Dynamical Systems and Control

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Lecture – 42
Feedback Control - IV

Hello viewers. Welcome to the fourth lecture on the feedback control. In this lecture, we will complete the proof of the theorem which we have started in the previous lecture.

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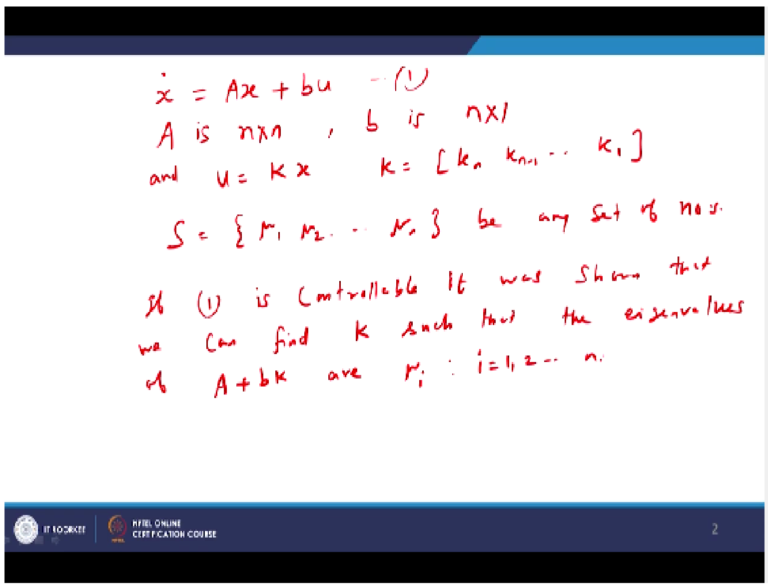

In lecture II of the feedback control, we have seen the following result, that is if you consider the equation x dot=Ax+bu where A is n*n matrix and b is n*1, it is a column matrix and u is a feedback control where k is a column, kn, kn-1, k is a row vector which is unknown and S is a set mu 1, mu 2, etc., mu n be any set of numbers, real or complex number.

Then if the system is controllable, it was shown that if 1 is controllable, we have shown that we can find the feedback matrix k such that the eigenvalues of A+bk are mu i, i=1, 2, 3, up to. So this was the result which we have already proved. And the procedure how to find the matrix k was also shown earlier. So in this lecture, in the previous lecture, we have started the same theorem for the general case, that is where A is n*n matrix and b is a n*m matrix.

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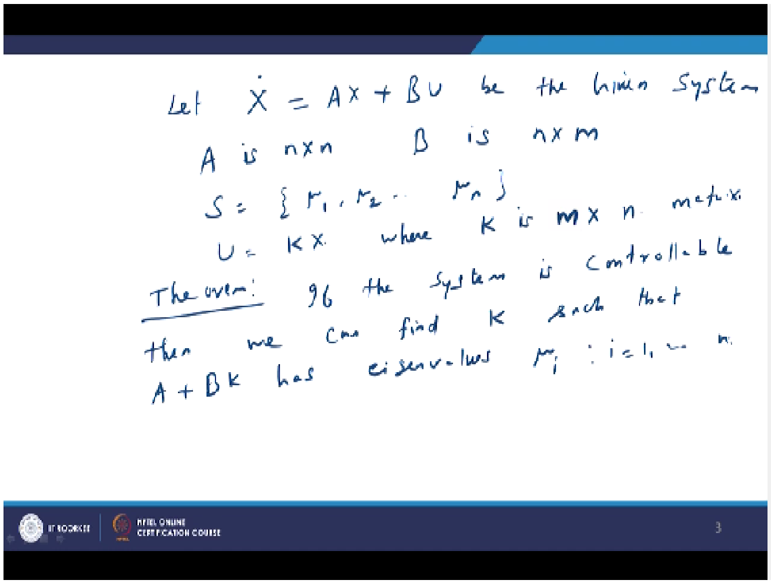

So that we will see. So we will see that let X dot=Ax+Bu be the system, be the given control system and here A is n*n. So same statement, only the difference is the B matrix is n*m. S is any set mu 1, mu 2, mu n and the feedback control u is k*x, here k is according to the size of the A and B, it is n*m matrix, m*n matrix, okay.

So the theorem statement is if the system is controllable, then we can find the feedback matrix k such that A+B*k has eigenvalues mu i. So there is no change in the statement except the matrix B. So for proving this, we have already proved a lemma in the previous lecture. It is in the last time, we have shown that.

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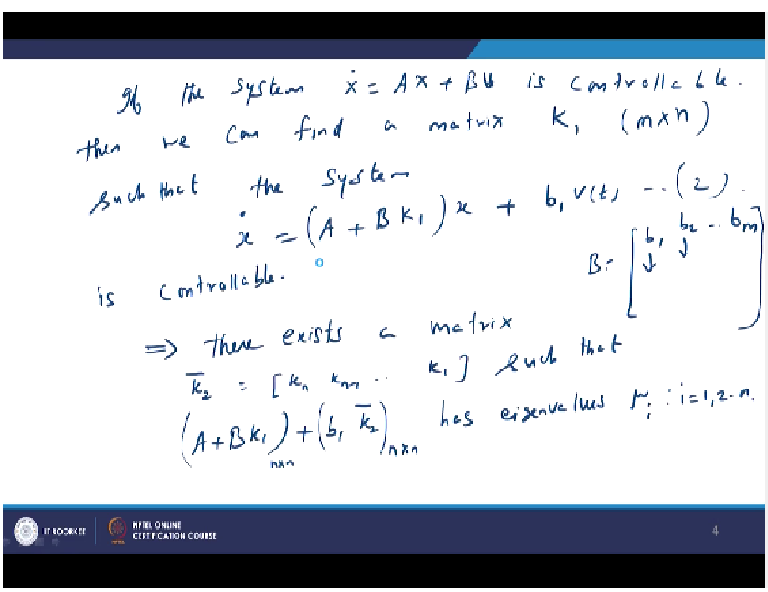

So if the system is controllable, if the system x dot=Ax+Bu is controllable, then we can find a matrix k1. In this case, it is of the size m*n, okay, m*n matrix such that the system; now we construct a new system, that is x dot=A=B*k1*x+b1*v, okay. We change the notation for control instead of u. Here u is a vector belonging to Rm and here v is the real number, v of t is a real number.

So it is a single input system. So what we have done is from the original system, what was given, we have converted into a system of this type the control matrix is a column vector only, okay. So this is in the form which we consider in the second lecture, like b is a column vector. So we have already shown that we can find a matrix k1 such that this system is controllable. Here b1 is the first column of the matrix B. So if B matrix is b1, b2, etc., bm.

So if you take, if the first column is non-0, we can convert the system into this expression. And we have already discussed if the first column is a 0 vector, then how to take another non-0 vector from the B matrix and then convert the system. So this was discussed. So we assume that b1 is a non-0 vector and so we convert the given system into this system. Now we can apply the theorem which was already proved for this type of system.

So the system 2 is controllable, already it is done and we have a matrix k1 also. Now since the system 2 is controllable, so this implies there exist or we can find a matrix, let us say k2 bar, a row vector so that is like kn, kn-1, etc., k1 as proved in the first theorem, that is if B is a column, then we can find a matrix k so that, such that. Now the matrix control, the state matrix is A+Bk1 and the control matrix which we are having is k2, so we can show that such that A+Bk1 is the matrix already given, +b1*k2 bar.

So it is a n*n matrix. This is also n*n, b1 is a n*1 matrix and k2 bar is a 1*n matrix, so the product will give n*n. So this has eigenvalues mu i, i=1, 2, 3, up to n. So by using the previous theorem, this theorem. So now we are interested in finding a feedback control for u, that is u=K*x type of thing.

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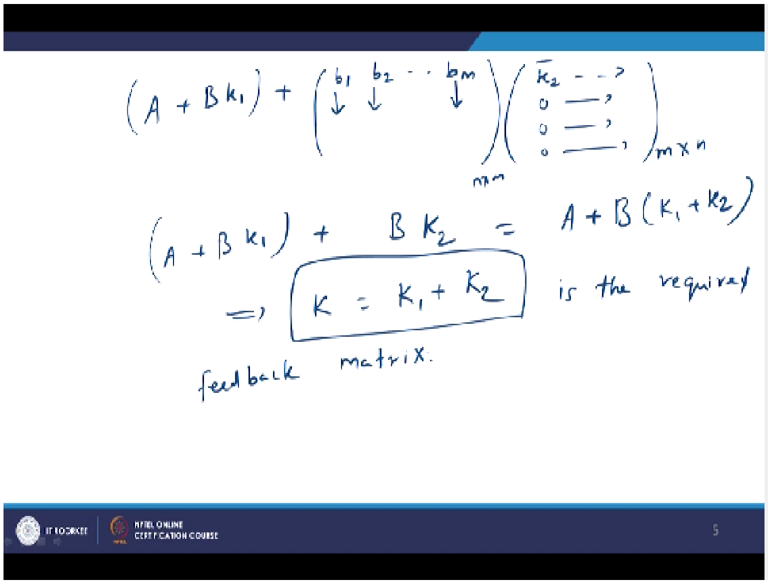

So for finding this K, we make use of this result. So we have A+Bk1+b1*k bar has eigenvalues this. So let us try to convert it into this form A+Bk1 is already there and in the place of b1, we will write, b1 is the first column, already there and we take all the other columns of b matrix, okay. And the first row is from this equation b1 is the column and k2 bar is a row vector. So let us take k2 bar as the first row and remaining rows are all 0.

We take a matrix like this. It is m*n matrix. It is n*m. So ultimately, the product will give, because all the second, third, all the rows are 0. The product will give the same effect as b1*k2 bar. So we can write it as B*some matrix like this. A+BK1 is as it is. This one can be written as B and this can be written as K2 matrix. so this can be written as A+B*K1+K2, okay. So this K1+K2 is the required matrix K. K is nothing but the K1+K2.

So this implies, so it is in the form A+BK. So K is K1+K2 is the required feedback matrix, okay. So this theorem is proved for the general case where A and B are.

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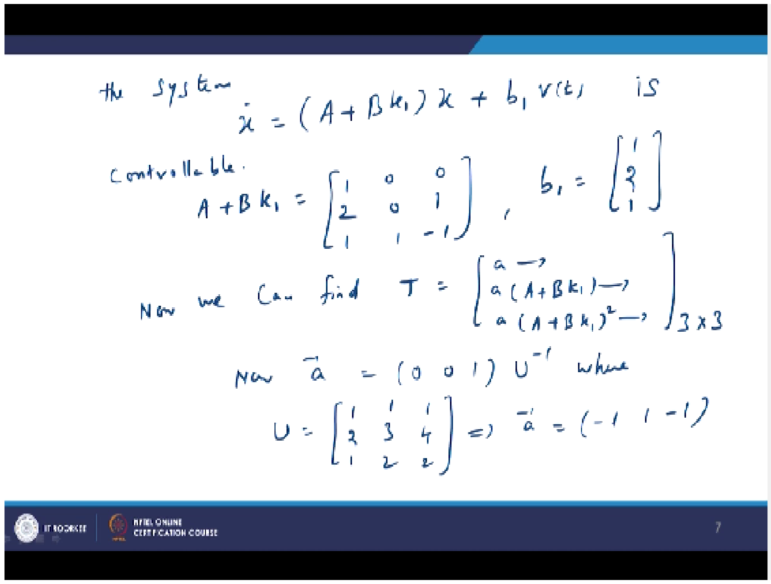

Now we will just illustrate it with an example quickly, that is in the example. So this we have already taken. This particular example was done in the previous lecture 1 0 1, and 0 1 -1 and B is the matrix 1 2 1, 0 1 1. So we can easily check that B, AB, A square B, the rank is 3. So it is controllable system. And let us take S to be any set, for example, -1, -2, -3. So now we want to find the matrix K such that the eigenvalues of A+BK are the same -1, -2, -3. So how to find this K, that was explained in the previous slides.

So quickly if you recall that, so we already have seen that in the last lecture that the matrix K, there exist a matrix K1 which is given by this matrix 0 0 0 1 0 0 such that the system x dot=A+BK1*x+b1, their first column of B matrix, *v of t is controllable. So this was already shown. So here A+BK1 matrix, if you calculate the matrix, we will get 1 0 0 and 2 0 1 1 1 -1 by substituting these matrices.

And b1 is the first column of the matrix b, it is 1, 2, 1. And so this system is controllable. Now by using the companion form and the theorem of feedback control, we can find the T such that, now we can find T which is of the form, the first row is a. The second row is a*A+BK1. The third row is a*A+BK1 square. It is a 3*3 matrix, non-singular matrix such that TA+BK1*T inverse will be the companion form.

And T*b1 is 0 0 1, in the standard companion form, it can be converted. Then we can find the expression, this a can be found out using this matrix. So now the a vector, the first row of this T matrix is nothing but 0 0 1*U inverse. What is U inverse? It is the controllability matrix obtained from this one, B, AB, etc. here. Where the matrix U is obtained from the first, that is b1. In the system, the control matrix is b1 and A+b1*b1, etc. we will get.

And A+BK1*b1 will give 1 3 2. And A+BK1 square*b1 will give the element 1 4 2. So this is of rank 3 and its inverse will exist. So this will imply that a bar, the vector, the first row of the T matrix is given by this entry, -1 1 and -1. We can multiply and then get this value like this. So once you have obtained the first row of T matrix; second, third row can be obtained from the formula.

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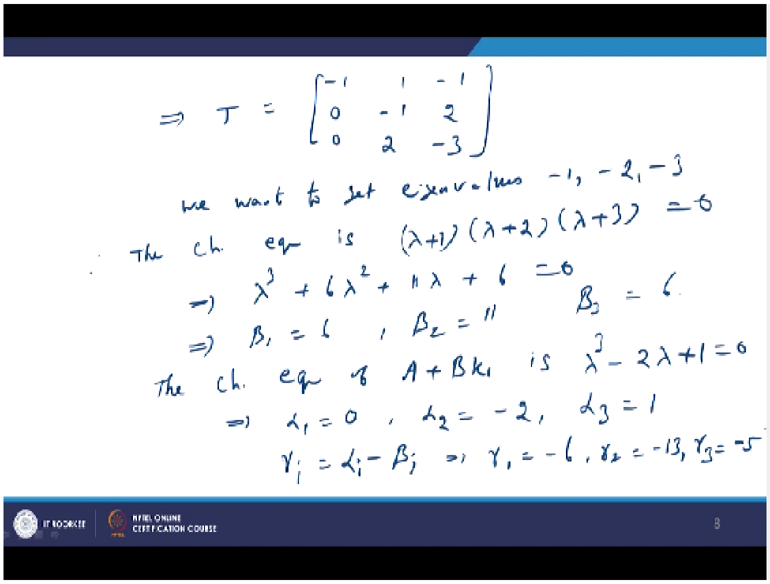

So that implies that T matrix is calculated as -1 1 -1 and 0 -1 2 and 0 2 -3. After substituting the proper values in T matrix, we get this one. Now we want to find this feedback control for this expression that is we want to find the K2 matrix so that we ultimately find K1+K2. So for finding K2, we adopt the following procedure. So we are interested in getting, we want to get eigenvalues -1 -2 -3 for the converted system.

So that implies the characteristic equation is lambda+1, lambda+2, lambda+3=0. So that will imply that we get, this is lambda cube+6lambda square+11lambda+6=0. So the notation of the previous lectures, we see that beta 1 is 6, beta 2 is 11 and beta 3 is 6 again. And the companion form will give, the companion form of this matrix A+BK matrix or you can say that the characteristic equation of A+BK1 matrix.

So that is given by directly if you calculate, you will get lambda cube-2lambda+1=0. So from the standard notation, we get alpha 1 is 0, lambda square term is not there, alpha 2 is -2 and alpha 3 is 1. So if you recall from the second lecture of this one, feedback control, we see that the expression gamma value, gamma i is alpha i-beta i. So that will imply that alpha 1 is -6, sorry gamma 1. So we get gamma is -6, gamma 2 is -13 and gamma 3 is -5.

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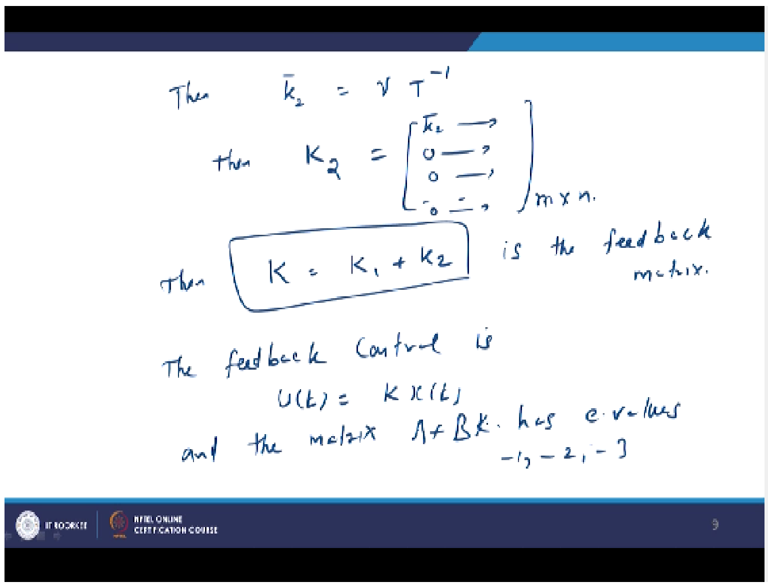

So from this calculation, we will get, finally we get then again recall the method which we have seen in the second lecture that is gamma*T inverse, the row vector K is given by gamma*T inverse. So once we calculate this, then we can calculate the K2 vector which we require actually. This procedure, we are defining this K2 vector where K2 vector is the first row is the matrix which we found and all the other rows are 0.

So after finding this K2 bar, we will put this K2 bar in the first row and then all the remaining rows are 0, is a matrix m*n matrix, okay. So then the required matrix is K1+K2. So this can be calculated by, because all the values are available. Gamma is there, T inverse is there. By substituting, we get the feedback matrix. So if we substitute this, so the feedback control, the U of t which belongs to Rm, that is K*x of t.

So the feedback control is given by U of t=K*x of t. And the matrix A+B*K has eigenvalues -1 -2 -3. So this can be easily verified by actually calculating the eigenvalues. So now if you see here, the main assumption in this theorem is that the system x dot=Ax+Bu is controllable. But in case the system is not controllable, then is it possible to find a feedback control in this similar form? So that we will briefly see because we have some procedure for controllable system.

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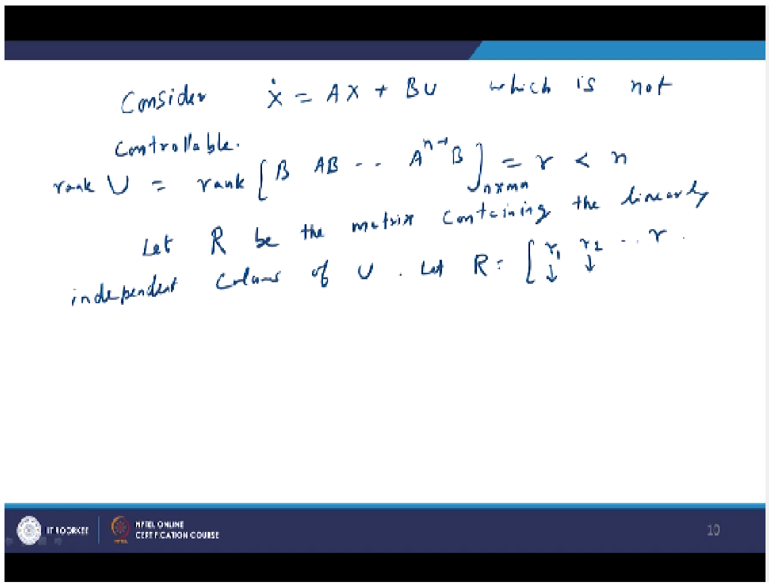

And now we will reduce the non-controllable system into a system which has 2 parts. One portion is controllable portion. Another portion is non-controllable portion. And then we can apply the same theorem for one portion that is which is controllable portion, okay. So consider the system X dot=Ax+Bu which is not controllable. So that means the rank of the matrix, the block, first block is B, second block AB, and A power n-1B, this rank is equal to, let us say r which is less than n.

So then can we apply a similar theorem? Can we find a feedback control that is the question here? Now let us consider the matrix like this. So let R be the matrix containing, because the rank is R here, we can select R columns from the matrix U. Let us call it as U matrix where U is B, AB etc. The size of the matrix is n*m*n but only R columns are linearly independent because rank is R.

So we select any R column which are linearly independent from that. And put it as a matrix containing. So let R be the matrix containing the linearly independent columns of U or any linearly independent columns we can select, R of them are available. So let R=the first column is r1, second column is r2 and r suffix r. This notation is not.

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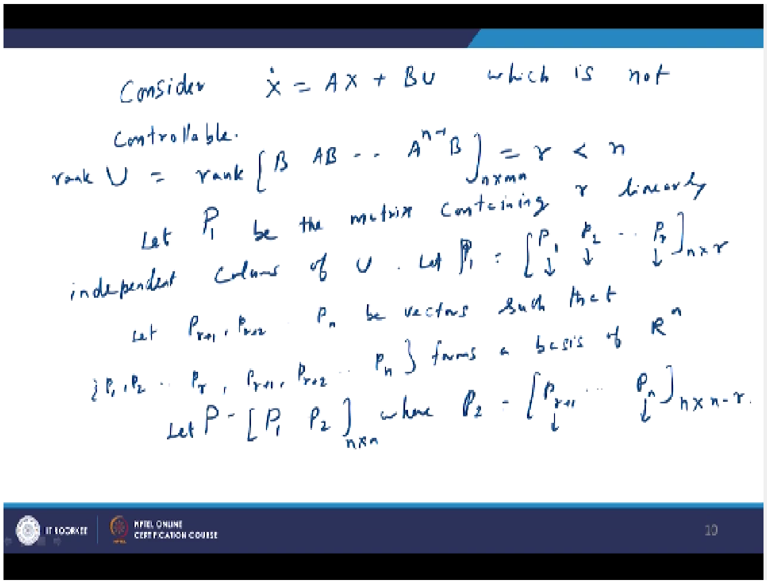

So now let us take a matrix P such that it contains the linearly independent columns are linearly independent columns given by, let us say P1, P2, etc., P suffix r. And so let us take Pr+1, Pr+2, etc., Pn are remaining linearly independent vectors which generate the entire space such that the entire set P1, P2, etc., Pr that is already taken from the U matrix and Pr+1, Pr+2, etc., Pn. And so this forms a basis of the space Rn.

So let us write the matrix as, let us consider the matrix P1, P2, okay, the block of this one. Here the notation we can consider, instead of P, we can consider as P1 and P2 is the remaining matrix, okay, where P2 is given by Pr+1, etc., Pn. So the column vectors are like this. So it is a matrix n*n-m matrix and this is a square matrix n*n matrix. And P1 matrix is n*r matrix. I think there is some notation problem. r+1 etc. rn, 1 minute. You can cut it little bit. n-r. So we have the matrix as P1 of this size and P1P2 forms a matrix P, okay.

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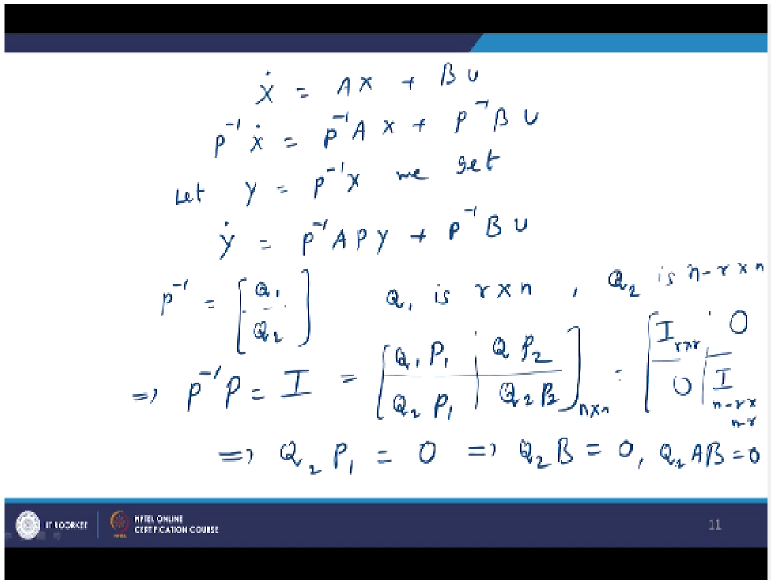

So now consider the equation X dot=Ax+Bu. If you multiply both sides by this P inverse matrix, we get P inverse Ax P inverse Bu and if you let y=P inverse x here, we get, from this equation, we get, y dot=P inverse A, x can be written as Py and +P inverse Bu, okay. So let us write this P inverse the matrix, another matrix Q1 Q2 as a block matrix. The first block contains r rows and the second block contains, so Q1 is a r rows and n columns and Q2 contains n-r rows and n columns.

So if P inverse and P, that is identity matrix and if you actually multiply P inverse and P matrix, we will get Q1*P1 and Q1P2 Q2P1 and Q2P2, these are block matrices. Q1 is of the size and P1, we have already seen. It is n*r. So the first one is the square matrix r*r matrix. So accordingly we can find the size of this one. So if identity matrix also should be divided into I. This will be r*r and this is the remaining n-r *n-r size.

The remaining are 0. So from this notation, what we get is Q1P1 is identity and Q2P2 is identity. The remaining are 0 matrices. So in particular, we are interested in this Q2P1 is the 0 matrix, okay of the proper size. Whatever size we have. Now if you observe, this P1 matrix contains all the vectors which are in the controllable matrix B, AB, etc. This equation, Q2*P1=0, it implies that all the columns of P1 and the rows of Q2, they are orthogonal to each other.

Because the row of Q2 multiplied by a column of P1 gives the 0 element. So it implies that Q2 is orthogonal to the columns of P1. So that can be utilized. And if we observe this U matrix, it contains, it is generated by the columns P1, P2, Pr. So all these matrix, AB, A square B, A power n-1 B, all these columns are generated by the elements of P1, P2, Pr because there can be only maximum r linearly independent columns for this.

All the columns are generated by P1, P2, Pr. So the matrix B, AB, A square B, all the columns are generated by the vectors P1, P2, Pr. So here Q2P1=0, it automatically implies the various other things.

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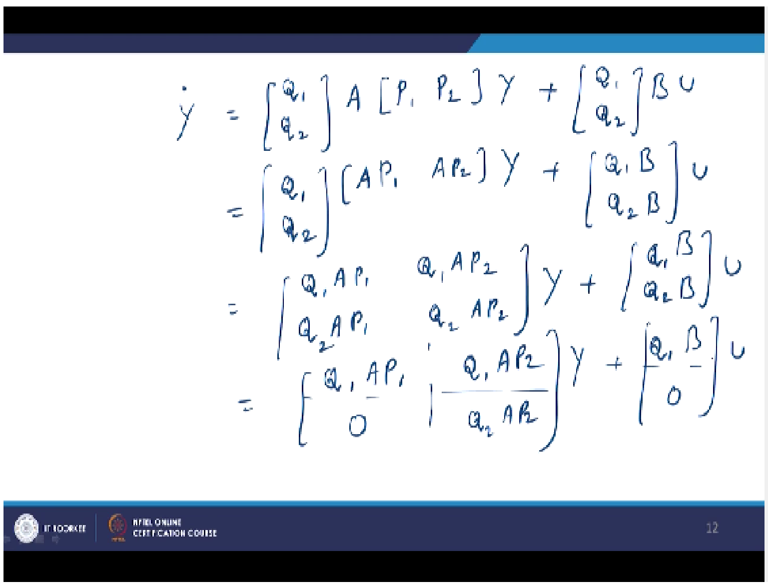

If we observe this equation y dot=P inverse APy, y dot=, P inverse is written as Q1Q2 and A is there. P is written as P1, P2*Y+, again P inverse that is Q1Q2, *B*U. So this is the equation in this notation introduced in this page. So this implies that Q1Q2, the first matrix, *matrix A*this block, y dot=Q1Q2 and this product will give AP1 is the first block and AP2 is the second block, *the vector y and this can be written as Q1B is the first block and Q2B is the second block, *the vector U.

So again, we multiply the Q1 block with the AP1. So this will give Q1AP1 and Q1*AP2, Q2AP1, and Q2AP2; *the vector y and this is as it is, Q2, sorry Q1B and Q2B, *U vector. So now we can, I make use of the thing here, Q2P1=0. It means the rows of the matrix Q2, they are all orthogonal to all the vectors P1, P2, etc. And so that implies, this step itself will imply that

Q2*B=0.

Because B also is part of the U matrix and all the columns of U matrix are generated by P1, P2, Pr. So if Q2 is orthogonal to all these P1P2, it should be orthogonal to B as well as AB, A square B, etc. So this implies that Q2*B is 0, Q2*AB will be the 0 matrix, etc. So we can easily get this expression Q1AP1 and Q1AP2. But here, Q2 is orthogonal to all the elements, all the columns of U. So AP1, AP2, all of them are columns of U.

So this will be the 0 block and here, it is Q2AP2, we get y. And here it is QB, but Q2*B is 0, okay, 0 block by the previous one. So this separates the system into very simple form like this.

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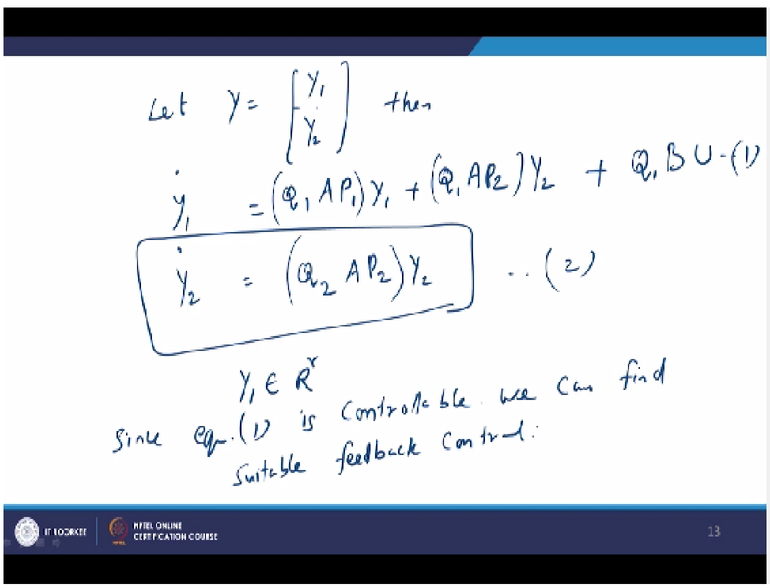

So if we write y to be, it is an Rn vector. We will write it as y1 vector and y2 vector, first R elements are called y1. The remaining n-r elements are called y2. So if you write like this, then we get from the previous page, we will get y1 dot, the derivative of y1 is Q1AP1*y1+Q1AP2*y2+, here Q1B*U vector we will get, okay. Similarly, y2 dot will be 0*y1, so that will not come.

Q2AP2*y2+0, so we will get the simple form y1 dot=that is Q1AP1y1. So y1 dot is Q1AP1*y1, this matrix, +Q1AP2*y2 matrix, +Q1B*U, okay. Similarly, y2 dot, it will be Q2AP2*y2 and +0. So this system, we can see that there is no control at all. Simply it is a separate system. It is a dynamical system and only first system is having a control. So this system is controllable, actually this y1, it belongs to a R dimensional vector space and we already saw that the system is controllable because the rank of the controllable matrix is R here.

It is equal to the control space. So this system is controllable. It is divided into 2 systems. The first system is controllable and the second system has no control at all. So it is not controllable. Now we can find a feedback control for it. Because the first one is controllable. So since equation 1 is controllable because we can assume that this y2 is a known quantity. We can easily solve this differential equation 2 and then substitute the value of y2 in the first equation.

So only y1 is the state variable for the first equation and U is the control variable and y2 is a known value because of the dynamical system. So this system is controllable due to the rank condition. One is controllable. We can find suitable feedback control using the previous theorem for the controllable systems. So we can make use of this theorem for controllable system and find the appropriate feedback control for any required eigenvalues of this thing. So I will complete this lecture with this example. Thank you.