Dynamical Systems and Control
Prof. N. Sukavanam
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 39
Feedback Control - I

Hello viewers. Welcome to the lecture on feedback control. In this lecture, we will consider linear control system and its feedback control. Let us see what is a feedback control. Any control which can be expressed in the form of or as a function of the current value of the state variable is called a feedback control. That is at any instant of time t, if the control function u of t is a function of the state variable x of t, then it is called a feedback control.

In our everyday life, we normally use feedback control in performing many tasks. For example, when we drive a vehicle, then the control on the vehicle is always based on the current position and velocity of the object and apart from various other observations. So such controls are called a feedback control. So we will consider linear feedback in this lecture. So let us see what is a linear feedback control.

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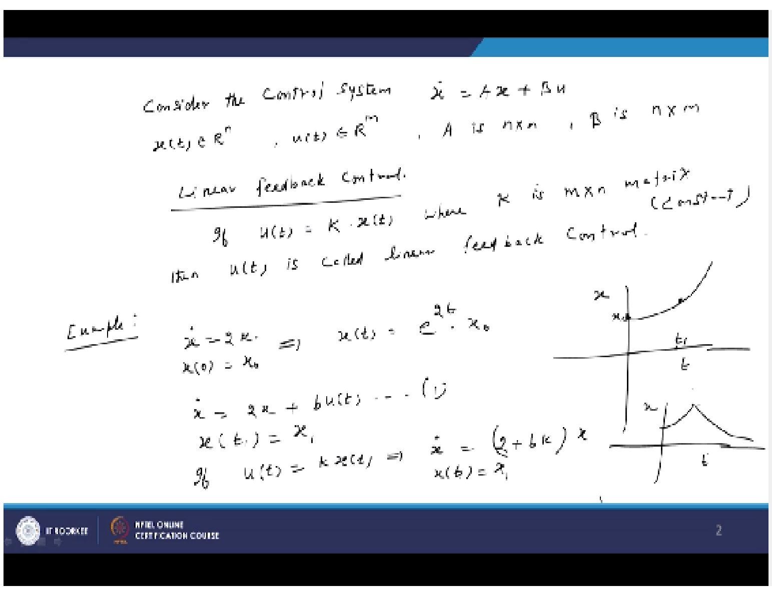

Consider the control system x dot=Ax+Bu. So as usual we will consider x of t is in Rn is the state space, u of t belongs to Rm is the control space and A is n*n matrix and B is n*m matrix. These are constant matrices. A given linear control system. Now what do we mean by a feedback control, linear feedback? So if the control function, control vector function u of t is written as some k matrix*the x of t where k is m*n matrix, constant matrix, then we call it as the linear control, linear feedback control.

Then u of t is called linear feedback control. So now how we use it in a practical situation because in many practical problems we use a linear feedback control in the following way. For example, let us consider x dot=2x and x of 0=x0, it is a very simple dynamical system. So the solution of this problem is x of t=e power 2t*x0. For example, let us say it is a population growth of some bacteria, okay, or any insect population or something like that.

So for a short period of time, this grows in an exponential manner if it is a model for the population. So if x0 is the initial population, t is this and the x is this. At t=0, the initial population is x0 and it grows in an exponential manner. Now if you want to control this population using some kind of pesticides or some medicine, etc. So if we control this system using some kind of controller u of t and then want to make this population to go to 0.

Instead of going to infinity, we want the trajectory should travel and come towards 0. So if the initial condition is let us say x1. At time t1, we see that the population is this much x1 and from there it should not grow exponentially but it should come down and make the population 0. If you are interested in this type of controlling the population, then we can apply a feedback control in the following manner.

So if you say u of t is some constant time x of t itself, whatever is the population, we take proportional to that the control function u of t. That will imply that the system, the control system will become x dot=, if you substitute u of t in this equation 1, then we will get this as 2+B*k of x of t with the initial condition xt1=x1.

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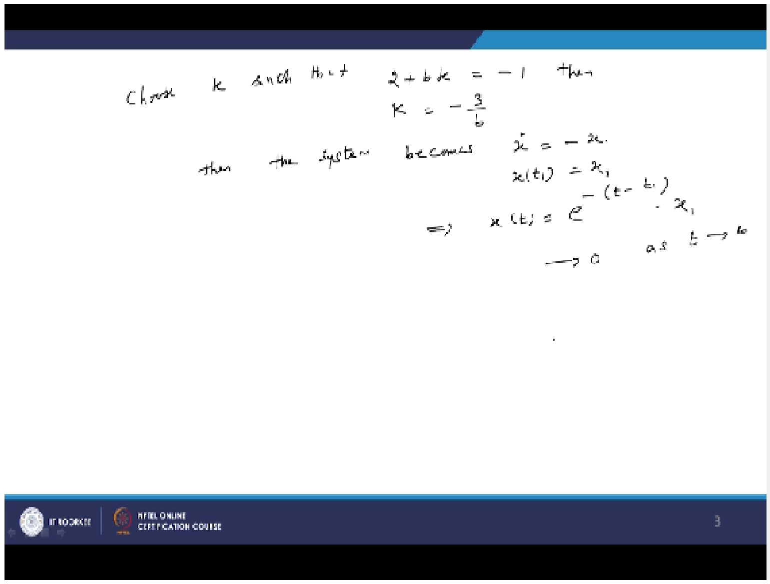

So if we select this k suitably so that 2+Bk is negative, so choose k such that 2+Bk; for example, let us say it is -1. Then that means you have to select k to be equal to -3/B where B is a known number already. If you put it like this, then the system becomes x dot=-1*x and x of t1=x1. So this immediately implies that x of t will be e power -t-t1*x1. So after the time t1, you can easily see that it tends to 0 as t tends to infinity.

So by selecting a feedback control, like this u=k*x of t, we will be able to control the system like this. So if without control it was going exponentially and with control, then it will start coming down and it will become 0, t versus x graph. So we will be able to change the behaviour of a system by using feedback control in a suitable manner. So that is the usefulness of the feedback control.

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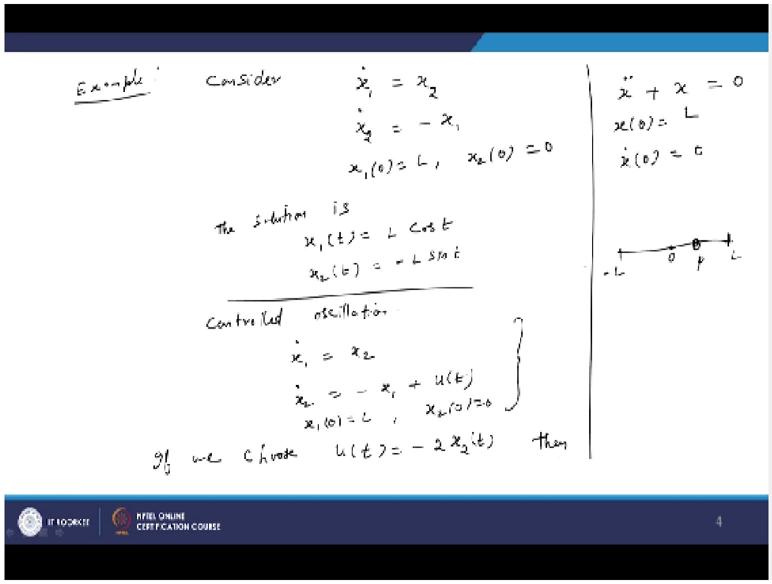

So for the second example, consider the harmonic oscillator x1 dot=x2 and x2 dot=-x1. Let us say this is the equation. So this you can easily see that it comes from the equation x double dot+x=0. if you convert this system into 2 first order equation, you will get this one. This is nothing but; for example, if x of 0=L and x dot of 0=0. So it is an oscillator MOS1. A particle of MOS1 which is oscillating having O as the center, the force of attraction is at O and then it is oscillating indefinitely from -L to +L.

Initially it is at L and the velocity is 0 and then it start moving in the, indefinitely like this. So we can easily see that if we solve this equation, so here if you convert this x of 0 as x1 of 0=L and x dot of 0 means x2 of 0=0. So with this initial condition, it is a harmonic oscillator which will move indefinitely. The solution is, if you solve this equation using the standard way of solving the dynamical system and substituting this, we will get L cos of t, okay, and x2 of t is -L sin t.

So it is a periodic solution keep on moving, never stops. Now if we apply a force on this particle, then it will become a forced harmonic oscillator and then we can change the behaviour of the motion as we wish. For example, if you want to stop this harmonic oscillator at one position, we can apply a suitable force, so controlled oscillation. So if we apply x1 dot=x2, it cannot be changed because the rate of change of displacement is velocity and rate of change of velocity x2 dot is here -x1.

So now if you add another force here, u of t on the particle, we will get the forced equation here, controlled equation with this initial condition. So by selecting this u of t properly, then we can control it in the following way. So if we select u of t=-2*x2 of t, for example, okay. It is a planned selection like this, if you do it like this, then we will get by substitution, x1 dot will be equal to x2, x2 dot is -x1-2x2.

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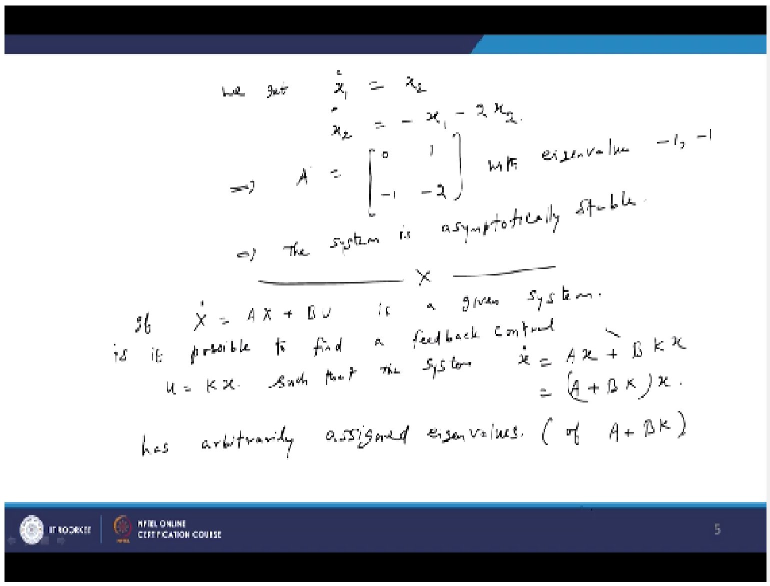

So this is obtained and this implies that the matrix involved in this system is 0 1 -1 -2 with eigenvalues are -1 and -1 repeated 2 times, okay. So when the eigenvalues are negative, both the eigenvalues are negative, this implies that the system is asymptotically stable. So this is a feedback control that is what we are doing. This control which we have obtained, it is a feedback control because we are writing u of t as 0 and -2 multiplied by x1 x2.

So it is like 0 -2 is the k matrix and x is the state variable. So u=kx is the feedback control which we have applied here. And by applying this, the system response, it has changed to, asymptotically stable property has been obtained here. So here we have just by trial and error, we have chosen a feedback control like this. So the question here is, if we have a general system, is a given system, is it possible to find a feedback control u=kx such that the system, resulting system x dot here equal to Ax+Bkx which is nothing but A+Bk*x.

By using a feedback, we are getting a new system x dot=A+Bkx. Is it possible that this A+Bk is chosen in such a way that it has any arbitrarily assigned eigenvalues? So the question is, is it possible to find a feedback control u such that the system, this has arbitrarily assigned eigenvalues? Eigenvalues for the matrix, eigenvalues of A+Bk.

So what is the importance of the eigenvalues of this matrix, eigenvalues will decide the nature of the solution because we have seen that if all the eigenvalues are having negative real part, the system will be asymptotically stable or even if one of the eigenvalue is positive, it will be unstable, etc. So the stability of the system is decided by the eigenvalues of the matrix involved in the system.

So if you are able to select a set of eigenvalues which we are interested using a feedback control, then it is an advantage, that is we can control the given system in a manner which we are interested in.

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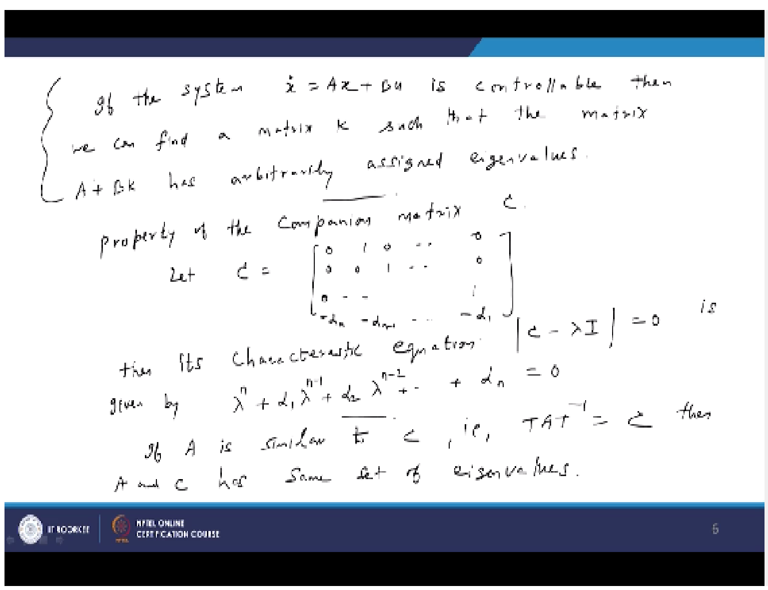

So we will see here that result. So if the system is controllable, then we can find a matrix k such that the matrix A+Bk has arbitrarily assigned eigenvalues. So this is the result which we will; so before going into this result, so let us first see the property of the companion form. Property of the companion matrix C. If C is in the form 0 1 0 0, 0 0 1 0 and 1 here, -alpha n -alpha n-1, -alpha 1.

So this is the standard form of a companion matrix. Then it can be easily seen that, then its characteristic equation that is C-lambda I determinant=0 is given by. These will be the coefficient of the characteristic equation, that is lambda to the power n+alpha 1*lambda n-1 alpha 2lambda n-2 etc. alpha n=0. So the characteristic value has the coefficient which are in the last row of the matrix C that can be easily verified.

If A is similar to C, A is a given matrix, C is a matrix if they are similar, that is TAT inverse=C which we have seen already in this particular form, then the set of eigenvalues of A and the set of eigenvalues of C, both are the same. Then A and C has same set of eigenvalues, okay. So this is a known fact from the linear as we draw that can be easily verified.

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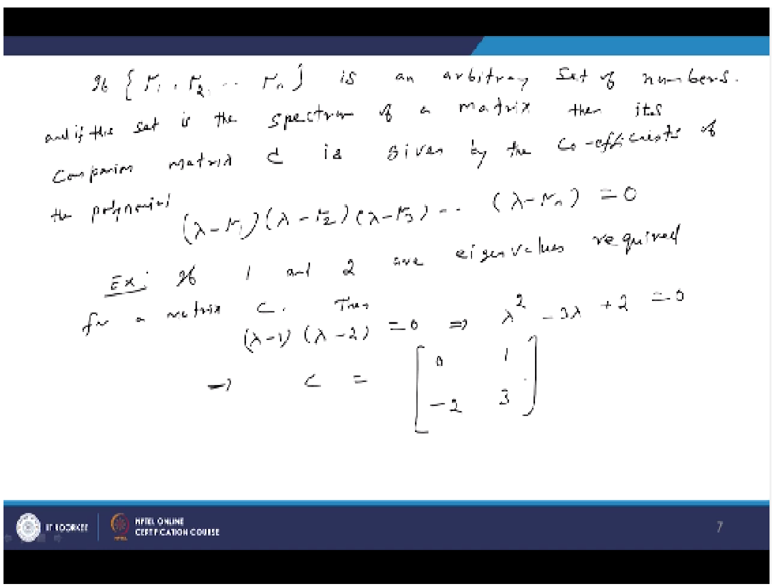

Now making use of this, we will see, that is, so if let us say mu 1 mu 2 etc. mu n is any given set, arbitrary set of numbers whether real or complex numbers, and if we want that this should be; and if this set is the spectrum of a matrix, any matrix, then its companion form, its C is given by the following, given by the coefficients of the polynomial lambda-mu 1 lambda-mu 2, etc. So if you calculate the coefficient of this polynomial and write them in the reverse order with a negative sign, we will get the last column of the C matrix.

Here -alpha 1 -alpha 2 etc. are the last column. So these numbers will be obtained by finding this expression. So for example, if let us say 1 and 2 are the eigenvalues required for a matrix C, then we will calculate lambda -1*lambda -2=0. This implies we will get lambda square-3lambda+2=0. So that implies that the companion matrix is of the form 0 1 and 2 here and -2 in the place. Is not it?

The coefficient of the characteristic equation is obtained here. Is not it? It is lambda square-3lambda. So it is 3 here, okay. The coefficient is given in this manner. So this thing will be used for proving the particular theorem, for proving the statement of the theorem given here. If the system is controllable, then it can be converted into the companion form.

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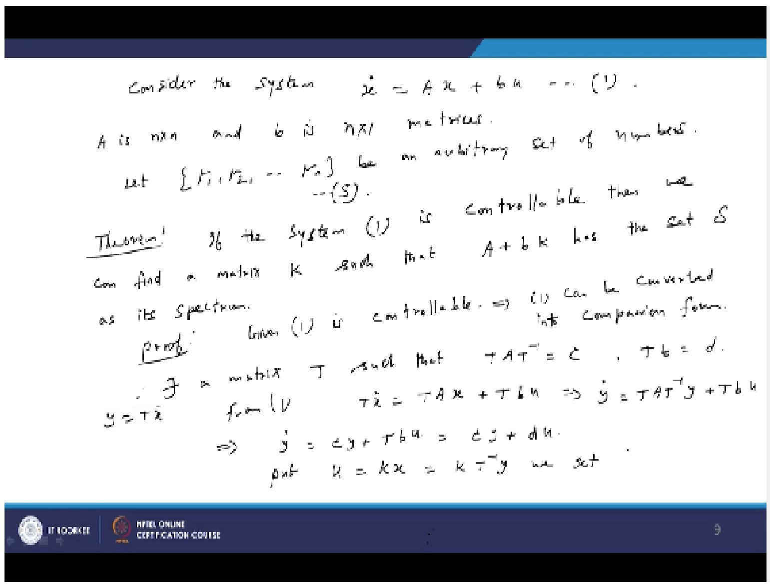

To see it very briefly, now we will consider the system x dot=Ax+bu where A is n*n and b is n*1 matrices. So for the general case, we will see in a later lecture. So now let mu 1, mu 2, etc., mu n be an arbitrary set of numbers. It can be real or complex number. So the question is, is it possible to find a k matrix so that A+bk has this set as the eigenvalue. So consider this system. Now the theorem is, so if you call this equation as 1 and this set as a set S, okay, the theorem is if the system 1 is controllable, then we can find a matrix k such that A+b*k has the set as S as its spectrum.

Spectrum is the set of all eigenvalues of the matrix. So the proof is the following way. It is given that the system is controllable. So this implies that one can be converted into its companion form. So there exist a matrix T such that TAT inverse will be C and T*b=D. So this we have already seen. System is controllable. We can get C and D matrix like this. So now as usual, we multiply both side with T and from 1, we will get Tx dot=TAx. So TAx+Tbu.

So that implies that if you substitute y=Tx, we get y dot=TAT inverse of y+Tb of u, okay. So now this implies that y dot=Cy+Tbu, Cy+d*u. So if you substitute, put u=kx and which is equal to kT inverse y, we will get y dot=Cy+dk*T inverse of y.

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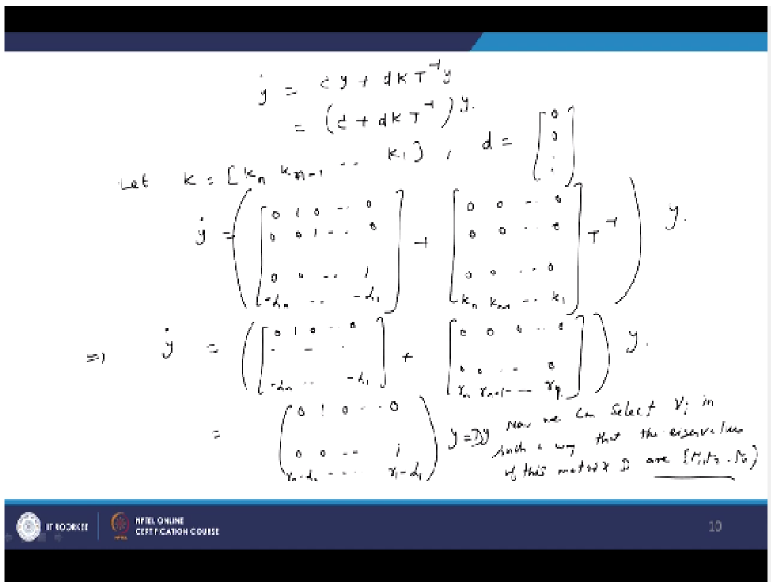

So this we will get it as C+dkT inverse of y. So now let k is k1, k2, it is kn, kn-1, k1 is a column vector. Already we have that d=0 0 1 and T inverse is the n*n matrix. So we will obtain, in this case, we will get y dot=, let us write the whole thing here. C is 0 1 0 0, 0 0 1 0, -alpha n to -alpha 1, +d*this k1 will give 0 0 etc., the last one, 1*kn, etc. will come, kn, kn-1, k1, *T inverse the whole thing*the y vector.

So we can get here, now again multiplying this matrix with T inverse, we will get all this, -alpha n, -alpha 1, +, again we will get all the first n-1 column, rows are 0. Only the last row multiplied by the matrix will give a non-0 row vector. Last row multiplied by T inverse, we get this expression*y. **"Professor - student conversation starts"** This will be gamma n. Gamma n-1 yes. **"Professor - student conversation ends."**

And this is gamma 1 here. So this implies, we will get the matrix as 0 1 0 0, if you add the element wise, up to n-1 rows, we will get like the companion matrix and the last element will be -alpha n+gamma n. So we will get gamma n-alpha n, etc. Last element is gamma 1-alpha 1, this*y. Now if you are interested in having mu 1, mu 2, mu n as the eigenvalues of the matrix, so now we can select this gamma values in such a way that the eigenvalues of the matrix, let us call it as some notation D*y.

The eigenvalue of this matrix D are mu 1, mu 2, etc., mu n which is always possible because the last row is nothing but the coefficient of the characteristic root of this polynomial. So the roots are given already, mu 1, mu 2, mu n are given. Then by finding the characteristic polynomial using this eigenvalues, we can get the coefficients which is in the last row. And alpha 1, alpha 2, alpha n are already known.

Only gamma 1, gamma 2, gamma n can be obtained from the given values of the roots here. So this theorem shows that using the controllability of the equation 1, we are able to get a companion form. And using the companion form, we are able to get the particular form D*y in the last line and because of that, we will be able to get all the values of gamma corresponding to the roots given here as mu 1, mu 2, mu n. And from here, we can calculate the control function, that is k*u because from gamma, we can obtain the k values.

From k value, you can obtain the control as k*x. So this completes the proof of the theorem on the feedback control of the system, okay. So the next lecture, we will see some example of how to compute the feedback control of a system and for the general case. Here we have seen that the result is proved for the particular case where B is a column vector. So the next one, we will see if B is a general matrix, the m*n matrix, how to get this feedback control for the system. Thank you.