Dynamical Systems and Control

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Lecture – 04
Existence and Uniqueness Theorem - II

Hello friends, welcome to this lecture, in this lecture, we will continue our study of existence and uniqueness of dynamical system.

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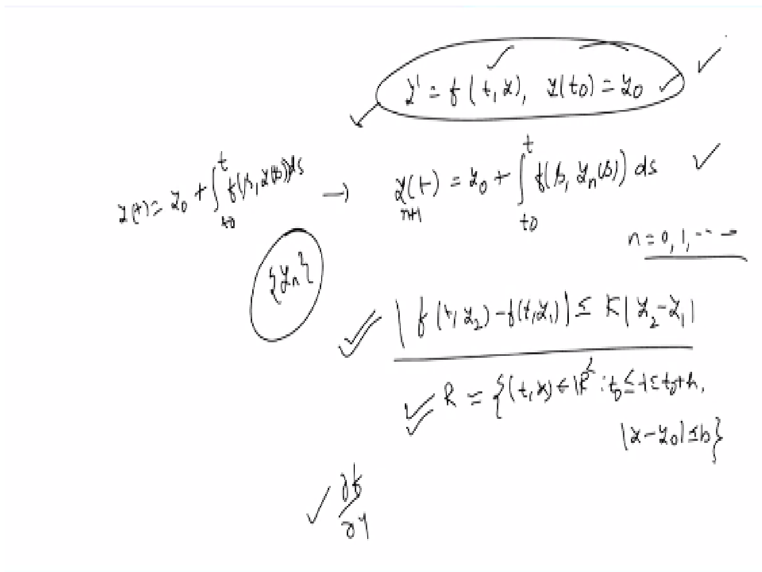

So, if you will recall in previous lecture we have discuss this initial value problem that y dash = f ty yt0 =y0, we are discussing the existence condition on f such that this initial value problem has a solution and not only solution, it has a unique solution. So, what we have done, we have shown that this initial value problem is equivalent to an integral equation that is yt = y0 + t0 to t f of s y of s here.

And then with the help of this integral equation, we have defined the sequence of approximation solution that is y and +1 t = y0 + t0 to t f of s yNS ds for N = 0, 1 and so on. So, once we have these sequence yn then we try to find out whether this will converge to some solution or not for that we we need some additional condition or f ty, one such condition is the Lipschitz condition and we have shown that f satisfies the Lipschitz condition if it satisfy the following condition that is f t, y2 – f t y1 is < = K times modulus of y2 – y1.

And here this t and y belonging to some close rectangle R and it is defined like this and we have shown that if dou f dou y is continuous and define on this close rectangle R, then f already satisfying the Lipschitz condition that is a content of the previous lecture. Now, in this lecture we try to show that this yn, it converging under the condition that f satisfy the Lipschitz condition or dou f/ dou y is continuous on this rectangle R.

And not only it will converge, it will converge to a solution yt which is the solution of this initial value problem that is the content of this lecture.

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Before we do anything with these successive approximations, we must show that they are defined properly. This means that in order to define y_j on some interval I, we must first know that the point $(s, y_j(s))$ remains in the rectangle R for every s in I.

$I = [t_0, t_0+h]$

Lemma 4

Choose any two positive numbers a and b, and let $R := \{(t, y) \in \mathbb{R}^2 : t_0 \leq t \leq t_0 + a, |y - y_0| \leq b\}$ be the rectangle. Compute $M = \max_{y \in \mathbb{R}} |f(t, y)|$ and set $h = \min\{a, \frac{b}{M}\}$. Then

$$|y_n(t) - y_0| \leq M|t - t_0| \text{ for } t_0 \leq t \leq t_0 + h \quad (13)$$

$|y_n(t) - y_0| \leq Mh \leq b$

Above lemma states that the graph of $y_n(t)$ is sandwiched between the lines $y = y_0 - M(t - t_0)$ and $y = y_0 + M(t - t_0)$ for $t_0 \leq t \leq t_0 + h$. These lines leave the rectangle R at $t = t_0 + a$, if $a < \frac{b}{M}$, and at $t_0 + \frac{b}{M}$ if $\frac{b}{M} < a$.
Therefore, the graph of $y_n(t)$ is constrained in R for $t_0 \leq t \leq t_0 + h$.

So here let us move, so before we do anything with these successive approximation, we try to show that they are defined properly in the sense that all these approximating solution belongs to a rectangle R, so it means that we must first know that the point s, yj s remains in the rectangle R for every s in the interval I, here I is defined as t0 to say t0 +; now, that is a content of this lemma 4.

Choose any two positive numbers a and b, and let define R as ty belongs to R2, t is lying between t0 to t0 + a, y – y0 modulus of y – y0 is < = b, be the rectangle, this is the rectangle we have defined and we assume that f is continuous on this rectangle R, so it means that it is bounded and let us define M as the maximum value of y belongs to R modulus of f ty and then we can find out the quantity h that is minimum of a, b/M, so then we say that modulus of y and t – y0 is < = M t – t0 for all t lying between t0 to t0 +h.

So, this lemma says that all the approximating sequence yn t satisfy this condition 13 and this means that it belonging to rectangle R, how we can say that; since t – t0 is bounded by h, so we can say that yn t – y0 is < = M and t – t0 is lying in say in h, so we can say M of h and h is what; minimum of a, b/M, so we can say that it is writing as in this, so it means that yn t – y0 is < = b, it means that all these yn t belongs to this rectangle R here.

So, what this lemma says that the graph of yn t is sandwiched between the lines y = y0 – M t – t0 and y = y0 + M t – t0, for t lying between t0 to t0 + h and these lines leave the rectangle R at t = t0 + a, if a is < b/M and t0 + b/M, if b/M < a. Now, so it means that the graph of yn t is contained in R for all t between t0 to t0 + h, so let us try to prove this lemma, proof of this lemma is given with the help of mathematical induction.

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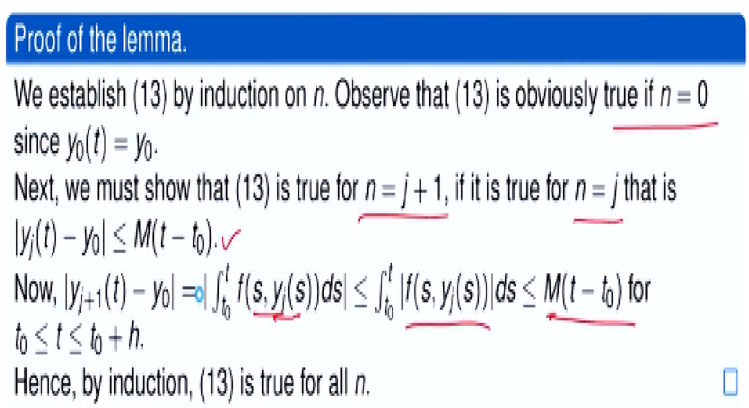

And we say that this is obviously true if n = 0, see it means that for n = 0, you can simply say that this is modulus of y0 – y0, so this is automatically less than this quantity. Next we assume that 13 is true for n = j and we try to show that it is also true for n = j +1 so, it means that we assume yj t – y0 is < = M t – t0, it means that this y jt belongs to the rectangle R, now try to find out modulus of yj +1t – y0.

And we can say that if this can be written as t0 to t f of s, yj s d of s, now since yj s belongs to rectangular R, so it means that s yj s is an element of rectangle R, so it means, and we already know that f is bounded on that rectangle by capital M, so we can say that this modulus of t0 to t f of s yj s ds is further < = t0 to t modulus of f of s yj s ds. Now, this is bounded by capital M, so we can write down that this is bounded by M * t – t0.

So, it means that yj +; modulus of yj + 1t + y0 is bounded by M t – t0, so it means that it satisfy the condition we have assume, so it means that this 13 is true for n from 0, 1, 2 and so on, so and this shows that all these t yn t belongs to the rectangle R, so it means it is properly define. Now, let us move to the convergence of this yn t.

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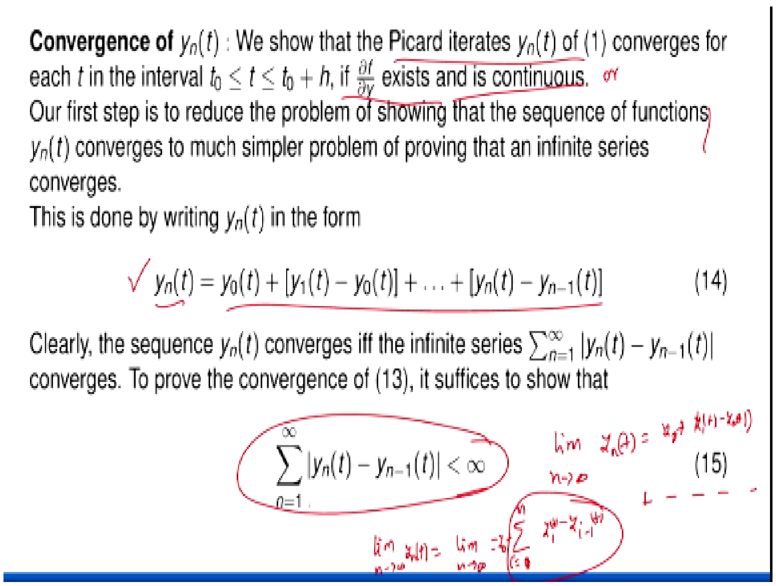

So, convergence of yn t and to show this that Picard iterates yn t, we call this yn t as Picard iteration or Picard iterate and we show that this converges if this dou f / dou y exists and is continuous or that f satisfy the Lipschitz condition and we have shown that if this exists and continuous then it automatically satisfy the Lipschitz condition. Our first step is to reduce a problem of showing that the sequences of function yn t converge to a quite simpler problem of proving that infinite series converges.

So, here these two problems equivalent, if we can show that yn t can be written as following manner and this is not very difficult to visualise that yn t can written as y0 + y1 – y0 + so on yn – yn – t, if we look at then these will cancel out each other and we will have yn t, so we rewrite yn t as this so partial sum that is y0 t + y1 t – y0 t + 1 and we say that limit n tending to infinity yn t will be what; this is nothing but y0 t + y1 t – y0 t and this goes on.

So, it means that this limit exist provided that this infinite series is basically convergence series and we can summarise this as writing that this implies what; that summation I can write it yi – yi – 1t, i = say here it is 0 to in fact it is 1 to n y0 here and then we are looking at limit n tending to infinity. So this we, this is limit and tending to infinity, yn t is basically this. So, we to show that this limit exists provided that this limit exist.

And this limit means this limit is < infinity, so our aim is to show that this infinite series basically converge.

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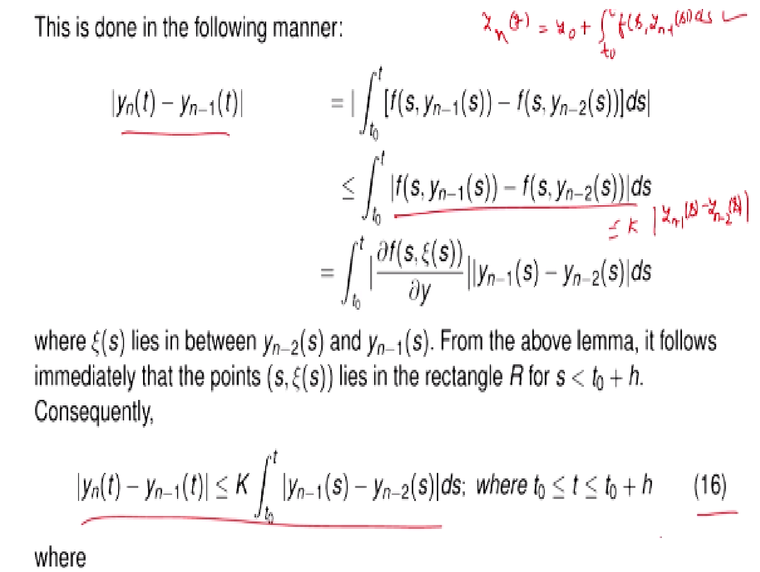

So, this is done in the following manner, so let us look at yn t – yn – 1t, so here we try to show that this; let us look at this quantity and yn; modulus of yn t – yn - t and we already know that yn t is given by Picard iteration, so it satisfy, so yn t is basically what; yn t is basically y0 + t0 to t f of s yn – 1 s ds, so using this expression of for yn t and yn – 1 t, we can write down this as modulus of yn t- yn – 1 t = t0 to t f of s yn – 1 s – f of s yn – 2s ds.

This can be further < = t0 to t modulus of f of s yn – 1s – f of s yn – 2s ds, now here either you use that dou f/dou y exist and continuous on that rectangle R or you simply use that f satisfy the Lipschitz condition, in both way you can say that this further < = K times yn – 1s – yn – 2s ds, this quantity is less than this quantity and we can get this thing; modulus of yn t – yn – 1t < = K times t0 to t modulus of yn – 1s –yn – 2s ds, where t is lying between t0 to t0 +h.

So, here we have a recurrence relation between yn - yn -1t, so we can use this 16 repeatedly and we can get the bound for this quantity, this infinite series.

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$$K = \max_{(t,y)\in\mathbb{R}} \left|\frac{\partial f(t,y)}{\partial y}\right| \qquad (17)$$

For $n = 2$,

$$|y_2(t) - y_1(t)| \leq K \int_{t_0}^{t} |y_1(s) - y_0(s)|ds$$
$$\leq KM \int_{t_0}^{t} (s - t_0)ds$$
$$= \frac{KM(t - t_0)^2}{2}$$

So, let us look at n = 2, for n = 2, we can write y2t – y1t < = K times t0 to t y1 – y0 ds, now this quantity we have already obtained using lemma, this can be written as M times t – t0, now here it is I can write it s – t0, here we are using this lemma that yn t – y0 is < = M t – t0, this is true for n = 0 onward, so we can write it n = 1th modulus of y1 t – y0 t is < = M times t –t0. So, using this we can write it here.

And we can say that it is < = K times M t0 to t s - t0 ds and when we simplify we have this thing, so for n = 2, modulus of y2 t- y1 t is < = KM times t – t0 whole square.

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This, in turn, implies that

$$|y_3(t) - y_2(t)| \leq K \int_{t_0}^{t} |y_2(s) - y_1(s)| ds$$
$$\leq MK^2 \int_{t_0}^{t} \frac{(s - t_0)^2}{2} ds$$
$$= \frac{MK^2 (t - t_0)^3}{3!}$$

Similarly, by induction, we see that

$$|y_n(t) - y_{n-1}(t)| \leq \frac{MK^{n-1}(t - t_0)^n}{n!} \text{ for } t_0 \leq t \leq t_0 + h. \qquad (18)$$

Similarly, for n = 3, we can write as modulus of y3 t – y2 t is < = K times t0 to t y2; modulus of y2 s - y1 s ds and for this, we have already obtained and we can plug in this value and we can have this, so by induction we can say that we have this estimate that is modulus of yn t – yn – 1t is < = MK n – 1; MK to the power of n – 1 t – t0 to the power n divided by factor n for t lying between t0 to t0 + h.

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Therefore, for $t_0 \leq t \leq t_0 + h$; we have

$$|y_1(t) - y_0(t)| + |y_2(t) - y_1(t)| + \ldots$$
$$\leq M(t - t_0) + \frac{MK(t - t_0)^2}{2} + \ldots + \frac{MK^{n-1}(t - t_0)^n}{n!} + \ldots$$
$$\leq Mh + \frac{MKh^2}{2!} + \frac{MK^2h^3}{3!} + \ldots + \frac{MK^{n-1}h^n}{n!} + \ldots$$
$$= \frac{M}{K}\left[Kh + \frac{(Kh)^2}{2!} + \ldots + \frac{(Kh)^n}{n!} + \ldots\right]$$
$$= \frac{M}{K}(e^{Kh} - 1)$$
$$< \infty$$

Hence, $y_n(t)$ converges for all $t \in [t_0, t_0 + h]$. Let the limit be $y(t)$.

So, once we have bound for this, now let us look at this y1 – y0 + y2 – y1 and so on, so this term is bounded by this, second term is bounded by this and so on, now so it means that this series is bounded by this series; infinite series. Now, t – t0 is bounded by h, we have already assume that t is lying between t0 to t0 + h. So, t – t0 is bounded by h, so we can write, replace this, we can look at the upper bound.

So, Mh + MK h square upon factorial 2 and so on, now here we just do some more simplification, we divide by K, multiply by K, so we can write M/K, Kh + Kh square upon factorial 2 and so on and if you look at this, this can be simplified in this manner that we can write it M/K, this thing is nothing but infinite series e to the power KH - 1 and we can say that since it is a converging series, so it means that this quantity is < infinity.

So, it means that this infinite series is bounded by infinity, so if this infinite series is bounded by infinity, so if we add by y0, we say that it is still less than infinity, so it means that our series is absolutely convergent, so it means that not only it is convergent, it is absolutely convergent, so here we can say that this series, this series is absolutely convergent what we have shown here, so if it is convergent, let us find the limit say y team.

And this convergent for some t lying between t0 to t0 + h, now let us find out the limit yt and we try to show that this limit is the required solution.

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We will show that $y(t)$ satisfies the initial value problem:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds \text{ and } y(t) \text{ is continuous.} \qquad (19)$$

Recall that

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds \qquad (20)$$

Taking limits on both sides of (20), we have

$$y(t) = y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, y_n(s))ds \qquad (21)$$

To show that the right hand side of (21) is equal to $y_0 + \int_{t_0}^{t} f(s, y(s))ds$, we must show that

$$\left| \int_{t_0}^{t} f(s, y(s))ds - \int_{t_0}^{t} f(s, y_n(s))ds \right| \to 0 \text{ as } n \to \infty.$$

So, we will try to show that yt satisfy the initial value problem, this yt = y0 + t0 to t f of s ys ds and yt is continuous because we have shown that if a y is a continuous solution of this, then y satisfy the initial value problem, so we have to show y is continuous and y satisfy the initial value problem; integral equation given by this. So, first we show that it satisfy the integral equation.

For that recall that y and +1 t = y0 + t0 to t f of s yn s ds, now take the limit on both side, we have limit n tending to infinity yn + 1t that is given as yt = y0 + limit n tending to infinity t0 to t f of s yn s ds, so what we want to show is that this quantity is nothing but t0 to t f of s y of s, so this we want to show, it means that this limit can go inside the integral and inside your f also. So, it means that if this is equal to this, it means that we want to show that t0 to t f of s ys ds – t0 to t f of s yn s ds is tending to 0 as n tending to infinity.

So, this can be rewritten in this way that as n tending to infinity, this quantity is tending to this limit.

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To prove this, we first observe that $y(t)$ lies in R, since it is the limit of $y_n(t)$ whose graphs lies in R.

Hence

$$\left|\int_{t_0}^{t} f(s,y(s))ds - \int_{t_0}^{t} f(s,y_n(s))ds\right| \le \int_{t_0}^{t} |f(s,y(s))ds - f(s,y_n(s))|ds$$

$$\le K\int_{t_0}^{t} |y(s) - y_n(s)|ds$$

Since

$$y(s) = y_0 + \sum_{j=1}^{\infty}[y_j(s) - y_{j-1}(s)]$$

$$y_n(s) = y_0 + \sum_{j=1}^{n}[y_j(s) - y_{j-1}(s)]$$

To prove this, we first observe that yt lies in R because each y and t lies in R, so it is a limit will also belongs to this in rectangle R, you can see like this yn t – y0 < = Mt – t0, let us take the limit inside, limit tending to infinity, it is independent of n, so no problem and since modulus is a continuous function, you can write it that yt – y0 < = M t - t0, so it means that yt belongs to the rectangle R.

So, it means that t yt belongs to R, so that is what we want to know, so once it is done, look at this quantity, modulus of t0 to t f of s ys ds – t0 to t f of s yn s ds and this further < = t0 to t f of s ys ds – fs yn s ds, now using Lipschitz condition or say that dou f / dou y is continuous, we can write that as this small < = K times t0 to t ys – ys t. so, if we can find out the bound of this, then we can talk about this quantity.

So, ys is basically what; ys is y0 + j = 1 to infinity modulus of yj as y js – 1s and yn s is y0 + j = 1 to n, y js – yj – 1s, so using this find out modulus of ys – yn – 1s.

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So, we have

$$|y(s) - y_n(s)| = \left| \sum_{j=n+1}^{\infty} [y_j(s) - y_{j-1}(s)] \right|$$

Consequently, from (18), we have:

$$|y(s) - y_n(s)| \leq M \sum_{j=n+1}^{\infty} \frac{K^{j-1}(s-t_0)^j}{j!}$$

$$\leq M \sum_{j=n+1}^{\infty} \frac{K^{j-1} h^j}{j!}$$

$$\leq \frac{M}{K} \sum_{j=n+1}^{\infty} \frac{(Kh)^j}{j!} \qquad (22)$$

And we can write in this quantity that modulus of ys – yn s = j = n + 1 to infinity, yj s – yj – 1s, now we already have found the bound of this y js - yj – 1s, so using the bound here, we can write down that ys – yn s, in fact we have find the bound of yj; modulus of yj s – yj – 1s, in this bound, we have obtained in basically here in equation number 18 basically, so we have bound the; bound of this modulus of yn t – yn – t as MK n – 1 t – t0 to the power n divided by factor n.

So, using this we can write down modulus of ys – yn s is < = M times j = n + 1 to infinity K power j – 1 s – t0 to power j divided by factorial j. Now, here again this s – t0 is bounded by h and we can further simplify and we can write down this M/K j = n + 1 to infinity Kh to the power j upon factorial j now, if you look at this is the tail of e to the power Kh. Now, if it is a converging series then tail is tending to 0 as n tending to infinity.

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and

$$\left|\int_{t_0}^{t} f(s, y(s))ds - \int_{t_0}^{t} f(s, y_n(s))ds\right| \leq M \sum_{j=n+1}^{\infty} \frac{(Kh)^j}{j!} \int_{t_0}^{t} ds$$

$$\leq Mh \sum_{j=n+1}^{\infty} \frac{(Kh)^j}{j!} \to 0 \text{ as } n \to \infty$$

Hence,

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s, y_n(s))ds = \int_{t_0}^{t} f(s, y(s))ds \text{ and } y(t) \text{ satisfies equation (19).}$$

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

So, it means that ys; modulus of ys – yn s is tending to 0 as n tending to infinity, so it means that modulus of j0 to t f of s ys ds – t0 to t f of s yn s ds < = M times j = n + 1 t0 infinity KH power j upon factorial j t0 to t ds, so and this can be written as bounded by h here and as we pointed out that this tail is tending to 0 as n tending to infinity then, this quantity is tending to 0 as n tending to infinity and this implies that that limit tending to infinity t0 to t f of s yn s ds = t0 to t f of s ys ds.

And this implies that yt satisfy the equation number 18 that is yt can be written as y0 + t0 to t f of s y of s ds, now only thing we had to prove now is that y is a continuous function because when f is; y is continuous solution of this integral equation, then why is a solution of initial value problem.

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y(*t*) **is continuous:**

By the Weierstrass M-test, the series $\sum_{j=0}^{\infty} |y_{j+1}(t) - y_j(t)|$ uniformly converges on the interval $[t_0, t_0 + h]$. In view of the definition of the $[y_{j+1}(t) - y_j(t)]$, this implies the absolute (and uniform) convergence on $[t_0, t_0 + h]$ of the series $\sum_{j=0}^{\infty} [y_{j+1}(t) - y_j(t)]$.

Since $y_j(t) = y_0(t) + \sum_{m=0}^{j-1} [y_{m+1}(t) - y_m(t)]$, this also proves the convergence of the sequence $\{y_j(t)\}$ for every t in the interval $[t_0, t_0 + h]$ to some function of t, which we call as $y(t)$. Since each y_n is a continuous function and convergence to y is uniform, therefore the limit function $y(t)$ is a continuous function.

So to show that yt is continuous, we can use Weierstrass M test and we can say that if you look at this quantity j = 0 to infinity yj + 1t – yj t, uniformly converges on this interval t0 to t0 + h, so how we say that it is uniformly convergence? Again, look at your equation number 18, 18 is given by this, so if you call this as say Mn and we already know that summation Mn is basically < infinity.

So, it means that summation yn t – yn – 1t is converging uniformly, so this; so this is Weierstrass M test that if yn t - yn – t is bounded by Mn and summation Mn is < infinity then this infinite series is uniformly convergent, so using this we can simply say that this convergence is uniform, so this implies that in the view of this definition that this infinite series j = 0 to infinity yj + 1t –yj t is converging absolutely and uniformly in the interval t0 to t0 + h and we already know that yj t is given as says.

And this implies that yj t is continuous function and y jt is converging to yt and this convergence is uniform, so the limit will also be continuous, so y jt is continuous implies that yt is continuous because your convergence is uniform, so it means that therefore the limit function yt is a continuous function. So, now we have shown that the limit function is a continuous function and satisfy the initial integral equations.

So, it means that it is also the solution of the initial value problem that is what we wanted to show here, so if we summarise, we have proved the following theorem.

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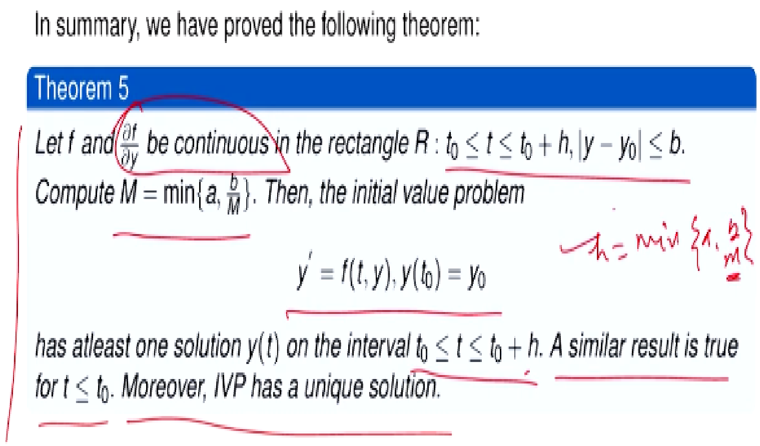

Let f and dou f/ dou y be continuous in the rectangle R and the rectangle R is defined like this that t is lying between t0 to t0 + h modulus and y – y0 is bounded by b and M is given by a minimum of a, b/M then the initial value problem y dash = f ty, y t0 = y0 has at least one solution yt on an interval t0 to t0 + h, here h is what; h is minimum of a and b/M, where M is the bound of ft y.

So, here we say that similarly, we can prove that this similar kind of result is also true for t < = t0 and now we want to show that not only this it has a unique solution so, how we show that it has a unique solution in fact, this theorem is given for this condition that f and dou f/ dou y be continuous but we have already shown dou f/ dou y continues in this rectangle R implies the Lipschitz condition and we have proved the theorem for Lipschitz condition only.

So, we can say that let f is continuous and satisfy the Lipschitz condition then it has a solution in this interval t0 to t0 + h, where h is minimum of a, b/M.

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Uniqueness of the solution: Earlier theorem guarantees the existence of atleast one solution $y(t)$ of $y' = f(t, y), y(t_0) = y_0$. Let $z(t)$ be another solution of the above differential equation, then

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$

$$z(t) = y_0 + \int_{t_0}^{t} f(s, z(s))ds$$

$\Rightarrow$

$$|y(t) - z(t)| \leq K \int_{t_0}^{t} |y(s) - z(s)|ds \tag{23}$$

Claim: y(t) - z(t) = 0 that is y(t) = z(t).

Now, let us prove the uniqueness of the solution so, the previous theorem, guarantees the existence of at least one solution of this y dash = f t y y t0 = y0, so one solution is guaranteed, let us assumed that we have 2 solution, let us say zt and yt, so yt satisfy this, zt will satisfy this, so we wanted to show that yt is identically = z of t, so for that let us consider modulus of yt – zt, then it is bounded by K times t0 to t modulus of ys – z of s ds.

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Lemma 6

Let $w(t) \geq 0, \forall t$ with $w(t) \leq K \int_{t_0}^{t} w(s)ds$. Then, $w(t) = 0$.

Proof.

Let $U(t) = \int_{t_0}^{t} w(s)ds$, then $\frac{dU}{dt} = w(t) \leq K \int_{t_0}^{t} w(s)ds$,
$\Rightarrow \frac{dU}{dt} \leq KU(t) \Rightarrow e^{-K(t-t_0)}U(t) \leq U(t_0) = 0 \; \forall t \geq t_0 \Rightarrow U(t) = 0$ □

Now, from equation (23), we have

$$0 \leq w(t) \leq K \int_{t_0}^{t} w(s)ds = KU(t) = 0 \Rightarrow w(t) = 0.$$

This shows the uniqueness of the solution.

Now, we want to claim that yt - zt is that is = 0 that is yt = zt, so for that we use the following lemma that let wt is non-negative for every t with satisfying the condition integral; inequality that is wt is < = K times j0 t ws ds, then wt is identically = 0 here, so using this we try to prove our result, so first let us prove this lemma and then we will use it, so we cannot directly differentiate this.

So what we try to do here let us assume this quantity as say Us, so let us say Ut has t0 to t ws ds, so Ut is given here, then we can differentiate this and we can write dU/dt as wt, so we already know wt is < = K times t0 to t ws ds and this thing, so now but this is what; this is your Ut, so we can write dU/dt as < = K times Ut, so dU/dt – K of Ut is < = 0 basically, now this can be written as using indicative factor, I can rewrite this as d/dt of e to the power –Kt Ut is < = 0.

Now, integrate between t0 to t and we can write down this thing that e to the power –K t – t0 Ut is < = Ut0 that we have to integrate between t0 to t, and we can say whether it is true or not then we have this e to the power -K times t – t0 Ut < = Ut0, now what is Ut0; Ut0, you can check from this that it is coming out to be 0, so it means that e to the power -K times t – t0 Ut is < = 0, this quantity is always non negative.

So, it means that Ut is < = 0 but Ut is already > = 0 because this ws is non-negative, integral t0 to t is positive, so Ut is > =0 and this shows that Ut is < = 0, this is possible only when Ut is identically = 0, so it means that if Ut is identically = 0, then you can put it here and you can say that t0 to t ws ds is identically equal to 0, so it means that this is so, 0 < = wt < = K times t0 to t ws ds, this quantity is 0.

So, it means that wt is identically = 0, so this shows the approve of this lemma, now using this proof of this lemma, we want to show that this solution is unique, so here we assume that wt = modulus of yt – zt, so when you assumed this when yt is non-negative, then this wt satisfy this quantity, wt is < = K times t0 to t ws ds and just now, we have proved that if wt is non-negative, sorry, wt is non-negative and satisfying this inequality then wt is 0.

So, it means that this implies that is tending to 0 and this implies that yt is and = zt and that proves the uniqueness of the result, so it means that not only it has a solution, it as a unique solution. Now, let us summarise whatever we have done so far.

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Consider the following first order nonlinear differential equation

$$\frac{dy}{dt} = f(t, y) \quad (24)$$

$$y(t_0) = y_0 \quad (25)$$

where f is a given function of t and y, defined and continuous in some neighborhood of the point (t_0, y_0). We can define a sequence of associated approximating solution as follows.

$$\begin{aligned} y(t_0) &= y_0, \\ y_{j+1}(t) &= y_0 + \int_t^{t_0} f(s, y_j(s))ds, \quad j = 0, 1, 2, 3.... \end{aligned} \quad (26)$$

So, here we have consider this initial value problem that is dy/dt = f ty, yt0 = y0, where f is a given function of t and y defined and continuous in some neighbourhood of the point t0, y0, we define the sequence of associated approximating solution as follows, yt0 = y0, yj + 1 t = y0 + t0 t0 t f of s y js ds, j form 0, 1, 2, 3 and so on.

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Then, existence and uniqueness theorem is given as follows:

Theorem 1

If $f(t, y)$ is a continuous function of t and y in a closed and bounded rectangle R and satisfies the Lipschitz condition in R. Then the successive approximations y_j, given by (26), converge(uniformly) on the interval $J = \{t : |t - t_0| \leqslant \alpha\}$, to a solution y of the differential equation (24) that satisfies the initial condition (25).

Example 7

Given that

$$y' = t(1 + y) \quad (27)$$

$$y(0) = -1$$

Then existence and uniqueness theorem is given as follows that if f ty is a continuous function of t and y in a close and bounded rectangle R and satisfies the Lipschitz condition in R, then the successive approximation yj we have defined in this equation number 26, converges uniformly on the interval j in this interval j that is t is lying between t0 to t0 + alpha in fact, we can prove the other way round also that is modulus of t – t0 is < = alpha to a solution y of the differential equation 24 that satisfy the initial condition 25.

So, we have proved the existence and uniqueness theorem, let us consider one simple example based on this, consider y dash = t times 1 + y with initial condition y0 = -1.

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Solution. Let $R : \{(a, b) \in \mathbb{R}^2 : |t - 0| \leq a, |y + 1| \leq b\}$ and let $f(t, y) = t(1 + y)$. Then f is continuous on R and

$$|f(t, y_1) - f(t, y_2)| = |t||y_1 - y_2|$$
$$\leq a|y_1 - y_2|$$

i.e. f is Lipschitz on R so by existence and uniqueness theorem, there exists a unique solution of (27) on $|t| \leq h = \min\{a, b/M\}$, where $M = \max_{(t,y)\in\mathbb{R}} |f(t, y)|$
We may observe that $y(t) = -1$ is a solution of (27) satisfying the initial condition.
So by existence and uniqueness Theorem, $y(t) = -1$ is the only solution of (27).

So, here we can check consider a rectangle R which consists this initial condition that is 0 and -1, so we have this rectangle centred at 0 and -1, so that is R and let f ty = t times 1+ y and we can easily check that this satisfy the Lipschitz condition that is modulus of f ty1 – f t y2 = modulus of t times y1 – y2 and we can say that ft; modulus of t is bonded by a, so this is bounded by a times y1 – y2, so it satisfy the Lipschitz condition with the Lipschitz constant a here.

So, it means that here by existing and uniqueness theorem, this problem has a unique solution and here we can easily observe that this yt equal to -1 satisfy the initial condition as well as the differential equations so, it means that since we have a unique solution and we already know that yt = -1 is satisfying the initial condition as well as the differential equations, so it means that it is the only solution of the differential equation 27.

So, it means that this problem as since it has a unique solution and yt = -1 satisfy the initial condition as well as the differential equation, this is the only solution available for this differential equation, so this initial value problem has the unique solution that is yt = -1.

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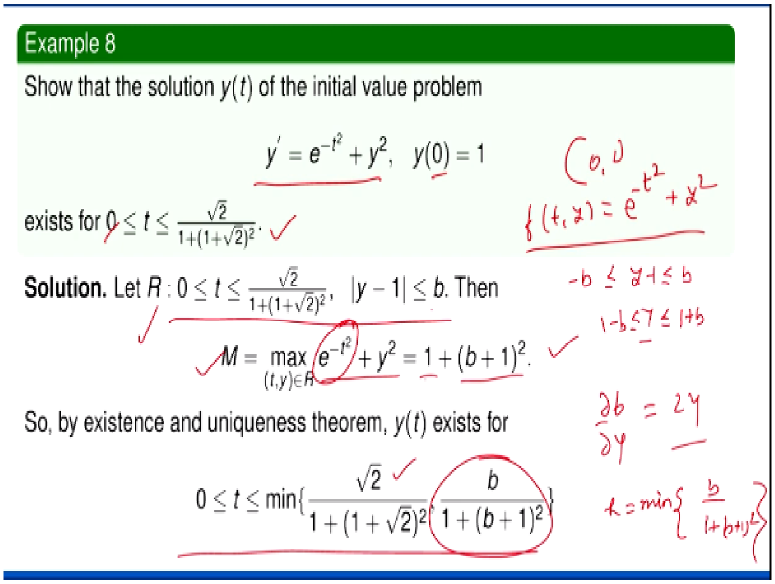

So, now let us consider one more example and show that in this example, showed that the solution yt of the initial value problem y dash = e to the power -t square + y square with y = 1 exists in the said interval, so for that let us consider a rectangle R that is t lying between 0 to this root upon 1+1+ root 2 whole square y – 1 < = b, so only thing is that here we consider rectangle centred at 0 and 1, so that is what we have considered here.

Then we can; since it is a close rectangle, we can find out the bound of f ty here, f ty is e to the power –t square + y square, so we can find out the bound of this, so make some of ty belongs to R e to the power -t square + y square, this quantity is bounded by one because t is lying between this 0 to this, so it is bound by one and y is bonded by b + 1 that we can easily check y -1 is bounded by –b and b and so y is bounded by 1 + b.

So, y is bounded by 1 + b, so y square is bounded by b + 1 whole square, so M is already obtained, so we have already shown that this f ty is this and we can easily check the dou f / dou y exist and continuous in this rectangle R, so existence and uniqueness theorem, assumptions of existence and uniqueness theorem verified and solution exist in an interval between t0 to t0 + h, here t0 is 0 here and h is given by minimum of a; a is given by this.

And b/M; M is we have already calculated as 1 + b + 1 whole square, so it means that solution exist in this interval, t lying between 0 to minimum of this quantity, so here this is some already number; given number, you want to look at that how much so it means that what is a value of this b, so here b is an arbitrary value here.

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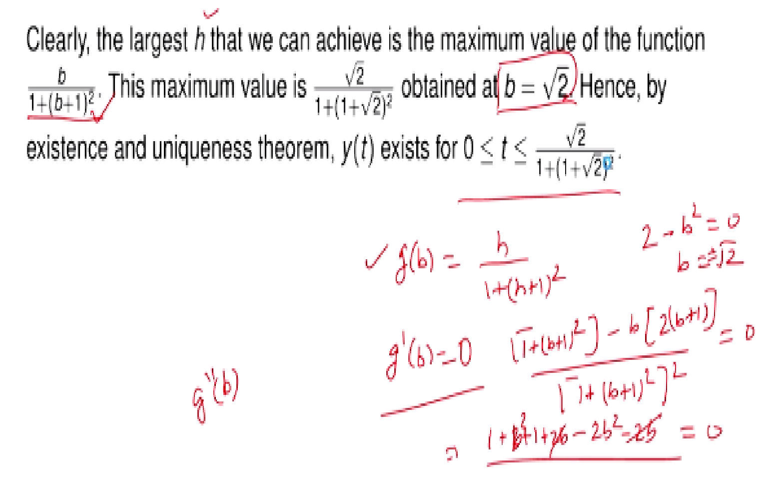

So, we can say that the largest h that can we can achieve is the maximum value of the function this b upon 1 + b + 1 whole square and so let us to find out the maximum value of this, let us consider gb as b upon 1+ b + 1 whole square, to maximise this let us consider g dash b and you can say that 1 + b + 1 whole square, so 1 + b + 1 whole square, sorry this can be done as - of this, okay that is.

So, we can write it here, so here we can write it b times 2b + 1 - this is what is your b, so here we simply write b + 1 whole square and this = 0, so g dash b = 0, so this implies that you simplify it 2b square + 2b – b square -1 – 2b, this is 0 because here numerator is; so this can written as b square = 1, so it is b =, sorry it is something; so to find out the maximum value of this gb, we need to find out b for which g dash b = 0.

For that let us calculate this, here is square, whole square and this is 1 + b + 1 whole square – here b and this is what 2b + 1 here and this = 0, so here we want to show, this is what 1 + 2b square +1 + 2b – 2b square – 2b = 0, so here if you simplify this can cancel out, so we have 2 - b square = 0, so we have b as + - root 2. Now, b is a positive value, so we have to take b = root 2, so it means that this can be maximise if we take b = root 2 that you can check that g double dash b is negative for this value.

So, here we can say that the maximum value of b upon 1 + b + 1 whole square, it achieved for b = root 2, if you put b = root 2, we can have that the maximum value of this quantity is given by root 2 upon 1 + 1 + root whole square, so it means that this prove that the solution of this initial value problem exist in this interval that is t from 0 to root 2 upon 1 + 1 + root 2 whole square that is what we have shown here.

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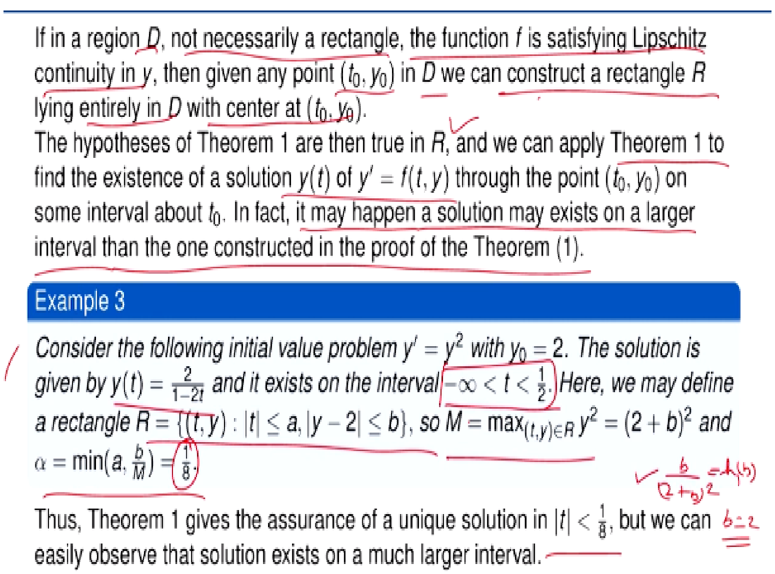

So, what we have shown one example where we shown that solution exist in a particular interval, now suppose consider a region D not necessarily a rectangle, the function f satisfy Lipschitz continuity in y, then given any point t0 y0 in D, we can always construct a rectangle R lying entirely in D with centre at t0, y0 and this is the hypothesis of existence and uniqueness theorem are then true in R.

And we can apply our existence and uniqueness theorem to find out the solution of y dash = f ty satisfying the initial condition and we can say that solution exist but here we want to observe one thing that in fact it may happen that a solution may exist on a larger interval then the one constructed in the proof of theorem 1, so it means that it may happen that solution actually exist in a larger interval but this existence and uniqueness theorem gives a smaller interval.

For example, consider this example, consider the following initial value problem y dash = y square with y0 = 2, the solution is given by yt = 2 upon 1 – 2t, this we can simply solve because it is separation variable form and we can say that solution exist in this entire interval – infinity to 1/2 but if we try to solve this using existence and uniqueness theorem, we construct a rectangle R that is modulus of t < = a, y – 2 is < = b.

We can easily calculate the maximum value of f ty that is y square is 2 + b whole square and we can calculate the h alpha that is minimum of a, b/M that is b upon 2 + b whole square and you can calculate the maximum value and it is coming out that this maximum value is coming out to be 1/8, so we can say that the if we apply existence and uniqueness theorem, I am leaving it to you how to find out the maximum value, you call it some h of; some say h1 of b.

And find out the value for which h1 dash b is 0 and h1 double dash b is negative and you can say that that b is coming out to be 2 basically, so you can say that maximum value is coming out to be 2. So, it means that by existence and uniqueness theorem, our solution exist in the interval that is between 0 to 1/8, so it means that here we can say that the theorem 1 gives the assurance of a unique solution in interval modulus of t < 1/8.

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Remark 1

1. Theorem 1 is a local existence theorem and discuss the existence only in a small neighborhood of initial point.
2. The proof of the above theorem required the Lipschitz continuity of the nonlinear function even when only existence is required.
3. Regarding the existence of a solution of (24), Theorem 1 is not the only and best result. We may have existence of solutions without uniqueness. One such important results are stated as follows:

Theorem 2

Suppose f is continuous on the rectangle R, and suppose $|f(t,y)| \leqslant M$ for all points $(t,y) \in R$. Let α be the smaller of the positive numbers a and b/M. Then there is a solution y of the differential equation (24) that satisfies the initial condition (25) existing on the interval $|t - t_0| \leqslant \alpha$.

But we have already shown that this actual solution exist in a larger interval that is – infinity to 1/2 yeah, so it means that this theorem is a local existence theorem, it means that this gives the existence in a small neighbourhood of the initial condition that is discuss the existence only in a small neighbourhood of an initial point. So, first is very important thing that it is a local existence theorem.

And the proof of the above theorem required the Lipschitz continuity of the non-linear function even when we need only existence, so it means that even when we need only half part that is existence then also this theorem requires the Lipschitz continuity. Now, regarding the existence of a solution of 24, theorem one is not the only one best results, so it means that if we want only the half part that is we are interested only in the existence of a solution part.

Then these theorem is demanding much because it is (()) (39:37) for Lipschitz continuity and it is much more than continuity, so here we have a one more result which gives only existence, may not give the information about the uniqueness and that is your theorem 2 and suppose f is continuous on the rectangle R and suppose that this modulus of f ty is bounded by M for all points t y in this rectangle R and alpha be the smaller of the positive number a and b/M.

Then there is a solution y of the differential equation that satisfy the initial condition existing on the interval t – t0 < = alpha, if you look at the theorem 2, the only part missing is the Lipschitz continuity rest everything is same, so it means that if we leave this part Lipschitz continuity is still we can guarantee the existence of solution. Of course, the proof of this theorem 2 is not the same as we have discussed the proof of theorem 1.

Because in theorem 1 when we prove, we prove the convergence using the Lipschitz continuity, so the proof of this theorem 2 is not the same as we have done, I am not giving the proof of this.

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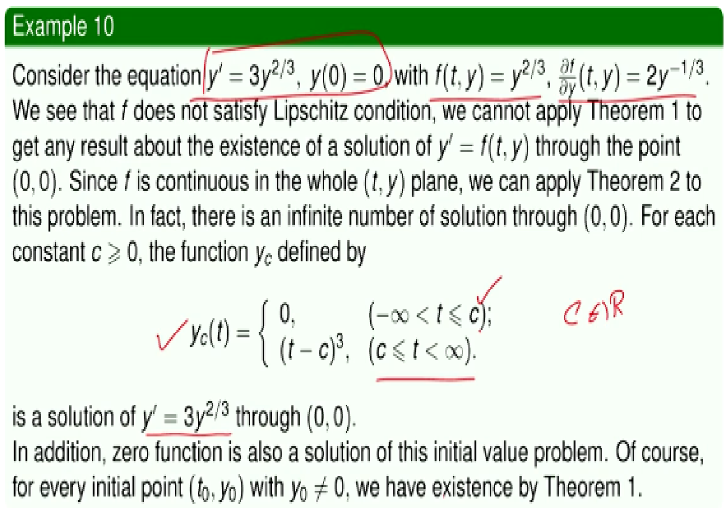

But just give one example based on this example, so consider the equation y dash = y to the power 2/3, y0 = 0, here f ty is y to the power 2/3, we can easily check that it does not satisfy the Lipschitz condition, so that we can check, so it means that we cannot apply the previous theorem and we simply say that it has no solution or we cannot apply this theorem to find out whether it has a solution or not.

But we can the check that it has a solution in fact not only one solution it has infinitely many solution and that we can done by separation and variable thing and we can say that if we define our solution properly like yc t = 0 between – infinity to c and t – c cube when t lying between c to infinity, then this is a solution of y dash = 3 y to power 2/3 passing through 0, 0 and you keep on changing your c and we have different infinitely many solution of this.

So, C belonging to R and we have infinitely many solution of this particular problem, so here this f ty does not satisfy the Lipschitz condition is still it has a solution not only one solution it has infinitely many solution, so this we can get it from the previous theorem that if f is continuous on the rectangle R and f ty is bounded by M and alpha be the smaller amount of a and b/M, then this differential equation has a solution lying in this interval modulus of t – t0 < = alpha.

So, this example shows that even ft y is not satisfy the Lipschitz condition but is continuous and bounded in the rectangle R then also it has a solution, it may not be unique solution that we have shown because this problem has a infinitely many solution here, so this here existence is given by theorem 2 not by theorem1 because theorem 1 is not applicable here.

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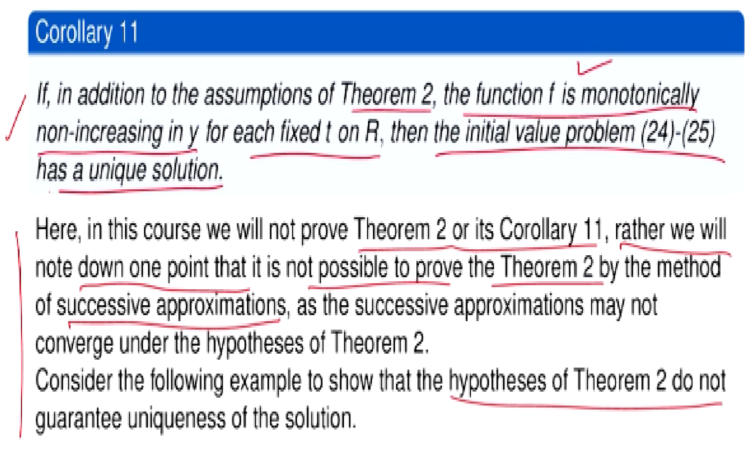

Now, one more corollary because here we are missing the uniqueness part, for that we consider the following corollary that if in addition to the assumption of theorem 2, the function f is monotonically non increasing in y for each fixed t on R, then the initial value problem, this has a unique solutions, so it means that if we lose the Lipschitz continuity and we consider only continuity and if in addition, if we assume that f is monotonically non increasing in y.

Then we have existence and uniqueness together, so here we are not providing the proof of theorem 2 and this corollary 11 but we observe this thing that in this course, we will not prove theorem 2 or its corollary 11 rather we note down one point that it is not possible to prove the theorem 2 by the method of successive approximations, so the proof of theorem 2 and corollary 11 is not depending on the successive approximation.

And one more example we can consider that hypotheses of theorem 2 does not guarantee uniqueness of the solution.

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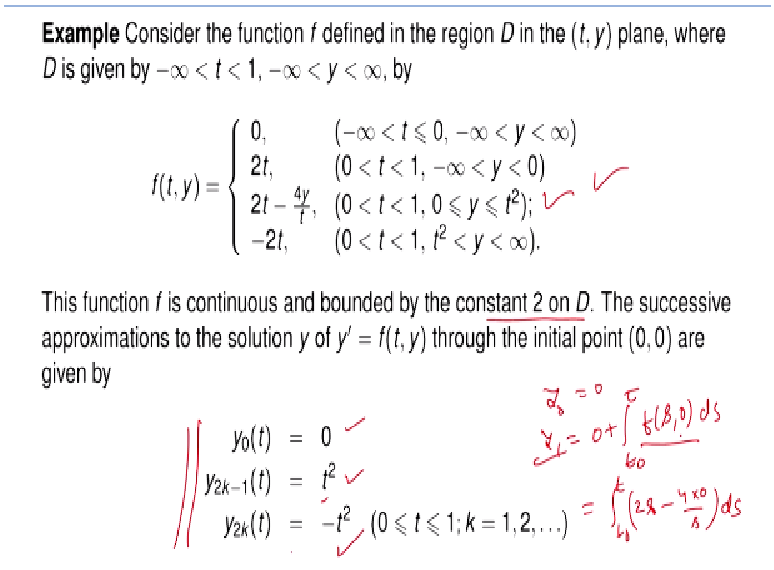

So, example is this, f ty is defined in this manner and here we say that f is continuous and bounded by the constant 2 on D that I am leaving it to you that you can verify that this is bounded by 2 and if you calculate the successive approximation and we can say that successive approximation is given by y0 t = 0 y2 K – 1t = t square y2 Kt = - t square that you can verify from this and that you can easily calculate y0 is given as 0, no problem, y1 you can calculate, 0 + t0 to t f of s 0 d of s and you can say that this lying here, here.

So, here this can be written as t0 to t and this is 2t 2s – 4 this is 0 upon s, if you simplify your y1 is coming out to be s square, so y1 t is coming out as t square and similarly, you can find out y2 n and so on and you can see that your approximation are coming like this but if you observe here, your odd terms; odd sequence will converge into t square, is nothing but t square and even is giving – t square.

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Thus the successive approximations alternate between t^2 and $-t^2$ and do not converge. Since the function $f(t, y)$ is continuous and bounded on D, therefore by Theorem 2, we have the existence of a solution. Also, since f is monotonically nonincreasing in y for every fixed t, Corollary 11 gives us the assurance of existence of a unique solution.

Thus we may conclude the following. $y' = x^{2/3}$

1. Assurance of merely existence of solution does not require Lipschitiz condition, only continuity of f is required in y.
2. Here approximations do not converge, but still we have a unique solution, thus continuity of f and uniqueness of solution do not imply the convergence of approximate solution.
3. The uniqueness results and convergence of successive approximation are two different independent phenomena.

So, it means that thus the successive approximation alternate between t square and – t square and do not converge and since the function f ty is continuous and bounded, therefore by theorem 2, we have the existence of solution, so existence of solution is given and not only this, f is monotonically non increasing in y that I am also leaving on you, you please verify, then we can say that corollary 11 gives the assurance of the existence of unique solutions.

So, it means that here in this particular problem your approximation; successive approximation solutions does not converge but it has a unique solution, so we can observe the following thing based on this example that assurance of merely existence of solution does not require Lipschitz condition that is guarantee by this that y to the power 2/3. Here, if you look at this problem here this does not satisfy the Lipschitz condition but is still it has a solution.

So, assurance of merely existence of solution does not require Lipschitz condition only continuity of f is required in y. Second; approximation do not converge but is still we have a unique solution that is done by previous example, thus continuity of f and uniqueness of solution do not imply the convergence of approximate solution in this problem, we have continuous function and here we also assume that solution is also unique.

But still we do not have convergence of approximate solution because approximate solution may not converge the limit, so this means that uniqueness result and convergence of successive approximation are 2 different independent phenomena, so that is this last 2 conclusion are based on the example given here and the first example; first conclusion is based on this example that here uniqueness does not require Lipschitz condition, sorry existence does not require Lipschitz condition only continuity is also sufficient.

So, with this, we end our lecture, so in this lecture what we have done, we have discussed the initial value problem and discuss the condition required for existence and uniqueness of the solution of the initial value problem, so will continue in next lecture, thank you very much for this.