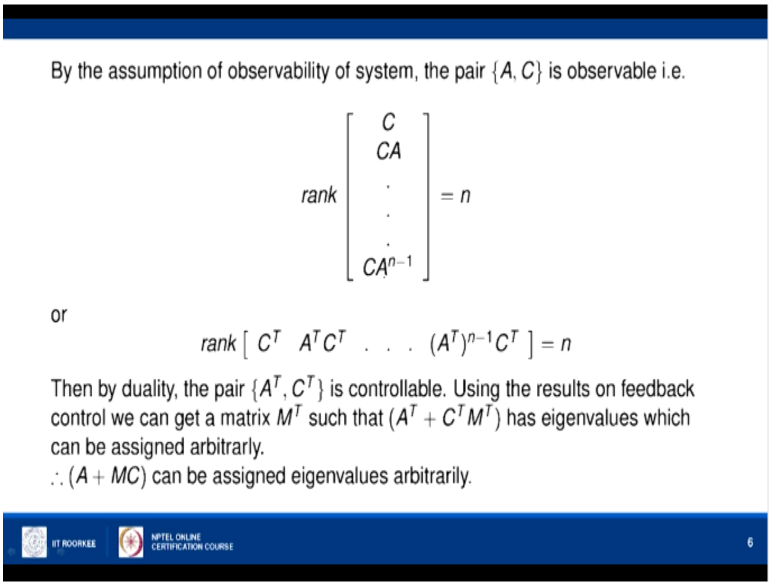

So this is what we have seen if the rank=n, we will be able to find this expression.

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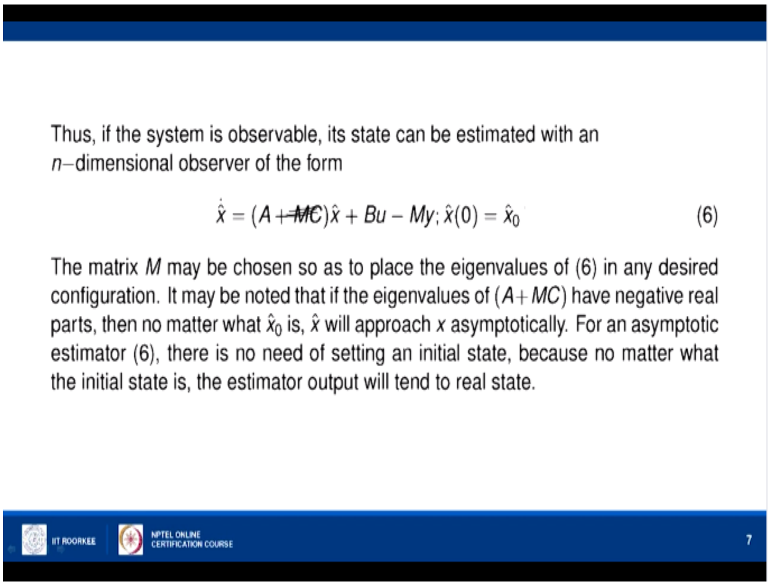

So now we will see, so we have in equation 6 the error system x bar dot=A+MC*x bar. So in this equation 6, the observer or the estimation of the state is given by x cap.dot=A x cap+Bu-My, so this thing there is MC has come extra okay this can be removed. So using this system whatever be the initial condition x0, this will converge to the exact value of the state x of t because of the equation 5 here.

Because we are selecting the matrix M so that all the eigenvalues of A+MC are having negative real part. The estimator will work properly for any initial condition x0 okay.

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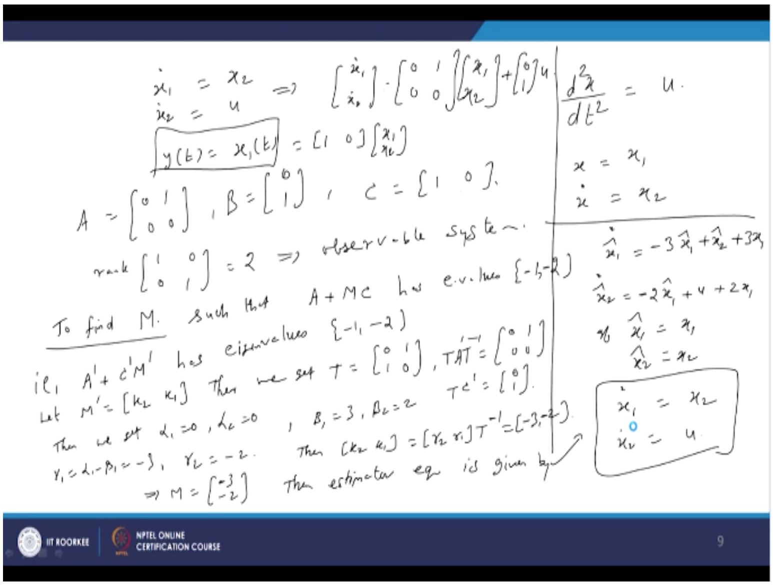

So now we will see for example a simple system which is familiar to us the usual Newton's equation mass*acceleration=force where mass is 1 here and the u is the force acting on the system. So this system if you convert it into a system of equation, if x is=x1 and x dot=x2, we will get the system as x1 dot=x2 and x2 dot=the force u and this will imply we will get to the system as x1 dot x2 dot as 0 1 0 0 x1 x2+0 1*u.

So let us assume that the observation, we will observe the position of the particle at each instant of time x1 of t we are observing, so is it possible to observe the, get the velocity from this expression. So if the measurement is only x1 of t can be observed both the states that is what here, so to estimate this state the entire state x1 and x2 but x1 is already measured, only x2 has to be estimated in this case.

So how to proceed with the equations here, so here the matrix A is 0 1 0 0 and the matrix B is 0 1 and the matrix C, so this can be written as 1 0 x1 x2, so the matrix C is 1 0 and it can be easily seen that the system is observable because the rank of the C matrix is 1 0 and C*A if you multiply it will be 0 1, so the rank is 2. So this implies the system is observable. So now the estimator equation we can easily calculate.

Now we want to find the matrix M such that A+M*C has any arbitrary eigenvalues. For example, eigenvalues if you select let say -1 and -2 any two negative eigenvalues we will select. Then, we will be able to find the matrix in the following way. Instead of finding directly A+MC as we have seen earlier, we will try to find this in the form of A dashed+C dashed M dashed okay has eigenvalues -1, -2.

So we are interested in finding M dash in such a way that this has the eigenvalues -1 and -2, so that can be calculated in the following way. If you assume that M dashed=K2 K1 a row matrix. Then, we can get the T matrix all this can be calculated using the feedback control procedure which we have seen earlier. So we will get the T matrix to be like this and TAT inverse will be in the TA transpose T inverse is in the companion form 0 1 0 0 and T*the B matrix or in other words the C dashed matrix is 0 1 here.

So this is the procedure for finding the companion form of this pair A dashed C dashed and using this T matrix we will get then we get this companion form will give alpha 1 and alpha 2 in the last row. We get alpha 1 is 0, alpha 2 is 0 and from these eigenvalues if you find the characteristic equation for -1 and -2 that is lambda+1*lambda+2=0. So that will give the characteristic equations coefficient as beta 1 and beta 2 value as 3 and 2.

So that will give the gamma values, gamma 1 is alpha 1-beta 1. So these values are -3 and similarly gamma 2 is alpha 2-beta 2 is -2 and then the K2 K1 that is what we require M dashed matrix that is gamma 2 gamma 1*T inverse. Using this procedure, we get the value to be M dashed is=-3, -2. So M matrix is so this implies the required matrix M is the column matrix -3, -2 and the equation 3, the estimator equation 3 here x cap dot=A x cap+Bu-My bar.

So if we substitute all the values here, the M matrix value, etc here, so then the estimator equation is given by this equation. Let us write here x1 cap dot=-3 x1 cap+x2 cap+3 x1 and x2 cap.dot=-2 x1 cap+u+2 x1. So from the equation 3, estimator equation 3 using the matrix M, we will be getting this estimator equation. So if you solve this equation, we will obtain because x1 is the observation.

As given here, the observation is given by x1, so using x1 how to estimate x2? Here it is estimating x1 also but it is already measured here as the measurement. So x2 bar can be obtained. So if we can verify whether this equation is proper or not, so if you substitute the exact value for the x1 cap and x2 cap as x1 and x2 itself, so that will imply that we will get so if x1 cap=x1 and x2 cap=x2 in this equation what we get is the original equation x1 dot=x2 and x2 dot=u is the original equation from this itself.

So it verifies that the system given in equation 3 can estimate the state variable in a proper manner. So this is about the estimator of the state variable.

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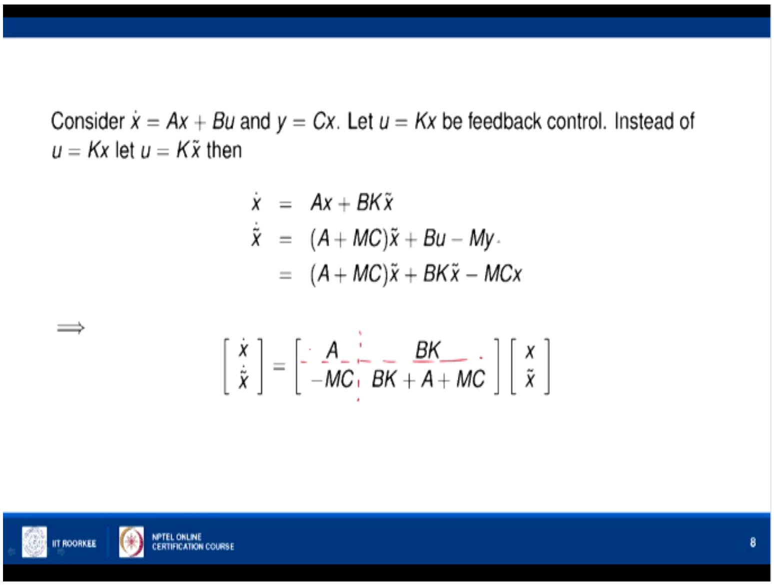

Now we will consider the system x dot=Ax+Bu and the observation y is=Cx. So the previous problem is to estimate the state variable x and so in this section what we will see is how to find the feedback for the given control system x dot=Ax+Bu. So usually when we consider the feedback u is=Kx where K is suitably chosen matrix so that A+BK will behave in a desirable manner.

That means we will assign a fixed set of eigenvalues and we are interested in finding K so that A+BK has that set of eigenvalue. So for that it is required that we should substitute the exact value of x for the feedback control but in practical problems as we have already seen, it may not be possible to get x but we will be able to get an estimate value x cap. So let us try to substitute the control u in the form of K*x cap where x cap is the estimated value of x.

So now the control which we are applying is not the one which we desired that is u=Kx but it is slightly different, u=K x cap. So when we substitute that control, the system is x dot=Ax+BK x cap and the second equation is it should be x tilde dot, that is the equation 3 which is the estimator equation=A+MC. So now if you substitute the control u=K x tilde or K x cap, we get the equation as x dot=Ax+BK x cap and x cap dot=A+MC which is the equation 3 that was the estimator equation.

Now from these two equations, if you combine these two equations as a single control system, we can see that the first equation, the first row*this column will give Ax+BK x cap and the second equation x cap dot will give the equation. So in the matrix form we can write like this where these are the block matrices A, BK, etc.

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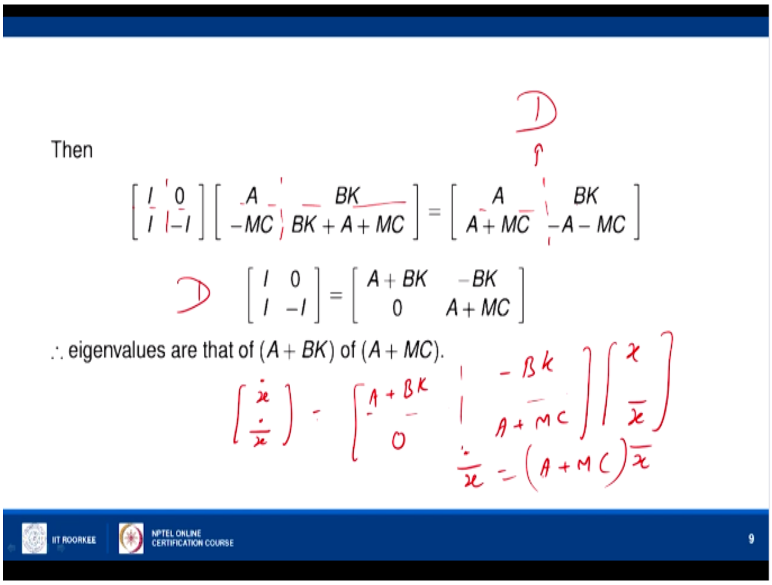

Now pre-multiplying the block matrix I 0 I –I, the sizes are compatible, then we will get this expression A BK in the first block and A+MC –A–MC after multiplication. Now again multiplying this matrix by I 0 I -1, this multiplied by so let us call this matrix as some D matrix and if you put D here that is it is missing here.

So D*I 0 I –I we will get this expression and if you multiply the left hand side also in the same manner with these two matrices I 0 I –I in the left side, again I 0 I –I in the right hand side for this one we will get the expression to be like this. So we get the equation to be x dot and x cap.dot okay, instead of this left hand side we will get this expression and the right hand side will be as given here A+BK and –BK block 0 and A+MC*x and x cap.

So we will get a system of equation in this manner. Now we can see that this is making it into two compartments, so they are even though this system looks as if they are combined system. Now after multiplying left and right using these matrices, we will get the system to be like this. So the first equation will give x dot=Ax+BK*x and –BK*x bar, the error in the estimation.

The second equation gives x bar dot is simply so the second equation x bar dot=A+MC*x bar. So if you are selecting the eigenvalues of A+MC properly that is all negative eigenvalues, then it is obvious that x bar will tend to 0 as t tends to infinity. So when x bar tend to 0, it automatically implies that x cap will tend to x as t tends to infinity.

So whether we use in the feedback control K*x or we use K x cap, both will give the same effect as t becomes larger and larger because x cap is going to tend to the value x here. So the effect of control will be the same as t increases. So that is advantage of the estimator here. We estimate x and then use it in the feedback control for driving the system. So in this lecture, we have seen how to estimate the state variables using the observation and how to make use of the estimate in the feedback control of the system.

So in the next lecture, we will see observable and unobservable systems and how to find the unobservable subspace and then estimate the observable states of the system. Thank you.