Dynamical Systems and Control
Prof. N. Sukavanam
Department of Mathematics
Indian Institute of Technology- Roorkee

Lecture – 36
Observability – II

Hello viewers welcome to the lecture on observability. We will see if a system is unobservable then what portion of the state of the system can be observable so this results we will see in this lecture.

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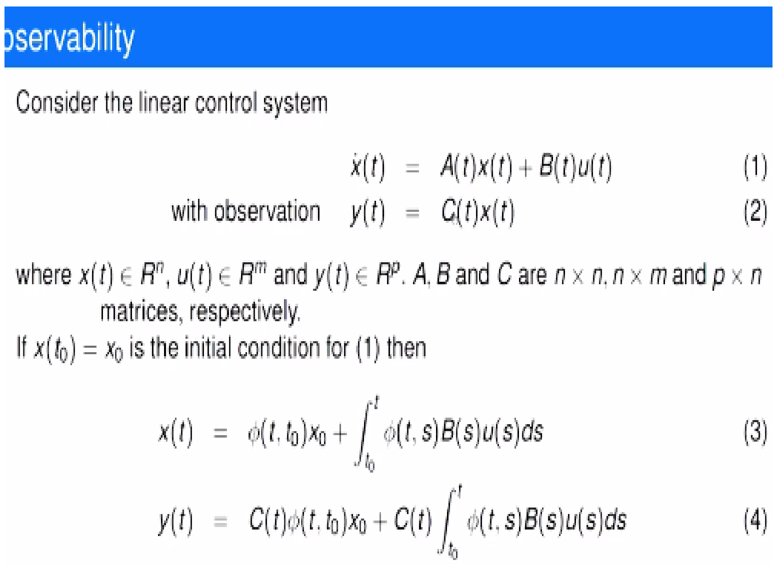

Let us consider a control system dx/dt = A of t x of t + B of t u of t and the observation or measurement of the system is y of t = c * x of t. Here the matrix A is a n x n matrix B is n x m and C is a p x n matrix and the state of the system x belongs to R n. That means the x1 x2 xn are the state of the system and y1 y2yp are the observation of the system. So, now we will see that if x of t0= x0 is the initial condition of the system.

Then the solution of the 1st equation is written as x of t = in terms of the state transition matrix Pi t t0 as given in the equation 3. Then the observation y of t is given by this expression now we assume that the equation 4 is available with us. That is y of t is available from that is it possible to get x of t?

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Definition

Consider the system $\dot{x} = Ax + Bu$; and the observation or measurement of the system $y(t) = Cx(t)$ in a time interval $[t_0, T]$. The system system is said to be observable if the knowledge on the input $u(t)$ and the obervation $y(t)$ for $t \in [t_0, T]$ is sufficient to determine the initial state $x(t_0)$ uniquely.

By obtaining $x(t_0)$ uniquely we can get unique $x(t)$ for all $t \in [t_0, T]$, from equation (3).

Observability Grammian Matrix

$$M(t_0, T) = \int_{t_0}^{T} \phi'(t, t_0) C'(t) C(t) \phi(t, t_0) dt \tag{5}$$

That is the question of observability of the system so actual definition of observability is given here if you consider the system x dot =Ax + Bu the observability of a system is in terms of the grammian matrix observability grammian matrix given by M of t0T= the integral given in the equation 5 in terms of the state transition matrix and the matrix C here.

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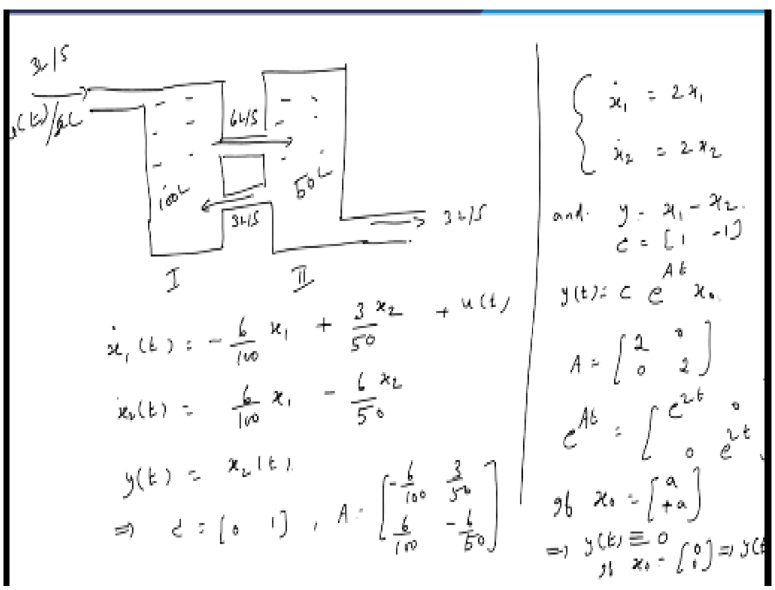

So, let us see for example the mixing problem let us say there are 2 tanks tank1 and tank2 with salt water in both the tanks. Tank1 contains 100 litres of salt water tank2 50 litres and 3 litres per second is entering into the tank1 and 3L per second is leaving tank2 and 6litres per second is entering from tank1 to tank2 and 3 litres from tank2 to tank1 so they are circulating. We have earlier seen that the mathematical model of the amount of salt.

Present in tank1 tank2 can be written like this if x1of t is the amount of salt in tank 1 at time t x2 of t is in tank2 at time t then the dynamical equation is written in this way where u of t is the amount of salt entering per second in the tank1. So, this is a control system the observation here is from tank2 that is y of t is x2 of t, so we are measuring only the salt content of tank2. So, is it possible to get the information about x1 of t that is the question here.

Because the state variables are x1 and x2 the observation is x2 only so is this system observable that means can we get the information about all the variables? We will see that the system is observable and the condition for observability we will see here. And from here itself we can also see that if x2 is known then by substituting in the 2 equations we can get the value of x1 also, so this system is observable.

So in the 2nd example here we have x1 dot =2 x1 x2 dot is2 x2 and the observation y is given by x1–x2 so in this case if you convert it into the matrix form A is 2002 the diagonal matrix and the C is 1 and -1 if you compare it with the equations 1 and 2. So, from here the solution of the problem is given by e to the power At * x0 and multiplying it with C we get the observation y of t = C * e to the power At * x0.

So, we can see that if the initial condition x0 is like a and -a or both the values are same saorry a and a then it is very clear that if you substitute it here then we will get this implies that y of t is identically = 0 in this case because e power At is given by this diagonal matrix multiplied by a and a and multiply with C matrix, we will get the value to be 0 for all t. So, for this initial condition a and a we get the observation to be 0 for all values of t.

But the same observation will be obtained even if x0 is say 0,0 then also we will get y of t identically = 0 if we substitute here. So, for 2 different initial conditions we are getting the same observation. So, through this observation we will not be able to get the information about the initial condition uniquely. That is what is seen from this example, so this example is not observable. This system is not observable so we will see mathematically what condition.

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Definition

Consider the system $\dot{x} = Ax + Bu$; and the observation or measurement of the system $y(t) = Cx(t)$ in a time interval $[t_0, T]$. The system system is said to be observable if the knowledge on the input $u(t)$ and the obervation $y(t)$ for $t \in [t_0, T]$ is sufficient to determine the initial state $x(t_0)$ uniquely.

By obtaining $x(t_0)$ uniquely we can get unique $x(t)$ for all $t \in [t_0, T]$, from equation (3).

Observability Grammian Matrix

$$M(t_0, T) = \int_{t_0}^{T} \phi'(t, t_0)C'(t)C(t)\phi(t, t_0)dt \tag{5}$$

So, set M t0 T is the observability grammian matrix given by this expression in 5

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Theorem

The system (1)-(2) is observable in the time interval $[t_0, T]$ iff the symmetric matrix $M(t_0, T)$ is nonsingular.

So, the theorem is the system 1 2 here the equation 1 and 2 is observable if and only if matrix M is non-singular.

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uality Theorem

Consider the following two systems

$$\left.\begin{aligned} \dot{x} &= Ax + Bu;\ x(t_0) = x_0 \\ y &= Cx \end{aligned}\right\} \qquad (6)$$

and the corresponding dual system

$$\left.\begin{aligned} \dot{x} &= -A'x + C'u;\ x(t_0) = x_0 \\ y &= B'x \end{aligned}\right\} \qquad (7)$$

We know that if $\phi(t, t_0)$ is the state transition matrix corresponding $A(t)$ then $\psi(t, t_0) = \phi'(t_0, t)$ is the state transition matrix for $-A'(t)$.

Now using this result we can prove the following duality result, so this previous result is valid for whether it is time varying system or time invariant system. Now the same thing can be used for a simplified condition will be obtained for time invariant system that is autonomous system.If A and B are constant matrices and x dot=Ax +Bu is the system y =Cx is the observation. And initial condition is given like this.

Then the corresponding dual system is defined in this manner in the place of A we replace it with -A transpose in the place of B replace it with C transpose and C is replaced with B transpose so the given system and the dual system are considered.

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Now if the system (6) is observable then it was shown in the above theorem that

$$\int_{t_0}^{T} \phi'(s, t_0) C' C(s) \phi(s, t_0) ds \qquad (8)$$

is nonsingular.

Now the condition for controllability of (7) is

$$\int_{t_0}^{T} \psi(t_0, s) C'(s) \left(C'(s)\right)' \psi'(t_0, s) ds \qquad (9)$$

must be nonsingular.

But (8) and (9) are one and the same. Thus we get the duality theorem.

Now we can show that if the original system.

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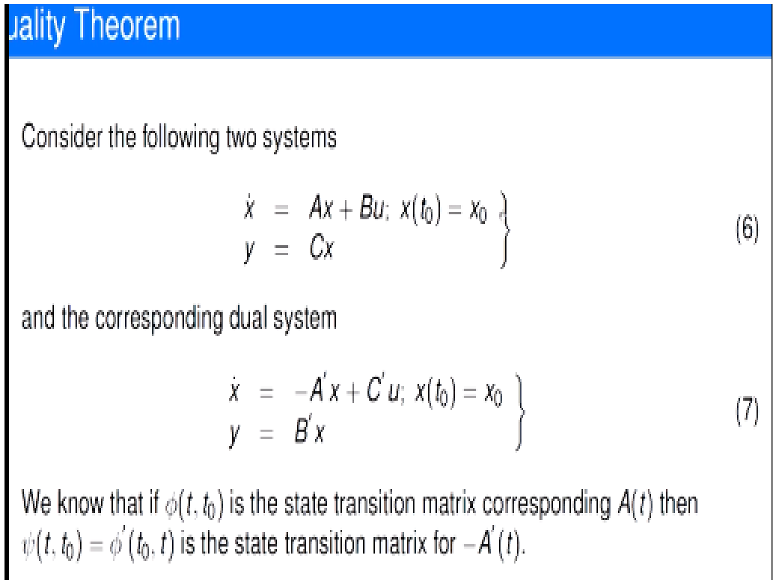

That is the system 6 is controllable then the dual system is observable similarly if the system 6 is observable then system dual system 7 is controllable so we get the duality theorem. That is the system 6 is observable if and only if 7 is controllable and vice versa. So, for proving that we make use of the theorem already proved so let us first consider the system 6 is observable if the system 6 is observable we get the condition M is non singular.

That is integral t 0to T Phi dash C dash C so the system 6 is observable means the integral given in the equation 6 is non singular. The observability grammian matrix is non singular. Now we observe that if Phi t t0 is the state transition matrix for the matrix A then Psi t t0 which= Phi transpose t0 t is the state transition matrix for-A transpose this result was earlier proved while proving the existence theorem for the solution of this type of systems.

So we can recall that result and make use of that one so the system 7 is controllable if this condition is satisfied using the state transition matrix of the equation 7 that is -A transpose using Psi matrix we write the condition to be integral t0 to T Psi of t0 S C dashed C transpose and Psi dashed ds it should be non singular. So, now from this 2 equation 8 and 9 we can see that both of them are the same equation because Psi of t t0is nothing, but Psi dashed of t0 t.

So, if a substitute here we get the same thing so both the conditions are same so the condition 8 is the condition for the observability of system 6 and condition 7 is the condition for controllability of the equation 7. So, both of them are the same this shows that the observability of 6 implies the controllability of equation 7. Similarly, we can prove that the controllability of 6 implies the observability of equation 7.

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Theorem:

System (6) is observable if and only if system (7) is controllable. Similarly system (6) is controllable iff (7) is observable.

Now using Kalman condition for autonomous system the controllability condition for system (7) is

$$rank \begin{bmatrix} C' & A'C' & (A')^2C' & \cdots & (A')^{n-1}C' \end{bmatrix} = n$$

which can be written as

$$rank \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \qquad (10)$$

So, from here we come to this conclusion using the Kalman condition for autonomous systems the controllability of the system 7 can be written as is guaranteed under this condition rank of Cdashed Adashed Cdashed Adashed square Cdashed etc, because the system has -A dashed for the state matrix and C dashed as the control matrix. So, using this 2, we write the Kalman condition now the same thing can be written in the, if you take the transpose of the matrix.

We get this expression. So the condition for observability of the equation 6 the system 6 is given by this expression 10, so this proves the duality theorem through duality theorem we obtain the condition for observability of the system 6 here.

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nobservable Systems

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (10)$$

and the observation

$$y(t) = Cx(t) \quad (11)$$

where A, B and C are respectively $n \times n$, $n \times m$ and $q \times n$ real constants matrices.

So, now we will consider a system which is not observable in case the system is not observable then how much of the state we can actually observe if all the states of the system may not be observable. But some of their state maybe observable depending on the matrices AB C etc so let us consider the system xdot =Ax + Bu and y =Cx with these matrices ABC given by this thing.

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We know that if a system is not observable, it is not possible to establish uniquely the state x of the system from the output y. However some of the state variables can be observed using a canonical form.

The unobservable subspace of linear time-invariant system (10) is the linear subspace consisting of the state x^0 for which $y(t) = 0$ for $t \geq 0$.

For the systems (10) and (11) with $u(t) = 0$, we have

$$\begin{aligned} y(t) &= Ce^{At}x^0 \\ \Rightarrow \int_0^{t_1} e^{A't}C'y(t)dt &= \int_0^{t_1} e^{A't}C'Ce^{At}x^0 dt \\ &= M(0, t_1)x^0 \end{aligned}$$

$M(0, t_1)x^0 = 0$ if x^0 belongs to null space of the observability grammian matrix

If the system is not observable, then it is not possible to establish the uniquely the initial state of the system or it is not possible to get uniquely this entire state of the system. So, how to observe some of the state of the system possible so now we consider the equation y of t = C e to the power At * x0 e to the power At * x0 is the solution of the given system 10 under the condition u of t is identically =0.

See the observability definition says that with the knowledge of y of t and u of t if you are able to find the state of the system so we assume without loss of generality the control u of t is identically equal to 0 if it is not 0.

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Thus from duality theorem we infer that if A and C are constant matrices then (10) is necessary and sufficient for the obsevability of the system (6).

Then we can do in the similar manner as we have done in this case.

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oof: Sufficiency

Let $M(t_0, T)$ be nonsingular. From (4) we get

$$C(t)\phi(t, t_0)x_0 = y(t) - C(t)\int_{t_0}^{t} \phi(t, s)B(s)u(s)ds$$

Calling the R.H.S. as $\overline{y}(t)$ we get

$$\begin{aligned} \overline{y}(t) &= C(t)\phi(t, t_0)x_0 \\ \Longrightarrow \int_{t_0}^{T} \phi'(t, t_0)C'(t)\overline{y}(t)dt &= \left[\int_{t_0}^{T} \phi'(t, t_0)C'(t)C(t)\phi(t, t_0)dt\right] x_0 \\ \Longrightarrow y_0 &= M(t_0, T)x_0. \end{aligned}$$

If $M(t_0, T)$ is nonsingular then x_0 is determined uniquely $\Longrightarrow$ obsevability.

Here in this equation if you have u of t is not 0 then we take that information to the other side and then we can write the equation as y bar of t is C of t * Phi of t t0*x0. So, if A is a constant matrix Phi of t t0 is nothing but e to the power A t - t0. So, ultimately, we get so if u is not identically not = 0 then from this equation we take the integral the values to the left side. Finally we get y bar of t = C of t Phi of t t0x0.

So, if A is a constant matrix Phi of t t0 is nothing but e to the power At - t0 so that is what we are getting here.

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We know that if a system is not observable, it is not possible to establish uniquely the state x of the system from the output y. However some of the state variables can be observed using a canonical form.

The unobservable subspace of linear time-invariant system (10) is the linear subspace consisting of the state x^0 for which $y(t) = 0$ for $t \geq 0$.

For the systems (10) and (11) with $u(t) = 0$, we have

$$
\begin{aligned}
y(t) &= Ce^{At}x^0 \\
\Rightarrow \int_0^{t_1} e^{A't}C'y(t)dt &= \int_0^{t_1} e^{A't}C'Ce^{At}x^0 dt \\
&= M(0,t_1)x^0
\end{aligned}
$$

$M(0,t_1)x^0 = 0$ if x^0 belongs to null space of the observability grammian matrix

So, here without loss of generality we are assuming that u of t is identically = 0 so with that we get y of t = C e to the power e At * x0 we are assuming that the initial time is also = 0. So, now if you multiply both sides with e power A dash t * C dash and then integrate from 0 to t1. The final and initial time is 0 and the final time is 1 so we get the expression in this form where M 0 t1 is the observability Grammian matrix x0.

So, here if x0 is in the null space of the matrix M 0 t1 because the system is not observable therefore the matrix M is not invertible. So, the null space of the matrix M has some non 0 dimension, so we get M 0 t1 x0 = 0 if x0 is the null space so in other words we say that the null space of the matrix M are the unobservable states of the system and the range of the matrix M is the observable state of the system. So, instead of using the matrix M we can also use the matrix.

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$$V \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Thus the unobservable subspace of systems (10) and (11) is the null space of its observability matrix V and the observable space of this system is the range space of V.

V = C CA CA n-1 which we have seen earlier if the range of this matrix has dimension let us say P then the P dimensional matrix the subspace of rn is the observable subspace of the system.

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Observability Canonical Form

Theorem: Consider the n-dimensional linear time-invariant system (10) with (11). If the observability matrix of this system has rank $r(< n)$, then there exists an equivalent transformation $\hat{x} = Px$, where P is a constant nonsingular matrix, which transforms (10) into observability canonical form

$$\left.\begin{aligned} \dot{\hat{x}}_1(t) &= \hat{A}_{11}\hat{x}_1(t) + \hat{B}_1 u(t) \\ \dot{\hat{x}}_2(t) &= \hat{A}_{21}\hat{x}_1(t) + \hat{A}_{22}\hat{x}_2(t) + \hat{B}_2 u(t) \end{aligned}\right\} \qquad (12)$$

$$y(t) = \hat{C}_1\hat{x}_1(t) \qquad (13)$$

So, now we can see that if the system is not observable then we can convert the given system into a canonical form like this and one of the portion of the equation 12 is observable. And the 2nd portion is unobservable so if the state space is n dimension then we can split it into two parts a P dimensional. So, if the observability matrix has the rank r here then the r dimensional subspace will be the observable space.

And the reminding n- r dimensional subspace is the unobservable space here so here the 1st system x1 dot x1 bar dot = A bar A11 bar x1 bar + B1 bar u of t. That system will be observable and the second system x 2 bar dot etc., given in this equation is unobservable system and the dimension of x1 it belongs to a subspace of dimension r here. Because the rank of the observability matrix is r here, so the observation is given by y of t = C1 bar x1 bar. So, how this canonical form is obtained we will see in this following.

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and the *m*-dimensional subsystem

$$\dot{\hat{x}}_1(t) = \hat{A}_{11}\hat{x}_1(t) + \hat{B}_1 u(t) \tag{14}$$

$$y(t) = \hat{C}_1\hat{x}_1(t) \tag{15}$$

is observable.

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Proof: the observability matrix of (10) is

$$V = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

V has rank r, i.e., V possesses r linearly independent rows which span the observable subspace of (1). Let the row vectors $p_1, p_2 \cdots, p_r$ be a basis for this subspace. Furthermore let $p_{r+1}, p_{r+2}, \cdots, p_n$ be $(n-r)$ linearly independent row vectors which together with $p_1, p_2 \cdots, p_r$ span the whole n-dimensional space.

So, consider the observability matrix V = C CA CA power n-1 and we assume that the rank of this matrix is r which is strictly < n here so we can select r linearly independent rows from the matrix v. So, let us assume that p1 p2 pr they are the linearly independent rows of the matrix. The remaining rows will be dependent on these r rows p1 p2 pr. So the total of them the n such vectors which we collect here it forms a basis of n dimensional space.

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Define a nonsingular matrix

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$\text{where} \quad P_1 = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_r \end{bmatrix}; P_2 = \begin{bmatrix} p_{r+1} \\ p_{r+2} \\ \vdots \\ p_n \end{bmatrix}$$

Let $P^{-1} = Q = [Q_1 \quad Q_2]$

So, we define a matrix P as P1 P2 where P1 contains r rows p1 p2 pr and P2 contains n-r rows here and the inverse of this P is a non singular all the rows are linearly independent by the selection has given here. The inverse will exist so let us assume that P inverse is Q which is by Q1 block and Q2 block so Q1 contains r columns and Q2 contains n-r columns now P P inverse is identity.

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then

$$PP^{-1} = \begin{bmatrix} P_1Q_1 & P_1Q_2 \\ P_2Q_1 & P_2Q_2 \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\Rightarrow P_1Q_2 = 0,$$

$$P_1AQ_2 = 0$$

$$\text{and} \quad CQ_2 = 0$$

Therefore we will get if you multiply P * Q we get P1Q1 this one P1* Q1 P1 Q2 P2Q1 etc these are the blocks P1Q1 is a r cross r of order r and P2 Q2 is a a matrix of order n – r the remaining matrices are rectangular matrices so comparing that we get a and it should be identity matrix. We get the 1st block I is r cross r I matrix and the second I denotes n-r cross n-r matrix identity matrix.

And by comparing this we get the expression P1Q2 should be = 0 matrix and P1 it P1 contains the rows of C CA etc they are selected from this matrix only, So, P1Q2 0 we also get P1A *Q2 should be = 0 because this P1 are selected from this matrix and P1A also will be selected from this matrix we are just multiplying with A only similarly C*Q2 also should be 0 because P1 rows of C also so some of the row linearly independent rows of C are available in P1.

And that is orthogonal to Q2 matrix C*Q2 also should be = 0 so from this we are getting the information of this 3 equations now making use of that 1 let us make a transformation. X is our original state variable so Px is x bar. Let us take this transformation if you substitute this in the given equation 10 and 11 so x dot is given by p inverse x bar dot So wherever x is there to be substitute p inverse x bar.

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We now introduce a transformed state vector defined by

$$\hat{x}(t) = Px(t)$$

Substituting this into (10) and (11), we get

$$\dot{\hat{x}}(t) = PAP^{-1}\hat{x}(t) + PBu(t) \qquad (16)$$

$$y(t) = CP^{-1}\hat{x}(t) \qquad (17)$$

Using this transformation and then taking the p inverse to the right hand side finally we will get this equation x bar dot = PAP inverse x bar + PB *u because P inverse was there in the left side.

When it is taken to the right side there is a P coming in the 2nd term P*Bu and when y of t = C* x and x is replaced by P inverse x bar, so we get the equation like this now PAP inverse.

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It can be easily prove that

$$PAP^{-1} = \begin{bmatrix} P_1AQ_1 & 0 \\ P_2AQ_1 & P_2AQ_2 \end{bmatrix}; CP^{-1} = \begin{bmatrix} CQ_1 & 0 \end{bmatrix}$$

Q_1 has r and Q_2 has $(n-r)$ columns;

The second part of the theorem can also be proved easily.

It should also be noted that the observability canonical form is not unique.

If you calculate we will get P inverse is Q so if we substitute that 1 here, we get PA * Q is given by this expression and we can also see that this is PA * Q2 this expression PA * Q2 is 0 so because of that this block will become 0. Similarly C* Q2 is 0 therefore we get this expression. C * P inverse here is given by these two blocks 1st Block is CQ1 non 0 columns. And then CQ2 is completely 0 columns so we get this expression and from here.

We get the result of the theorem so if you substitute PAP inverse to be this matrix and if you call this as A11 and this is 0 the 2nd block the blocks in the second row is A21 and A22 and here we call it as this C1 0 matrix, so we get their statement of the theorem. We get the expression this 1 if we take x bar to be split into 2 parts the x1 bar and x2 two we get A11 x1 bar + B 1 u and similarly x2 bar is given by the 2 values A22 x1 bar+ A21 x1 bar + A22 x2 bar + B2 u.

That is from this expression. And then y of t is given by CQ2 x1 bar and the 2 nd term is 0*x2 bar so we get the statement of the result as it is here, and we can easily see that the 1st equation and y = C bar x1 bar this 2 system will be the observable system because of the rank condition which we can easily verify. So, from the unobservable system we can see that some states of the system is observable.

And the r states are observable and n – states are not observable because of the rank condition of the system. So, in this lecture we have seen various results about observability and some unobservable system and how to get the information about sum of the observable states of the system. Thank you.