Dynamical Systems and Control
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Lecture – 38
Companion Form

Hello viewers, welcome to the lecture on companion form. In this lecture, we shall see an important result which relates companion form and controllability of a linear system. So, to start with let us see what a companion form means.

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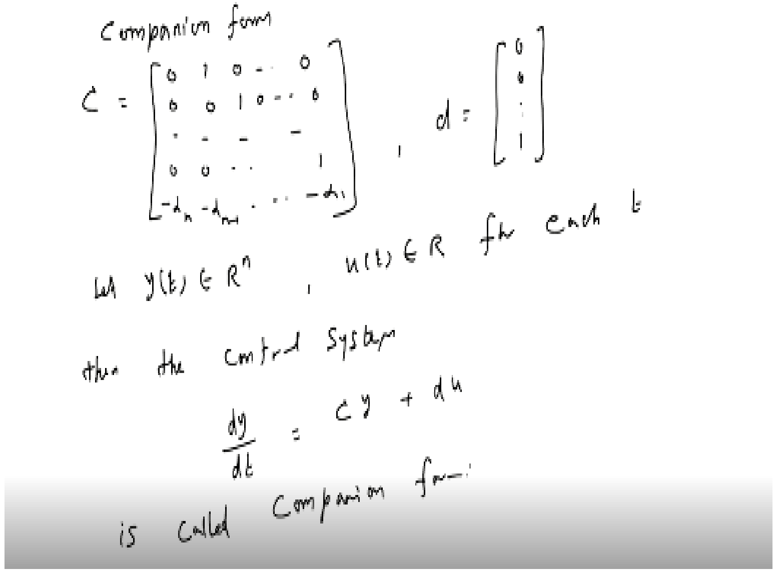

Companion form so let C be a matrix of order n given by 0 1 0 0 0 0 1 -alpha n -alpha n-1 -alpha. So, here the off diagonal elements are 1 and the last row can be 0 or non-zero elements and let us consider matrix d which is column matrix like this. And let y of t belongs to Rn and u of t belongs to R for each value of t. Then the control system y dot that is dy/dt=cy+du is called a companion form. So, any control system which can be written in this particular form is called a companion form of that control system.

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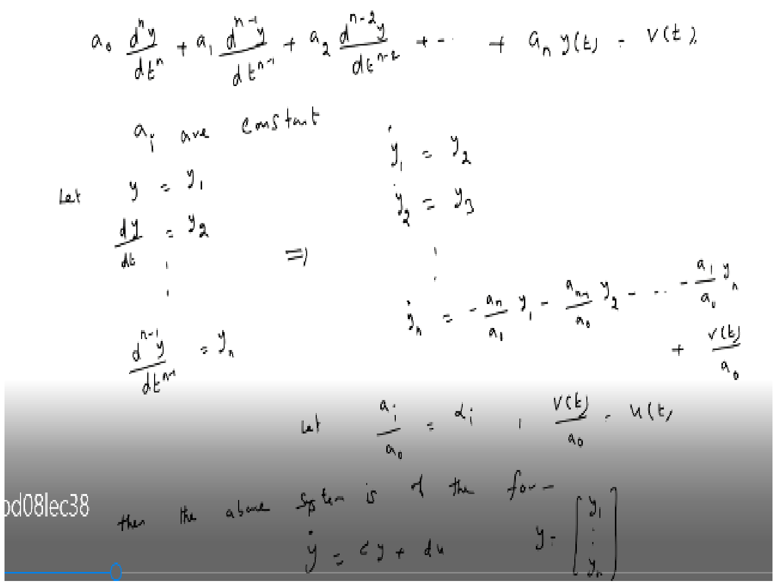

So, let us see from where we obtain the companion form so for that let us consider the differential equation of order n+ a to d n-2 y d n-2 etc., + an*y of t= sum function y of t. So, it is a nth order differential equation. Here a i are constants nth order equation with constant coefficient. Now if you convert this nth order and the equation into n first order equation by substituting.

So, let us take y=y 1 and dy/dt= y2 etc d n-1/d t n-1=yn so the we will get the derivative y1=y2 y2 dot=y3 here it is yn sorry and yn dot that is the nth derivative of y that is= the expression. If you take all this to the right hand side, we will get an/a0*y1-an-1/a*y2 etc., -an/a0 yn+v of t/a0. So, we get this expression so if you substitute ai/a0 as alpha i and v of t/a0=u of t then we get the companion form.

Then this system is written in the form of companion form then the above system is of the form y dot=cy+du where y is the vector y1 etc., yn and c and d are defined in the previous slide. So, from an nth order differential equation we can always obtain the companion form like this.

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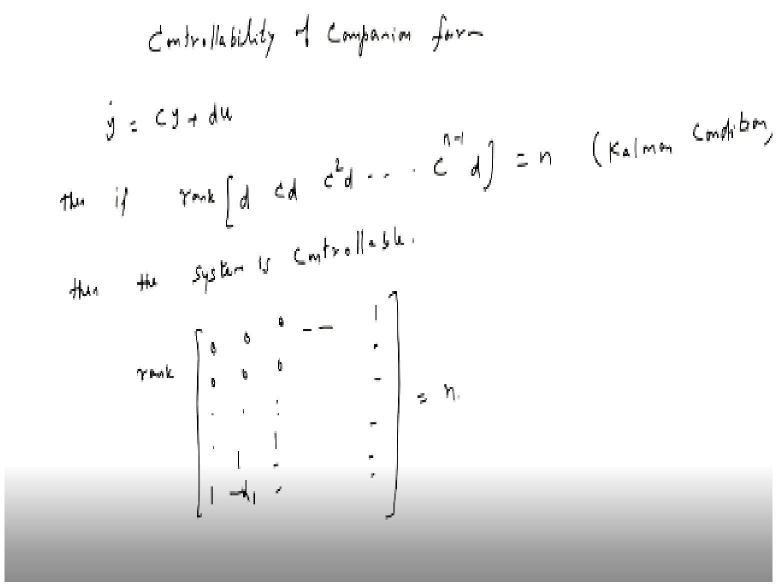

Now we will see that the controllability of the companion form. So, if we consider the system y dot=cy+du then the system is controllable if and only if the rank of the matrix d cd c square d etc cn-1 d if it is =n this is the Kalman condition then the system is controllable. But it is very clear that the rank of first column is dc that is 00 etc 1. And if you multiply c*d where c is given by the previous slide.

If you multiply c*d we will get the last but 1 element is 1 and the last element will be - alpha 1. So, we will get the second one is 0 0 1 and - alpha 1 will come. Similarly, the third 1 will be 1 and some two non0 elements 0 or non-zero elements are there and finally we will get 1 and various elements. So, it is very clear that the determinant of this matrix is non-zero that is 1 and therefore the rank =n.

So, any system which is in the companion form it is always controllable. So, you can say that if the differential equation is considered as a control system where the right hand side is considered as control then it is always controllable for any desired y y dot etc., up to the nth derivative we will obtain a particular control to steer the system.

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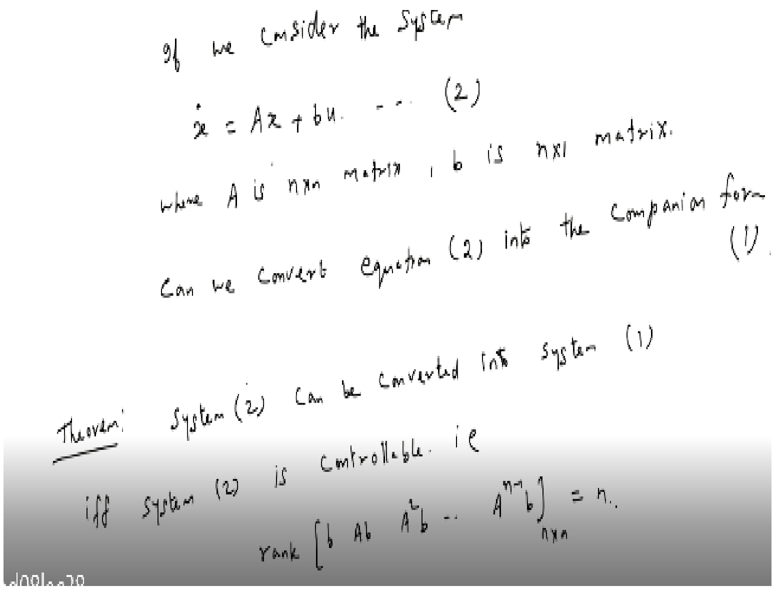

So, now we will come to the general theorem the question is if you consider general system if we consider the system of the form let us say x dot dx/dt=Ax+bu where A is any n*n matrix and b is n*1 matrix. It may not be the companion form. So, the question is can be convert into a companion form convert this equation to let us say into the companion form. So, let us call this as equation 1 companion form.

So, this is the main result that is if the system two is controllable the result is the theorem says that the system 2 can be converted into the companion form 1 if and only if the system 2 controllable. In other words, if and only if the rank of the matrix here the column matrix b Ab A square b etc A power n-1 b=n. So, here we consider A to be n*n b is a column vector only okay. We do not consider this result for any general that is n*m matrix b.

That we will consider in the later lectures so this matrix is a n*n matrix so it should be invertible the meaning is so the how to convert the system in to this particular form.

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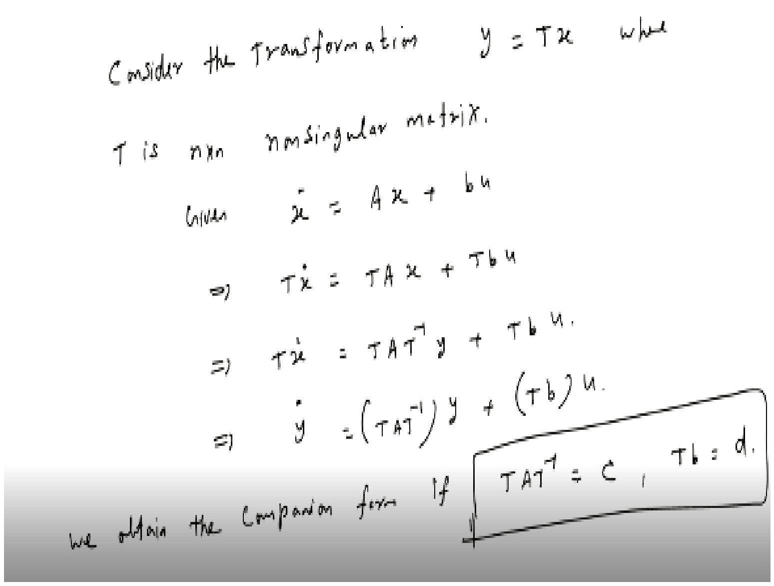

It is using a transformation so let us consider the transformation y=Tx here x is a column vector y is a column vector belonging to Rn an T is n*n non-singular matrix. Let us say T is any non-singular matrix just to take it transformation. Now given the system is x dot=Ax+bu if you multiply both sides with T Tx dot=TAx+Tb*u. And if T is invertible we have x T inverse y so that implies the T x dot = Ta and x can be replaced with T inverse y + Tbu.

And because T is a constant matrix we assume the derivative of y is same as T*x dot. So, this implies that y dot=TA T inverse*y + Tb*u. So, if this TA T inverse is in the form of c and Tb is in the form of T then we obtain the transformation T which converted into the companion form. So, we obtain the companion form if TAT inverse is the form of c and T*b is in the form of the d matrix.

So, if it happens then the result is true so what we will prove here is this particular thing can happen if and only if the system is controllable that this one. Rank of b Ab A square b=n then we will be able to find a matrix T satisfying the lost condition like this. So, we will first prove the necessary condition.

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Proof: Necessary condition

If the system (2) can be converted into the companion form

(1) then rank $[b \;\; Ab \; \cdots \; A^{n-1}b] = n \quad \cdots (3)$

Given $\dot{x} = Ax + bu$ is converted into $\dot{y} = cy + du.$

where $c = TAT^{-1}$ and $d = Tb$

Since the companion form is controllable we get

$$\text{rank } [d \;\; cd \;\; c^2d \cdots \; c^{n-1}d] = n \quad \cdots (4)$$

$$\text{rank } [Tb \;\; (TAT^{-1})(Tb) \;\; (TA^2T^{-1})Tb \cdots \; (TA^{n-1}T^{-1})(Tb)] = n$$

$$\Rightarrow \text{rank } [Tb \;\; TAb \;\; TA^2b \; \cdots \; TA^{n-1}b] = n$$

$$\Rightarrow \text{rank } [T] \, [b \;\; Ab \;\; A^2b \cdots \; A^{n-1}b] = n$$

$$\Rightarrow \text{rank } [b \;\; Ab \cdots A^{n-1}b] = n \quad (\because \text{rank } T = n)$$

So, the necessary condition is if the matrix so necessary condition if the system is if the system 2 can be converted into the companion form. Companion form 1 then we have to prove that rank of b Ab etc A power n-1 b is n. So, it means if can be converted into companion form then the system is controllable. So, it is given that x dot = Ax+bu is converted into y dot=cy+du where c is TAT inverse and d is T*b so this is given to us.

Now we have to prove that the rank of this matrix is n so now it is known that the companion form is always controllable. Since the companion form is controllable so we get the rank of d cd etc that is =rank =n. So, now we have to prove the other one we will call this as 3 and this is 4. And from 4 we have to prove 3 so what we do directly substitute the d value. So, this implies that rank of d is given by T*b and c is given by TAT inverse *d that is T*b.

And c square means T A square A T inverse and d is T*b etc and c power n-1 T A power n-1 T inverse d is T*b. So, this matrix has rank n it is given by 4 so this implies that rank of Tb and T inverse*T becomes identity. So, we will get TAb second third one will be T A square b etc T A power n-1* the rank=n. So, now T is the common n*n matrix in the left side of all the products so we will get that the rank of the matrix T.

Multiplied by the remaining matrix b Ab A square b A power n-1 b that is =n. Now rank of this is a product of two matrix T is one matrix and the remaining is the other matrix n*n matrices.

Already we know that rank of T = n so if the rank of the other matrix if it is <n then the product of these 2 matrix the rank will become <n. So, it will it implies that the rank of the b Ab etc A power n-1*b=n.

Since rank of T=n so it proves that the condition is necessary that is if the system is converted into companion form then the system has to be controllable. That is proved here as the necessary condition. Now we can prove the sufficient condition.

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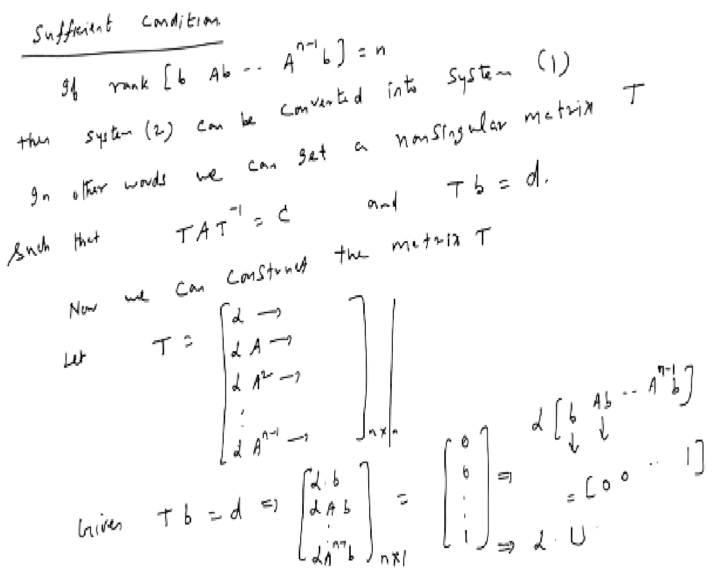

So, you rank of the matrix is n the system is controllable then we have to prove that then system 2 it is x dot Ax+bu can be converted into system 1 in a companion form. Or in other words we will get a matrix we can get a non-singular matrix T such that TAT inverse=c and Tb=d and we have to show that this is possible. So, now making use of the rank condition we can construct the matrix T.

Now we can construct the matrix T in the following way to let us take T= the first row is a unknown vector alpha 1 alpha 2 etc alpha n is the first row and the second row is alpha times A and alpha times alpha ia a row vector A is a n*n matrix. So, the product will be a row vector that is a second one similarly third fourth etc all of them are alpha times A n-1 is the last Vector okay.

So, we take a matrix T in this particular form now if you are able to find this alpha vector then it is clear we can get the entire matrix T. So, let us see how to construct this alpha matrix so first it is given that if at all there exist a matrix T then T*b should be = d so okay if the matrix exist then T*b= d so this implies that alpha* the v matrix that is the first element if you multiply T*b it will be single element.

The first element is alpha*b it is 1 element then alpha*Ab is another element alpha A power n-1 b is the last element. So, it is n*1 matrix but this should be = 0 0 0 1 because it is = d. So, this implies alpha*b is 0 alpha*Ab is 0 etc the last one is like this. So, this implies alpha times b so if you multiply write the b as the first column then Ab as the second column A power n-1 b that is nth column take alpha common outside.

So, we will get all the entries alpha*b that is first element is 0 alpha Ab if you multiply the second element is 0 the last element is 1. So, we get the expression like this so if we take okay this is let us say alpha *matrix u let us say the Kalman matrix b Ab etc if you denote it by the matrix u. So, alpha*u is nothing but 0 0 1 matrix. So, this gives because the rank is n it is invertible matrix u is invertible we get all alpha = 0 0 1 multiplied by u inverse.

So, it is a unique solution of the problem the algebraic system of equation because u is invertible we get a unique solution alpha. By substituting that alpha in the T matrix we get the n*n T matrix. Now we have to check whether TAT inverse is in the form of c or not. We got Tb=d we have already solved. Now if you put it here will it get will it give a matrix in the form of c.

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Now To Check if $TAT^{-1} = C$.

Let $T = \begin{bmatrix} \alpha \rightarrow \\ \alpha A \rightarrow \\ \vdots \\ \alpha A^{n-1} \rightarrow \end{bmatrix}$, Let $T^{-1} = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$

$$TT^{-1} = I \Rightarrow \begin{bmatrix} \alpha s_1 & \alpha s_2 & \cdots & \alpha s_n \\ \alpha A s_1 & \alpha A s_2 & \cdots & \alpha A s_n \\ \vdots & & & \\ \alpha A^{n-1} s_1 & \alpha A^{n-1} s_2 & \cdots & \alpha A^{n-1} s_n \end{bmatrix}_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ \vdots & & & & \\ 0 & 0 & \cdots & & 1 \end{bmatrix}$$

$$\text{Now } TAT^{-1} = TA\begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \alpha A \rightarrow \\ \alpha A^2 \rightarrow \\ \vdots \\ \alpha A^n \rightarrow \end{bmatrix} \begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ \cdots & & & \cdots & 1 \end{bmatrix} \text{ form of } C.$$

So, now to check if TAT inverse = the form of c matrix so that can be easily proved here. So, let T is already given that is alpha first row alpha*A etc alpha A power n-1. And let us assume T inverse if you find because we do not know this alpha value exactly so just to give you a notation here the first column is s1 the second column is s2 and the nth column is sn and T is T* T inverse will give identity.

So, that implies if you multiply T and T inverse first element is alpha *s1 the second element is alpha*s2 the first row*second column etc alpha*sn. Second row first column will give alpha As1 and second*second alpha As2 etc alpha A sn and the last one alpha A power n-1*s1 alpha A power n-1 s2. So, this is a n*n matrix it is the identity matrix okay 1 0 0 0 1 0 so we can easily compare this alpha*s1 has to be 1.

And alpha*s2 has to be 0 etc this comparison is there now if you take TAT inverse that is nothing but TA multiplied by s1 s2 etc first column s2 second column sn is nth column. So, we will get the expression like this the first element is the first 1 okay again we have to write alpha times A the first element is alpha is the first row multiplied by this matrix will give the first row. The second row is alpha A multiplied by A that is alpha square etc.

Alpha A power n-1 is the last row multiplied by A that is alpha A power n. Okay that is so this is T*A multiplied by T inverse s1 s2 sn. So, it can be easily seen that the first row into the first column that is alpha A*s1 that is the first element okay that is here alpha As1 is 0. So, what we observed here is the first row whatever we calculate that will come out to be the second row of the identity element.

Similarly, the second row alpha A square s1 etc will come out as the third row of the identity matrix. So, ultimately with the first row of TAT inverse is the second row of the identity matrix the second row of TAT inverse is the third row of the identity matrix etc. So, till the n-1 okay we will get like this but the last one we do not know what it is because there is no comparison in the equation.

So, the last row can be whether it is a 0 value or non-zero value we do not know but it is in the companion form. It is in the form of the c matrix. So, if you calculate T using this expression alpha = 0 0 1*u inverse and substitute it in the T matrix. We will obtain T and from that T if you calculate TAT inverse we will get a c matrix in this form. So, we will be able to and this construction is unique T is a unique construction.

Because of the equation here alpha*u= this expression so we get it in a unique manner. So, this prove the sufficient condition that the system has to be controllable for a conversion of the companion.

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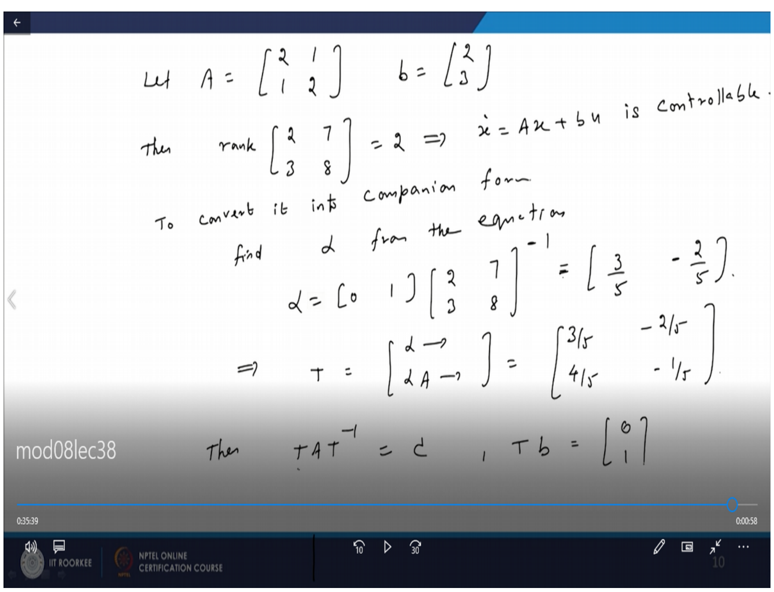

Okay now we will quickly see the following example if A is the matrix let us say 2 1 1 2 and b is the matrix let us take any vector like this then it is controllable the system x dot = Ax+bu is controllable because the rank of b is this and Ab is 7 8 A*b will give this thing this is = 2. So, this implies x dot = Ax+bu is controllable matrix is it not. Now can be convert into the companion form.

So, we have to find the value of alpha from the equation alpha = 0 1 because it is a 2/2 matrix* the inverse of the matrix 2 3 7 8 the inverse. So, once we get the value of alpha so we can see that alpha value is given by 3/5 and -2/5 so we can calculate like this implies T is the first row is alpha second row is alpha*A. So, if you calculate this expression we will get 3/5 -2/5 is the first row.

Second row will be 4/5 and -1/5 is a T matrix then we can obtain this TAT inverse if you multiply we will get the c matrix and T*b we will get 0 1 in the companion form of the expression. So, companion form is very important form in the control system we will see in the next lecture the application of the companion form in the feedback control of a linear control system. Thank you.