Dynamical Systems and Control
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Lecture – 37
Results on Controllability and Observability

Hello viewers, welcome to this lecture on the controllability and observability of the continuous control system, so in this lecture we will see some simple results on the controllability and observability of the given system and the corresponding canonical form system. So, let us first recall some simple results from the linear algebra.

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Theorem 1:

Let

$$P(t) = \begin{bmatrix} r_1(t) \rightarrow \\ r_2(t) \rightarrow \\ \cdot \\ \cdot \\ \cdot \\ r_n(t) \rightarrow \end{bmatrix}_{n\times n} \quad \text{and} \quad \underset{n\times n}{M} = \int_{t_0}^{T} P(t)P'(t)dt$$

the rows of $P(t)$ are Linearly Independent iff the matrix M has rank n (M is nonsingular).

So, if P of t is a matrix n cross n matrix with rows r1, r2, rn as functions of t and M is a constant matrix given by integral t0 to capital T P of t * P dashed t, so this also a n cross n matrix and then the rows of the matrix P of t are linearly independent if and only if the matrix M has rank n or in other words, the matrix M is non-singular. So, here the each row is a function of t, so here we say that the rows of the matrix as functions of t are linearly independent.

And here the constant matrix is non-singular; capital M is constant matrix, so the relation between the P of t and M is given by this expression.

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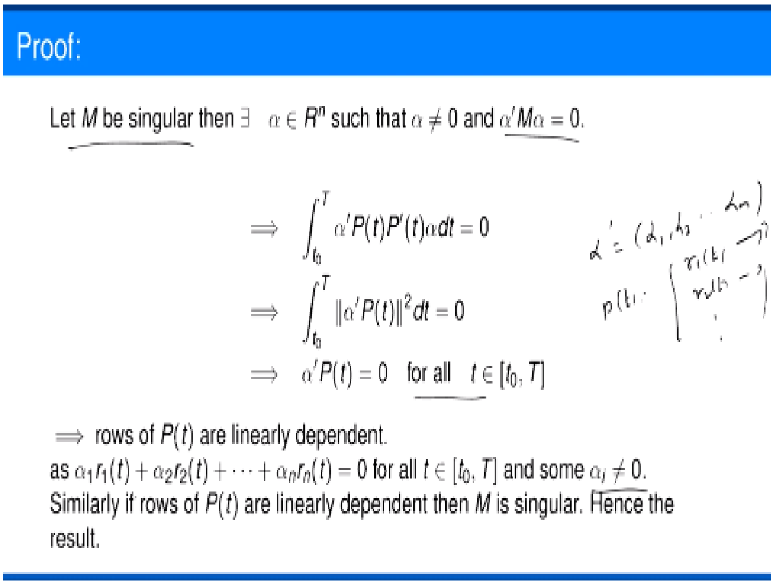

So, the proof is like this, if M is singular matrix, then we can find a vector alpha in Rn, so that alpha is non-zero and alpha dashed M alpha = 0, it is a standard result and so that implies by substituting the definition of M that is integral t0 to capital T, P of t * P dashed t gives the M definition and then we multiply; premultiply with alpha dashed and post multiply with alpha, so that = 0.

But this nothing but norm of alpha dashed * P of t square dt = 0 and the integral of the positive quantity is 0, it implies that function inside the integral alpha dashed * P of t = 0 for all t. So, this implies the rows of the matrix P of t are linearly independent because alpha; if you write alpha dashed to be alpha 1, alpha 2, alpha n, a row vector and if you multiply the matrix P of t; P of t is first row is r1 of t, second row is r2 of t etc.

So, if you multiply alpha dashed * P of t, we get this expression, alpha 1 * the first row + alpha 2 * second row etc., = to 0 for all t value and here, alpha is a non-0 vector, so some of the alpha i's with be non-0, so this implies that the rows of P of t are linearly dependent here. So if M is singular, the rows of P of t are linearly dependent and similarly, we can go back, if the rows are linearly dependent for P of t, then we can; the reverse order if we reach, then we will conclude that alpha dashed * M * alpha = 0.

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Theorem 2:

Let

$$P(t) = \begin{bmatrix} r_1(t) \rightarrow \\ r_2(t) \rightarrow \\ \cdot \\ \cdot \\ \cdot \\ r_n(t) \rightarrow \end{bmatrix}_{n \times n}$$

and $L(P(t)) = \bar{P}(s)$ which is $n \times n$ matrix in s variable. Rows of $P(t)$ are linearly independent for $t > 0$ iff rows of $\bar{P}(s)$ are linearly independent (as vector functions of s).

That implies M itself is singular, so it proves the theorem that if M is non-singular, then the rows of P of t also should be linearly independent, vice versa, if row of P of t are linearly independent, then M is non-singular. So, this result follows

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Theorem 3:

Consider the autonomous system $\dot{x} = Ax + Bu$ then the following are equivalent

- the system is controllable.
- $rank[B \;\; AB \;\; A^2B \cdots A^{n-1}B] = n$.
- rows of $e^{At}B$ are LI.
- rows of $(sI - A)^{-1}B$ are LI.

Proof: By taking $P(t) = e^{At}B$ in the above theorems (1)-(2) the results (3) and (4) follows.

Now, we use that result for the controllability problem, so if we consider the system x dot = Ax + Bu, where A is n cross n matrix and B is n cross m matrix, then the conditions; 4 conditions are equivalent; 4 statements, so the system is controllable and the second one is the rank of B, AB, A square B etc., A, n - 1B with the rank = n. So this we already proved, the equivalence of 1 and 2 has been already proved.

Then the controllability also for this constant matrix case, if A and B are constant matrix, we have also proved that the matrix 0; t0 to T of phi t t0 phi t0 – t * B B dashed e At0 – t dashed

* dt, so this is non-singular. We have shown that this controllability Grammian is non-singular. So, now we can easily prove that the second implies third because 2 implies this matrix is non-singular. Now if you take e to the power A t0 -t * B as the P of t matrix, we can assume that P of t is e to the power A t0 -t * B.

Then, we can show that the if this matrix M is non-singular, previously we have proved in the previous theorem that P of t is non-singular, the rows of P of t's are linearly independent, then the matrix M is non-singular and vice versa, so by taking P of t to be e to the power A t0-t * B, we can show that the rows of e to the power At * B are linearly independent because e to the power A * t0 is a constant matrix which can be; which is is non-singular, so we can prove 2 implies 3. Now 3 implies 4, also can be proved in the following way,

So, now if we take this theorem, let us assume that P of t is r1 of t, r2 t, rn t as earlier. Then the Laplace transform of this matrix P of t, we denote it by P bar of s, where s is the Laplace transform variable from t variable is transformed to the s variable here, and P bar of s is the Laplace transform of P of t, so if you assume that the rows of P of t are linearly independent, then the rows of P bar s are also linearly independent and vice versa, that can be shown because the Laplace transform is a linear operator.

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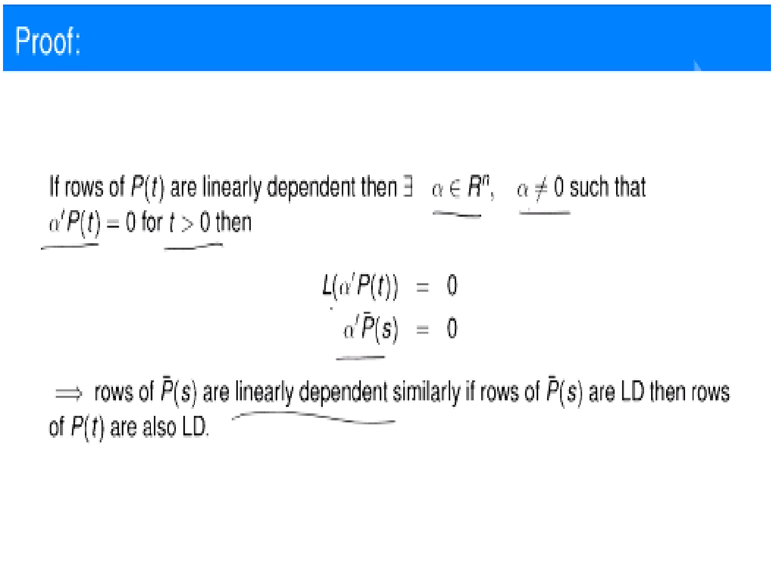

So first if we assume P of t, has linearly independent rows, then there exists a alpha non-0 vector such that alpha dashed * P of t = 0 for all t, then by taking Laplace transform of alpha dashed * P of t that will be = 0; that implies that because alpha dashed is a constant vector, alpha dashed * P bar of s is also = 0, so this implies that the rows of P bar a are also linearly independent.

So, similarly, the inverse Laplace transform is a linear operator, so we can easily proved in the reverse way, if row of P bar s are linearly dependent, we can show that the rows of P of t are also linearly dependent, so we proved. If rows of P of t are linearly independent, then the rows of P bar of s are also linearly independent. So using that result, we can prove that the condition; the statement 3 implies the statement 4.

And statement 4 implies the statement 3, both can be proved by taking P of t = e to the power At * B. In the previous 2 theorems, we can easily do the statement, one we have proved here. The statement 2 implies the statement 3 and now we can prove the statement 3 implies the statement 4 by taking simply the Laplace transform of e power At * B, so it will be nothing but SI - A inverse B, the Laplace transform.

So, if the rows of e power At * B are linearly independent, then the rows of the SI - A inverse B are also linearly independent. So this proves the result, so in order to prove the controllability of the system, x dot = Ax + Bu, we can prove any one of the statements either 2, 3, or 4, it will automatically imply the controllability.

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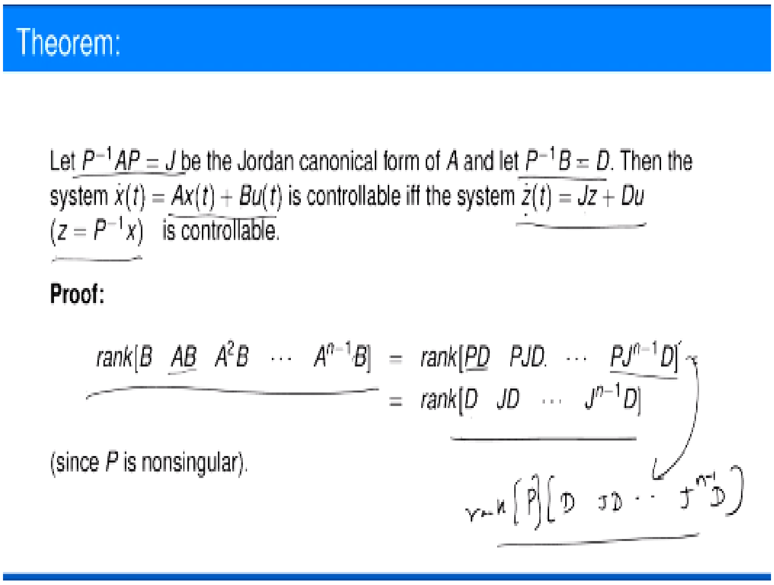

Now, we will prove some result for the Jordan Canonical form, so if A is the matrix, whose Jordan Canonical form is denoted by J, then there exist a non-singular matrix P such that P inverse A, P is the Jordan form, J. So, now let us assume that P inverse B is the matrix D and now, the control system x dot = Ax + Bu, this system is controllable, if and only if the system z dot = Jz + Du is controllable, where the transformed variable z is related with x/z = P inverse of X.

So if x is the state variable; original state variable, then the transformed state variable is z here which is related by z = P inverse of x and the A matrix is transformed from to the J matrix and B is transformed to D, where D is P inverse B here. So, the controllability of the system x dot = Ax + Bu is equivalent to the controllability of the canonical system, z dot = Jz + Du, so that can be easily proved by the controllability of x dot = x + Bu, condition is the rank of B, AB, A square B, A power n-1B is n.

Now A; this matrix B can be written as P * D and A can be written as PJ P inverse, so by substituting in the place of B, we get PD, in the place of AB, we get PJ * D etc. A power n-1B, we get PJ power n-1 * D, so the rank of this is same as the rank of D, JD, J power n-1 D because we can take this out from here, this is = this one, is the rank of the matrix P * D JD etc., because P is the non-singular matrix, so it can be written like this.

And the rank of this matrix will be same as the rank of D, JD, J power n-1 D, so if the AB system is controllable, then the JD system is also controllable from this expression. Similarly, we can go in the reverse order, if rank of D, JD, J power n-1D is n, then by multiplying with P matrix, we can conclude that the rank of B, AB, A2 square B etc. that is also n, so in both the ways we can show.

The controllability of the given system is equivalent to the controllability of the canonical system. So, now we will prove a useful result only in the form of the Jordan canonical form because we have already shown that the controllability of the given system and the Jordan canonical system both are equivalent. So, if you prove the result for the Jordan canonical form, then it will be the; it will be applicable to the given system x dot = Ax + Bu also.

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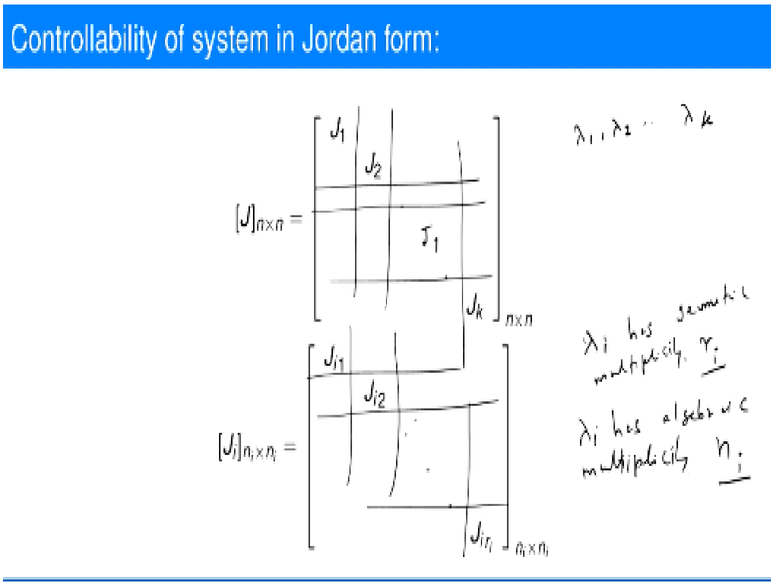

So, instead of the A matrix, let us assume its canonical form J, so J is an n cross n matrix in which we have the Jordan blocks given like this, J1 J2 and Jk. So, here the eigenvalues are lambda 1, lambda 2, lambda k, these are the k distinct eigenvalues and for each eigenvalue, there is a Jordan canonical block for the ith eigenvalue lambda i, we get Ji is the ith canonical; the Jordan block.

So, the Jordan block Ji is given by Ji1, Ji2 etc., Ji ri, so this J lambda i has geometric multiplicity Ri that means, for the eigenvalue lambda i, there are Ri linearly independent eigenvectors available but the algebraic multiplicity of that is given by ni, so in the characteristic polynomial, the root lambda i is repeated n suffix i times but it has r suffix i linearly independent eigenvectors, so it will have ri Jordan blocks like this

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$$[J_{ij}]_{n_{ij} \times n_{ij}} = \begin{bmatrix} \lambda_i & 1 & 0 & . & . & . \\ 0 & \lambda_i & 1 & 0 & . & . \\ & & & & & \\ . & . & . & . & . & 1 \\ 0 & 0 & . & . & . & \lambda_i \end{bmatrix}_{n_{ij} \times n_{ij}}$$

$$i = 1, 2, 3, \cdots k; \; j = 1, 2, \cdots, r_i \quad \text{and} \quad \sum_{i=1}^{K} r_i = n, \quad \sum_{j=1}^{r_i} n_{ij} = n_i.$$

Now for each J, this Jij, capital J, the Jordan form, i suffix small j, it is a nij x nij matrix, so the Jordan block for that eigenvalue, the Jth Jordan block will look like this, lambda i 1 0 0 0 lambda i 1 0 etc. The half-diagonal elements are 1 here and so this is repeated, nij times, this the size of the block is this one.

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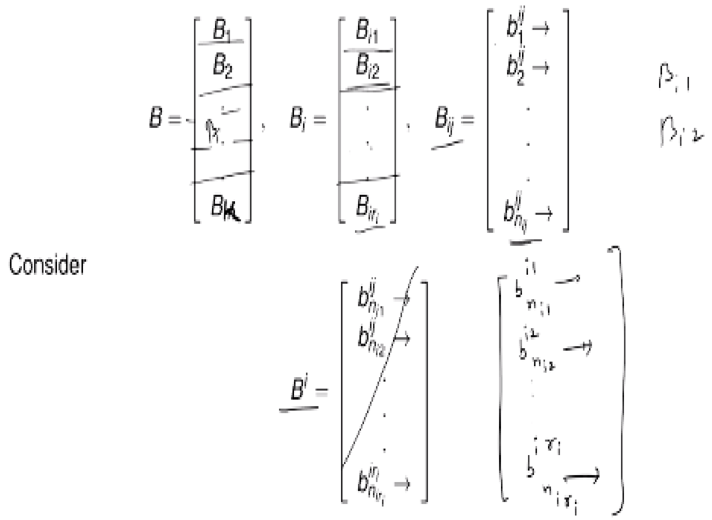

Now the matrix corresponding to B, it can be split into B1, B2 etc., whatever is the number of eigenvalues that is we have taken k, J1, J2, Jk, so correspondingly we have to divide this B matrix into that much form, so it should be Bk and for each i, the Bi block is having Bi1, is one block, Bi2 is the second block and Biri is the last block of this. So, corresponding to Jordan canonical form, we can split the B matrix in this form.

And then for each Bij, we can write the matrix in this form, so this superfix ij represents it is corresponding to the matrix Bij. So, the superfix ij and it has the rows, B1 is the first row, B2 is the second row and B suffix nij is the last row of the matrix Bij. Now, we take the new matrix, it is called B superfix i, a matrix which we define, it is like this, for each block here, Bi1, Bi2, Bi ri, the last row looks like this.

For example, for Bi1, the last row is small bi1 small b ij / nj1, so the matrix bij is written as bij, 1 is the first row, bij2 is the second row, bij, nij is the last row of the matrix bij. Now, we collect the last row of each such matrix that is B superfix i is the matrix defined as follows, so from the matrix Bi1, the last row of Bi1 is given by Bi1 as superfix and ni1 that is the last row like this.

And when we take the second block Bi2 here, its last row is Bi2 and the last row is given by ni2, this one, so in the place of j, if you put 1, 2, 3 etc., we will get all the rows of B superfix i, so it is Bi1 ni1 is the first row, Bi2 ni2 is the second row and Bi ri and n suffix iri is the last row of the matrix B superfix i. So, that means it is the last row of each such block, this last row is collected for each J, we get this one. So now, the result is as follows

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We know that the system

$$\dot{x} = Jx + Bu \qquad (1)$$

is controllable iff rows of $(sI - J)^{-1} B$ are linearly independent.

Now

$$(sI - J)^{-1} = \begin{bmatrix} (sI - J_1)^{-1} & 0 & & 0 \\ 0 & (sI - J_2)^{-1} & & 0 \\ & & \ddots & \\ & & & (sI - J_k)^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix}$$

If the row of this B superfix i, if they are linearly independent, this is a constant matrix and there are ri rows here, ni1, so here we see that 1, 2, up to ri rows, so if this r suffix i rows are linearly independent, then we say that the system is controllable, that is the theorem to be proved. So, here we consider the system x dot = Jx + Bu, where B is a matrix we can consider, it is a n cross n matrix and it is n x m matrix.

B is a n cross m matrix, u is m cross 1, so we know that this system is controllable if sI – J inverse B is linearly; if the rows of this matrix are linearly independent, that was the theorem proved earlier. So, we see under what condition this rows of sI – J Inverse B are linearly independent, so sI – J inverse can be written like this, because it has Jordan block J1, J2, Jk, so other blocks are zero blocks here we can write, they are all 0, depending on the size, we have to write the blocks here, okay.

And B has blocks B1, B2, as we have seen earlier Bk, now, we have to see if the rows of sI – J inverse B are linearly independent, then the system is controllable, so we will see under what condition the rows of sI – J inverse B are linearly independent. So if you multiply sI – J Inverse * the B matrix, we get this expression as given here.

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$$\therefore \quad (sI - J)^{-1} B = \begin{bmatrix} (sI - J_1)^{-1} B_1 \\ (sI - J_2)^{-1} B_2 \\ \cdot \\ \cdot \\ \cdot \\ (sI - J_k)^{-1} B_k \end{bmatrix}$$

$$(sI - J_i)^{-1} B_i = \begin{bmatrix} (sI - J_{i1})^{-1} B_{i1} \\ (sI - J_{i2})^{-1} B_{i2} \\ \cdot \\ \cdot \\ \cdot \\ (sI - J_{ir_i})^{-1} B_{ir_i} \end{bmatrix}$$

SI – J, here I is the identity matrix, n x n identity matrix, s is a variable; real variable and so we can write it like this, by multiplying these 2 matrices. Now each block sI – Ji Inverse * Bi, so if you take this one, it can also be written in the similar form, only thing is for each I, we take the corresponding blocks, Ji1, Ji2, Ji n suffix i are taken these blocks and similarly, Bi1, Bi2, Bi ri, so here si – Ji suffix i ri is there not ni, there is a correction here.

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$$\text{Now} \quad (sI - J_{ij})_{n_{ij} \times n_{ij}} = \begin{bmatrix} s-\lambda_i & 1 & 0 & 0 & \cdot & \cdot \\ 0 & s-\lambda_i & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & s-\lambda_i & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & s-\lambda_i \end{bmatrix}$$

$$(sI - J_{ij})^{-1}_{n_{ij} \times n_{ij}} = \begin{bmatrix} \frac{1}{(s-\lambda_i)} & \frac{1}{(s-\lambda_i)^2} & \cdot & \cdot & \cdot & \frac{1}{(s-\lambda_i)^{n_{ij}}} \\ 0 & \frac{1}{(s-\lambda_i)} & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \frac{1}{(s-\lambda_i)} & \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(s-\lambda_i)} \end{bmatrix}$$

Now we take one such block, si – Jij which is nij cross nij size matrix, which can be written in this particular way because the eigenvalues are lambda i for this and they are repeated nij times and so, sI – Jij is given by this matrix, now it is inverse, if you directly calculate, we can find the inverse of the matrix, sI – Jij inverse is given by this expression.

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$$(sI - J_{ij})^{-1}_{n_{ij}\times n_{ij}} B_{ij} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} b_1^{ij} \to \\ b_2^{ij} \to \\ \cdot \\ \cdot \\ \cdot \\ b_{n_{ij}}^{ij} \to \end{bmatrix}$$

Now $(sI - J_i)^{-1} B_i$ (order $n_i \times m$) has rows.

Now if we multiple with the matrix Bij with the inverse, we will get the following, sI – Ji inverse Bi, it has ni cross m rows and that can be written like this.

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$$\underset{n_i \times n_i \quad n_i \times m}{(sI - J_i)^{-1} B_i} = \begin{bmatrix} \frac{1}{(s-\lambda_i)} b_1^{i1} + \frac{1}{(s-\lambda_i)^2} b_2^{i1} + \cdots \frac{1}{(s-\lambda_i)^{n_{i1}}} b_{n_{i1}}^{i1} \\ \frac{1}{(s-\lambda_i)} b_2^{i1} + \dots + \frac{1}{(s-\lambda_i)^{n_{i1}-1}} b_{n_{i1}}^{i1} \\ \cdots\cdots \\ \frac{1}{(s-\lambda_i)} b_{n_{i1}}^{i1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{(s-\lambda_i)} b_1^{i2} + \frac{1}{(s-\lambda_i)^2} b_2^{i2} + \dots + \frac{1}{(s-\lambda_i)^{n_{i2}}} b_{n_{i2}}^{i2} \\ \cdots\cdots \\ \frac{1}{(s-\lambda_i)} b_{n_{i2}}^{i2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{(s-\lambda_i)} b_1^{ir_i} + \frac{1}{(s-\lambda_i)^2} b_2^{ir_i} + \dots + \frac{1}{(s-\lambda_i)^{n_{i2}}} b_{n_{ir_i}}^{ir_i} \\ \cdots\cdots \\ \frac{1}{(s-\lambda_i)} b_{n_{ir_i}}^{ir_i} \end{bmatrix}_{n_i \times m}$$

The product directly will give this expression, 1/s - lambda i * Bi1 suffix 1 + 1/s - lambda I square Bi1 suffix 2 etc. So, note that these are all row vectors, okay, Bi1, 1B, i12, Bi1, ni1, so these are all row vectors and the 1/s – lambda i etc., these are functions of s here. So, by adding all these terms, we will get a vector function in s variable. The second vector is also the linear combination of these vectors and it is a function of s.

So, the last one is the vector Bi1 and ni1 vector multiplied by 1/s – lambda i, so this expression, we get from this, from the previous slide, we multiply the last row with the matrix, we get this expression, I think here the correction is this one, there is no power here, the powers will be decreasing. The first term is 1/s – lambda i, the last term here is 1 / s – lambda i power nij.

But so, here the first block will have the last term 1/s – lambda i only multiplied by this row vector. Similarly, the second block if you take, the last row of the second block is given by 1/s – lambda i multiplied by Bi2 suffix ni2 and the last block is 1/s – lambda i multiplied by the row Bi-ri suffix ni ri block. So, this is the matrix which is of this size because this is of the size ni cross ni and this is of the size ni cross m, so this is of the size ni cross m matrix.

Now, we can see that this last rows of each block is made up of the rows of B superfix i matrix. So this matrix we have seen, this B superfix i matrix have this particular rows as given, so we have this matrix, the last rows like this.

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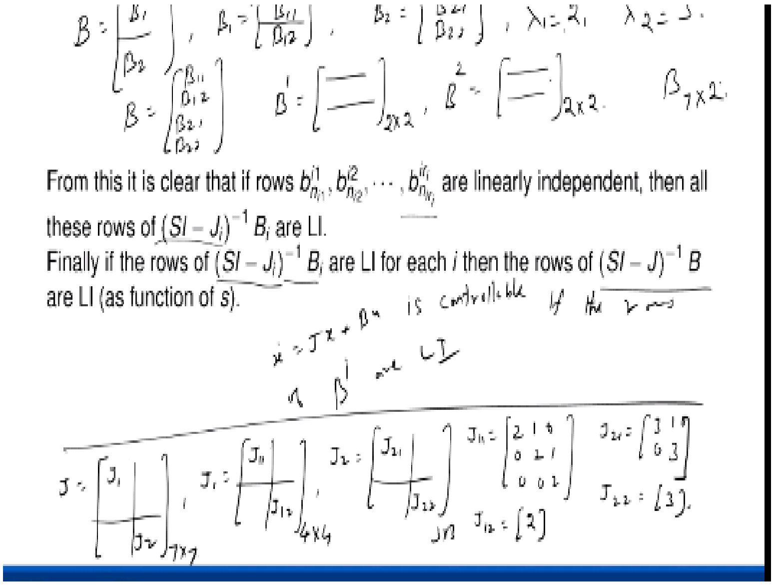

So, if we observe that if these rows bi1 suffix ni1, bi2 suffix ni2 and bi ri suffix ni ri, if they are linearly independent, then all the rows of si – J suffix i inverse * Bi are linearly independent, so this matrix is what is given in the previous, so this is the matrix, so if this rows are linearly independent, then automatically, all the rows of this matrix are linearly independent because we can see that none of the rows can be written as product of any of these rows.

But if any of this rows, the last row of the each block, they become 0, then obviously if any one of the row is 0 in a matrix, that matrix has linearly dependent row, so first thing is they should not be; should be non-zero, these rows, bi1 suffix ni1, bi2 suffix, ni2, so these rows should be first of all non-zero, then if they are also linearly independent, then the entire matrix ni cross m matrix, they are all linearly independent.

And if any of them become 0, then we can see that the entire matrix has 0, the linearly dependent rows and the condition, similarly, if for each i, SI – Ji inverse bi are linearly independent, then the total; the entire matrix SI – J Inverse b are also linearly independent, so that can seen from here, SI – J Inverse B is given by this one, so if the blocks, for each i, the rows of each blocks are linearly independent.

And since the J1, J2 they are all made up of different eigenvalues, the first block is made up of eigenvalue lambda 1 and second block we will have SI – lambda; sorry, SI - J suffix i inverse, so this will contain the element like 1/s – lambda i or its powers + etc. So, here we can see that each block contains different types of terms, like 1 / s – lambda 1 will be coming in the first block, 1/s – lambda 2 will come in the second block and its powers etc.

So, none of them can be linearly dependent on any of this blocks because of the nature of the different eigenvalues, so if separately if they have linearly independent rows, then the total rows; every row is linearly independent. The set of all rows of this matrix are linearly independent, so that is what is given here. For each i, if SI – J suffix i Inverse * Bi are linearly independent, then SI – J Inverse * B also will have linearly independent rows.

And this implies the controllability of the system, so the condition for controllability is given by this matrix, ultimately this B superfix i matrix is an important one, if the rows of this matrices are linearly independent, then these given system is controllable the system x dot = Jx + BU is controllable and if the Jordan system is linearly independent, then the original system will also be linearly independent by the previous theorem.

So here, we have seen the relation between the Jordan canonical form of a system and the system itself, how they are related by the controllability property, so now we have shown that the system is controllable, the system xi; x dot = Jx + Bu is controllable, if the rows of the B superfix i are linearly independent, so this what we have shown. So now, we will see an example in the following way.

So, let us consider the Jordan canonical form like this, J1 and J2, okay, so where J1 is having J11 and J12 and J2 is J21, J22, so in the previous notation as we have seen, so where J11 matrix is 2 1 0 0 2 1 0 0 2 and J12 is simply the 1 cross 1 matrix 2, and J21 is 3 1 0 3 okay J22 is matrix 3, so this total size of the matrix J is 4 6 7, so it is a 7 cross 7 matrix and J1; it is 4 cross 4 matrix and J2 is 3 cross 3 matrix and the corresponding matrices are given here, so we can divide now the matrix B into 4 parts.

It will contain, first it is B1, B2, corresponding to J1 and J2, we have B1, B2 and B1 has 2 blocks, B11, B12 and B2 have 2 blocks; B21, B22. Now, the matrix corresponding to the eigenvalues, so you can easily see that the eigenvalues are lambda 1 is 2 and lambda 2 is 3 and its geometric multiplicity is 2 for each one of them. Algebraic multiplicity of lambda 1 is; it is repeated 4 times.

Algebraic multiplicity is 4, but geometric multiplicity is 2 here and for lambda 2, algebraic multiplicity is 3 and geometric multiplicity is 2, the linearly independent eigenvectors will be there. Now, so, this will have 4 blocks, the matrix B finally has 4 blocks that is B11, B12, B21, B22. So according to the theorem, this theorem statement we have seen there, so this matrix B superfix i, for each eigenvalue lambda i, the last row of these blocks corresponding to that Bi, whatever is the last row of each block, they are linearly independent.

Then the system is controllable, so if we have this following example, this example, for B1, we have 2 blocks, so the last row of each block should be linearly independent, the collection of the last row. So for example, if we take B1, there are 2 blocks, so it will have first row and second row. The first row is the last row of B11 and the second row is last row of B12, similarly, B superfix 2, it means the last row of B21 will be taken.

And the last row of B22 is taken, so it is a matrix with 2 rows, so for example if we take 2 rows with 2 columns, we can say that the 2 rows can be linearly independent, so the theorem is, for each i, B superfix i has linearly independent rows, so minimum row we required here is; number of rows we required is 2 because there are 2 blocks, so the matrix B should be of the size, it should at least have 2 columns.

So, it will be 7 cross 2 at least, so for this example, if we take a matrix B as a column matrix simply, that is 7 cross 1 matrix, then the system will not be controllable, that is clear because of this Jordan canonical forms and the theorem that B superfix i should be having linearly independent rows, so we can see that if you select 2 rows, linearly independent for B superfix 1 and B superfix 2.

We can assure the controllability of the system given by this example with this Jordan canonical form. So this, in this lecture we have seen various results on the controllability of the system and its related Jordan canonical form, so similar result can be extended for the observability of the system also. Thank you.