Dynamical Systems and Control

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Lecture - 41
Feedback Control - III

Hello Viewers. Welcome to the third lecture on Feedback Control. In this lecture, we shall prove the general version of the theorem on feedback control which was discussed in the previous lecture. So today we will see the following result.

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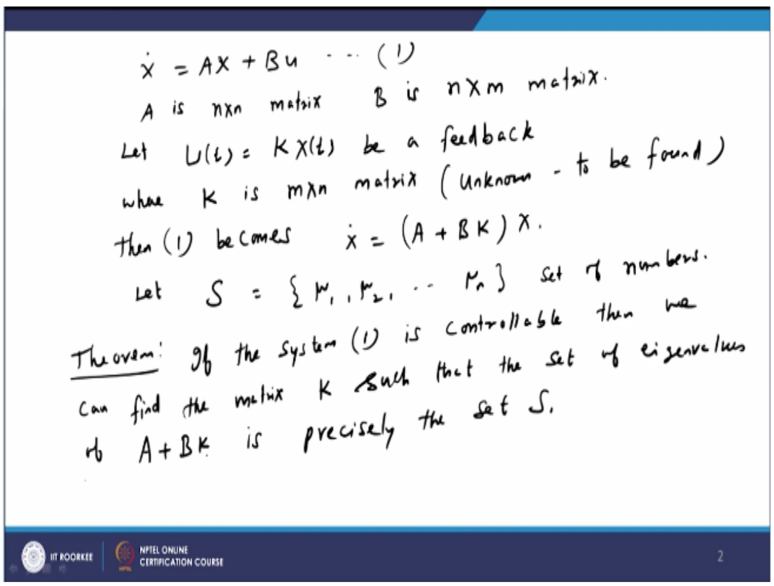

First let us consider the system dx/dt=Ax+Bu, where A is n cross n matrix, B is n cross m matrix. Let u(t)=k times x(t) be feedback control. This is a feedback control where this matrix k, it is a m cross n matrix which is unknown to be found. We have to compute it according to a particular goal. So u be the feedback control given by this. Then the system become let us call it as System 1, x dot=A+Bk times x.

So now let us consider a set, let S be a set of numbers Mu1, Mu2 etc., Mu n. Set of numbers real or complex. So now we want to prove this theorem. The last lecture it was proved for a particular case that is when the matrix B is a n cross n matrix a column matrix only, now we will prove this theorem for a general case where B is a n cross m matrix. So here the theorem is, if the system

(1) is controllable then we can find the matrix k such that the set of eigenvalues of A+Bk is precisely the set S.

So the theorem is any given orbitary set S, we can find a matrix k so that A+Bk has these eigenvalues Mu1, Mu2 etc., Mu n provided the system is controllable. So it is a very useful theorem in many practical situations as we mentioned earlier if you want to stabilize this system then we require that all the eigenvalue should have negative real part. So if already the system is not stable then by selecting this, all these Mu I S say negative number or with negative real part, we can make the system to be stable system. So now we will before proving the theorem.

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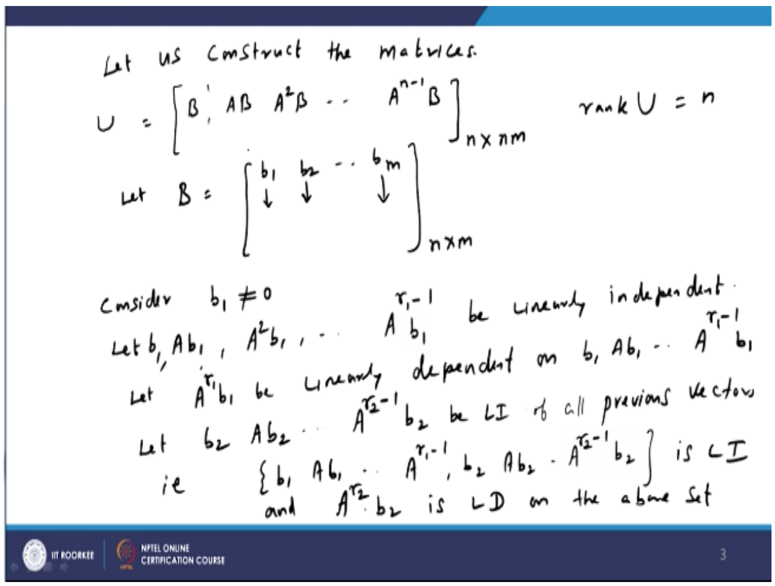

Let us construct the following matrices. First this matrix U, it is the first block is B, second block is AB, A square B etc., A power n-1 B. So this is already known to a; it is a n cross n*m matrix. The rank of U=n because the system is controllable. So that is already given fact. Now we will consider, so let the matrix B the first column is b1, the second column b2 and the bmth column is bm. So it is already given that it is a n cross m matrix, so this matrix is given.

So now let us consider the first vector b1 the column vector. If it is completely 0 then we will omit it and then go to the next one. But let us assume that b1 has at least one non-zero entry so consider b1 which is a non-zero vector, okay the first vector. Now we consider, let A*b1, A; see the first column of the first block b is b1. The first column of the second block Ab that is A* first column of b is b1, so A*b1 will give the first column of the second block. Then A square / b1 will give the first column of the third block.

So what we will do is we will select n such blocks; say n such column vectors from each block one by one. So let Ab1 A square b1 etc., A to the power r1-1. So we will go up to this step r1-1 like this. From each block we will select the first column like this. So this b linearly independent, b1 Ab1 A square b1 up to this be linearly independent.

And then the next entry A power r1*b1 the next one that we assume to be linearly dependent, so wherever it becomes linearly dependent we give that number r1, okay. A to the power r1*b1 be linearly dependent on all these column b1, Ab1 etc., A power r1-1. So for example let us say Ab1 itself is the multiple of b1 that means r1=1 in that case, okay so like that we have the number r1.

So we have now r1 such vectors b1, Ab1, A square b1 up to this one these are all linearly independent. The next vector has become linearly dependent. So what we will do we will start with the second column of the first block. B2 is the second column, so then we consider, let b2 Ab2 etc., A power r2-1*b, so the second column of each block now selecting. So let us assume that these are all linearly independent of all these thing, linearly independent set, okay, linearly independent of all the previous entries of all previous vectors.

Or that means we have the set b1 Ab1 A power r1-1 already and then we have b2 Ab2 etc., A power r2-1. So all these set, the entire set is linearly independent. Then the next entry is linearly dependent, that is if you take A power r2*b2 that will be linearly dependent on all these entries, okay. And A to the power r2*b2 is linearly dependent on the above set. So we cannot go to the next entry. What we will do we will take the third column of the first block, b3 and then third column of the second block Ab3 etc.

So proceed in the same manner we will get. So it cannot go beyond n such vectors. It will not go indefinitely, because total number of column is n*m. The rank of the matrix is n, so there will be only n linearly independent columns available for this matrix. So in this procedure we can select only n linearly independently such columns.

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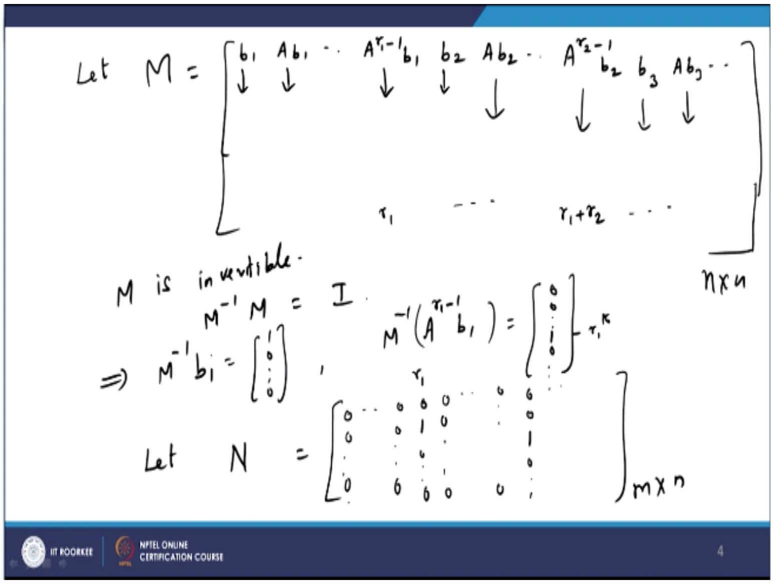

So we will construct a square m, M be equal to; its first column is b1, the second column is Ab1 and A power r1-1 * b1 is the r1th column. Then the next column is b2 then Ab2 A power r2-1 b2. So after that it will become linearly independent therefore we will start with b3 Ab3 etc. So we can go up to n such columns, okay all of them are column. So it is a n cross n matrix, square matrix here. And this will be invertible because the rank is n, we have all linearly independent columns, so M is invertible.

So M invertible exists. So this will imply M inverse M is identity, is not it. So that will imply M inverse * b1 will be; M inverse * first column will be 1 0 0 etc. So like that M inverse * for example the A power r1-1*b1, if you multiply the r1th; this the first column second this is r1th column and like this, this will be r1+r2th column etc., finally we will get the nth column. So if you multiply the r1 column with the M inverse we will get 0 0 1 0 at the r1th position, 1 will be in the r1th position.

Similarly, each column can be multiplied and we will get this vector is not it. So this matrix M is constructed in this fashion. So we note that if the first r1 is linearly independent we select b2, but if b2 is linearly independent of all the previous columns what we will do we will put here b3 and then Ab3 etc.,. So whichever column is linearly independent that we will put here, it is not necessary that after b1 we will put b2. Okay. So now using this matrix let us construct the matrix N, okay. How to construct the N matrix? We will write the r1th column.

We will put all the columns are 0s. And r1th column is 0 1 0 0. Okay. Because the next entry we are going to put b2 so we will put the second entry to be 1 and all the entry to be 0. But in case we are putting here b3 because b2 becomes linearly dependent, if we want to put b3 here and it is linearly independent, in that case you should put 0 0 1 here in this position. So whichever 2 or 3 whatever it is at that position 1 is return and remaining vectors are 0, column vectors.

So this is the r1th position. Then r1+r2th position what we get is, next is becoming b3 is not it. So in this position we will put 0 0 1. But if it is going to be b4 then we should put 0 0 0 1, so depending on what is appearing in this position we have to write okay so etc. So this is a m cross matrix, okay. The number of pros should be m and n column should be; because we are constructing it from the matrix m so there should be n columns, a numbers of rows we just put it as m. So using this n and m we will the construct the matrix.

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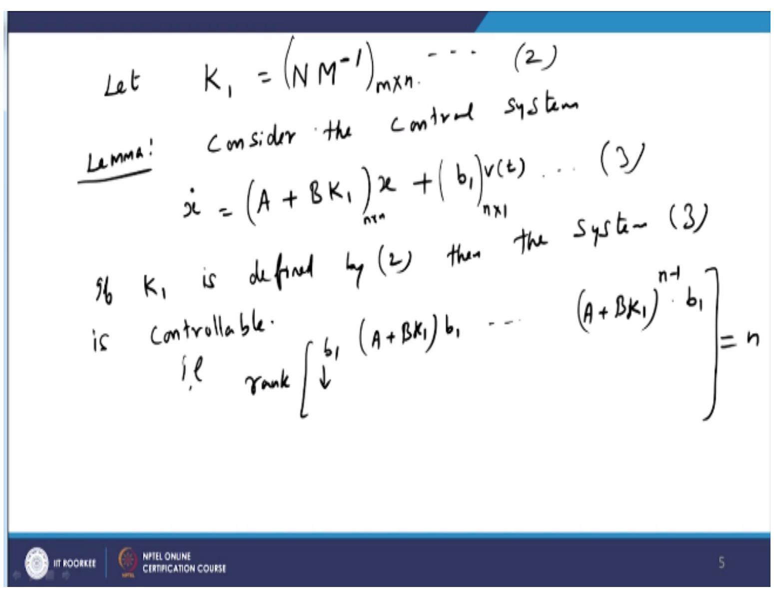

Now let a matrix K1 which is equal to n*m inverse, okay. Because the m is a invertible matrix we write like this, this matrix. Now we will prove the following result using these matrices, so we will write this following Lemma. Consider the control system. We construct a new control system which is given by x dot = A+Bk1*x + b1*v(t) okay. The control function is v and the; we normally have x dot = Ax+bu. Here in the place of A, we have A+Bk(1) it is a n cross n matrix, because here n is m cross n this is thing therefore this matrix is m cross n.

And b is a n cross m so the product will give n cross n after adding we will get a square matrix n cross n, okay. And then b1 is a column, simply the first column of the b matrix, so it is the n cross 1 matrix. So what we obtain is the thing which we have considered in the last lecture where the control matrix is a single column vector. Now we can make use of the previous theorem in this one, but before proving that we will prove this Lemma. Consider the control system this one.

So if K1 is (()) (19:36) equation 2, so if K1 is defined by (2) then the system, if you call it has (3) is controllable, okay. So this is the Lemma through which we will prove the main theorem, okay. So; or in other words what we want to prove we want to prove that, that is the rank of b1 the column b1 A+Bk1*b1 etc., A+Bk1 power n-1*b1. So we have n such column, this rank should be equal to n. If we prove this, then the system is controllable.

After this we will make use of the previous theorem to show the main result of the theorem. So first we will prove that this rank of this matrix is n. So the first column is b1, okay.

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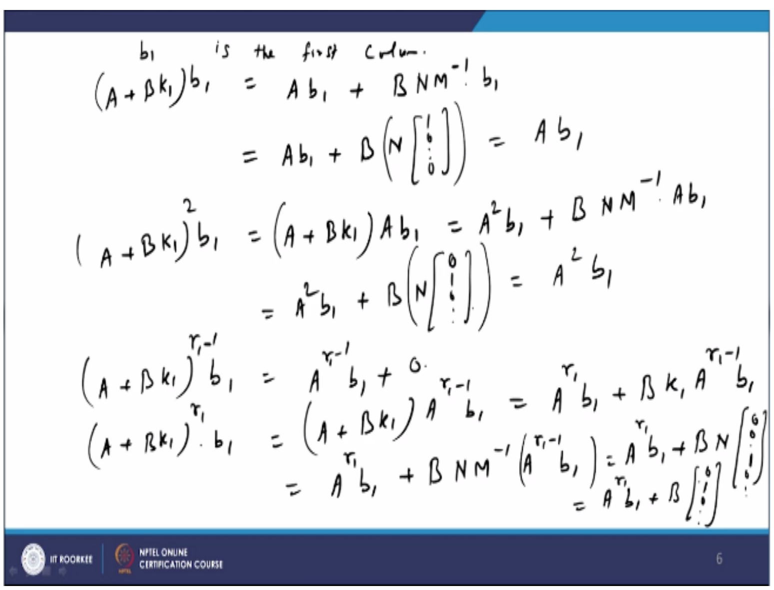

The second one is A+Bk1*b1. We want to show that this entry is linearly independent of the first entry b1, okay is the first column. This is the second column; we want this to be linearly independent of the first column. So if you multiply it, it is Ab1+Bk1 is n*m inverse* b1 and it is Ab1+B* the n matrix we know that m inverse * b1 which we saw; m inverse * b1 is 1 0 0 etc. Now if you multiply 1 0 0 with this n matrix; if you multiply here 1 0 0 etc., you can easily see that all the entries will be 0.

The product is 0. So N* this vector it is equal to 0 so ultimately we will get only A*b1 for the product. So we already know that b1 and Ab1 both are linearly independent from the construction. Now the next entry is A+Bk1 whole square * b1. So is it linearly independent of the previous two. So it is nothing but A+Bk1 multiplied by A+Bk1*b1 the previous one, so that is Ab1, okay. So that will give A square*b1+B N M inverse K1 is N*M inverse*Ab1.

So it is A square b1+B*N. M inverse into Ab1 that is given as 0 1 the second entry will be 1 here, all the other entries will be 0; 0 1 0 0 etc. So if you multiply 0 1 0 0 with n this matrix 0 1 0 0 we will see that the second place always 0 comes therefore you do not get anything, you will get again a 0 vector only. So this * this will be 0 therefore, we will get only A square b1. And already we know that A square b1 is independent of Ab1 and B1 etc.

So up to this stage A+Bk1 to the power r1*b1, if you see this we will get; this expression will be A to the power up to r1-1 we will get this expression same way. Now we will get A to the power let us say r1-1 b1 we will get A power r1-1*b1 only because if you do the indexation we will get square means here square we will get this expression plus the remaining things we will get, okay.

So again you can conclude that it will become 0. But the next one, A+Bk1*r1*b1 that will be A+Bk1*A power r1-1*b1 because up to the previous one we got this so only the 1 power we will take out. So this gives A power r1*b1+Bk1*A power r1-1*b1 is not it. Here we see that A power r1*b1 is dependent on all the previous entries, b1 Ab1 etc., is not it. So the second term has to be linearly independent otherwise if both of them are linearly dependent on the previous one it is not, then the system is not controllable.

So the first term is dependent. But let us show that the second term is independent of all the previous one. So it will be A to the power r1*b1 okay, because we have seen this matrix is constructed. A power r1*b1 is dependent that is why we did not include it here is not it. So this expression + B times K1 is N*M inverse multiplied by A power r1-1*b1. So if you see that M1*; this is the r1th column of the matrix m.

This one A power r1-1*b1 is the r1th column. So if you multiply m inverse * r1th column you will get in the r1th position the number 1 all other will be 0 is not it because of t he inverse property. So we will get it here A power r1*b1 as it is b is this n is this, but the product of m inverse * this will be 0 0 1 0 0. The r1th position 1 will up here. And then the product of this two will give what is n, n has; if you observe the second row in the r1th position you have 1 here, in the second row in the r1th column you have 1 others are 0.

And so if you multiply 0 0 1 in the r1th position when you multiply the second row and this column you will get 1 in that place is not it. So this will give A power r1*b1+B*0 1 0 0, okay. The second row and this column will give the element 1 other all 0 therefore you will get this expression. So if you multiply B*this you will get vector B2.

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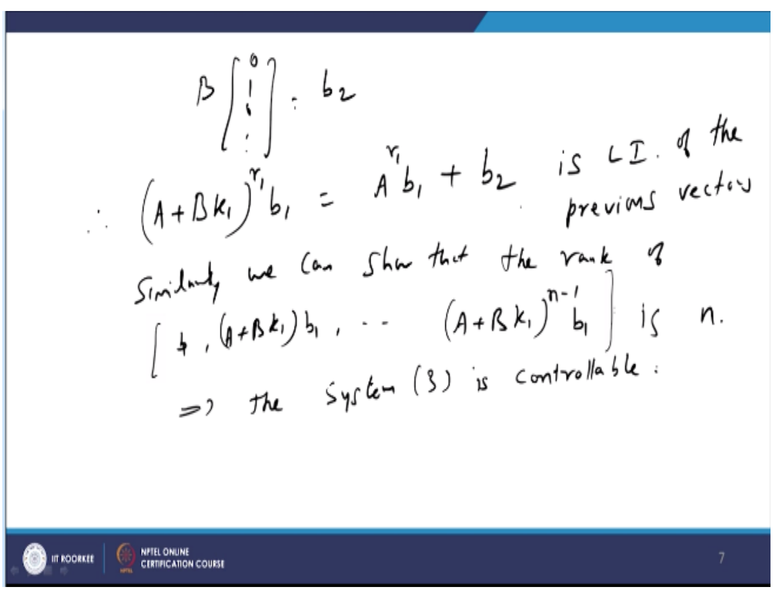

B*0 1 0 0 it is first column is B1, second is B2 etc., so finally we will get the vector B2. So what we get is A+Bk1 to the power r1*b1 is nothing but A to the power r1*b1+b2 we will get, okay, this expression. So even though this is linearly dependent on all the b1, Ab1 etc., but b2 is independent of all these. So this is linearly independent of the previous vectors, okay.

So now if you proceed further in the same manner you will simply get all the entries are similarly we can show that. So for example what we will get is r1+r2th position we will get a b3 in the similar procedure, so b3 will be independent of all the previous etc. so. So this implies, it show that the rank of b1, A+Bk1*b1 etc., A+Bk1 to the power n-1 b1 is n here, okay. So this implies the system is controllable, okay. System (3) is controllable system. So let us see here an example and then we will prove the theorem.

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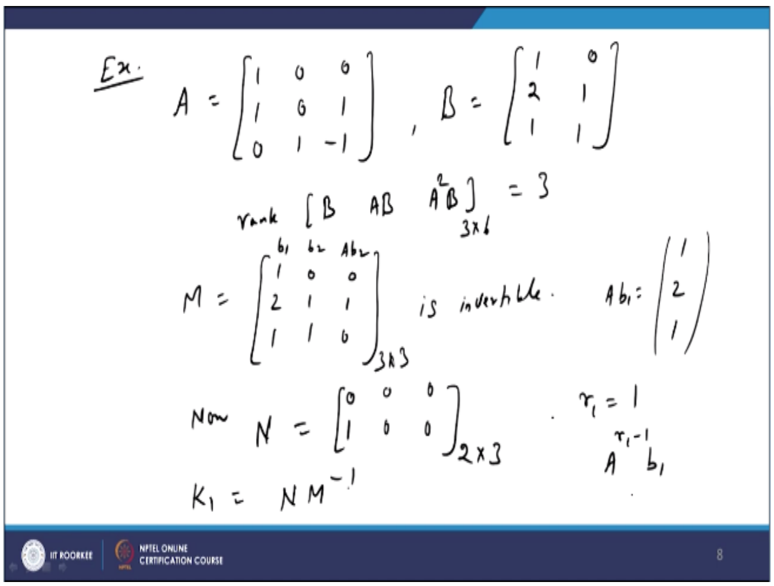

For example, let us consider the matrix A, A to be 1 0 0; 0 matrix is so 1 0 1 and 0 1 -1 okay. And let us say the B matrix is 1 2 1 and 0 1 1. So this system is controllable. You can say B Ab A square B. It is a 3/3 matrix, so we should see that rank of B this first block AB A square B = 3 because it is a 3 cross 6 matrix we will get and the rank is 3.

So this implies the system is controllable. Now we want to construct in the whatever is given in this matrix we want to construct this thing. We want to construct the M matrix first. So we will take b1 Ab1 A square b1 whichever is linearly dependent we have to omit them first. So the matrix M; the first column is b1 1 2 1. And if you multiply A*b1 we will get first row * first column is 1 and then the second is 2 then 3rd will get 1.

So A*b1 is like this, so it is same as b1 itself, okay the first one. Therefore, we should not take this A*b1 then we should put b2. 0 1 1 and these two are linearly independent b2 and b1. So we can take it as it is. Then we should put Ab2, if it is, it has to be linearly independent otherwise there is no column available. So A*b2 if you multiply it is 1, the first one, the second one 1, the third is 0, okay.

So it is a 3/3 matrix, and the rank is 3 so it is invertible. Now from here we have to construct N matrix. This will be a 2/3 matrix. We have to; it is a M cross N matrix so 2/3. Now you can see that r1=1 here in this case. Because b1 Ab1 itself has become dependent therefore r1=1 because A power r1-1*b1 up to this they are linearly independent.

So when you put this is r1 is 1 we get simply b1. So r1th position is 1 here therefore, we have to put 0 1, okay according to the construction of the N matrix the r1th position should have 0 1 if the next entry is b2. So r1th position is this. Then 0 0, the next position we have to put 0 0 1. But because there is no other third row available it should be 0 0 only. So n matrix is only this much. Now we have to construct K1 matrix. It is N*M inverse. So we compute this K1 matrix.

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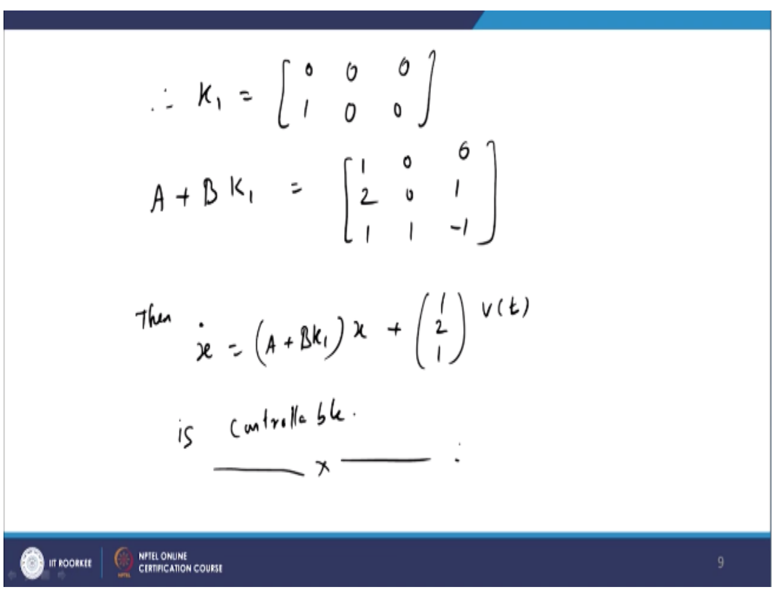

After computing all the inverse etc., we can get again this also to be the same 0 1 0 0 0 0. We get the similar as n itself we will get, okay. So now we will construct A+b*k1. A is given, B is given, k1 is given. So after doing all this multiplication we will get 1 0 0 then 2 0 1 and 1 1 -1 after computing all these values we get this one. And then we can easily check, x dot = this A+Bk1*x+b1, b1 matrix is 1 2 1 * the control; a notation for control we take it as v(t) so is controllable.

So we can check numerically the B1 and the Coleman condition can be checked for this, so it will be controllable. So this system, it is in the standard form that whatever we have proved in the previous lecture for the feedback control it is in the same form. So we can convert it into the companion form and then compute the control for the system. So the proof of the main theorem, we will prove it in the next class and then how to construct the control also we will see in the next lecture. Thank you.