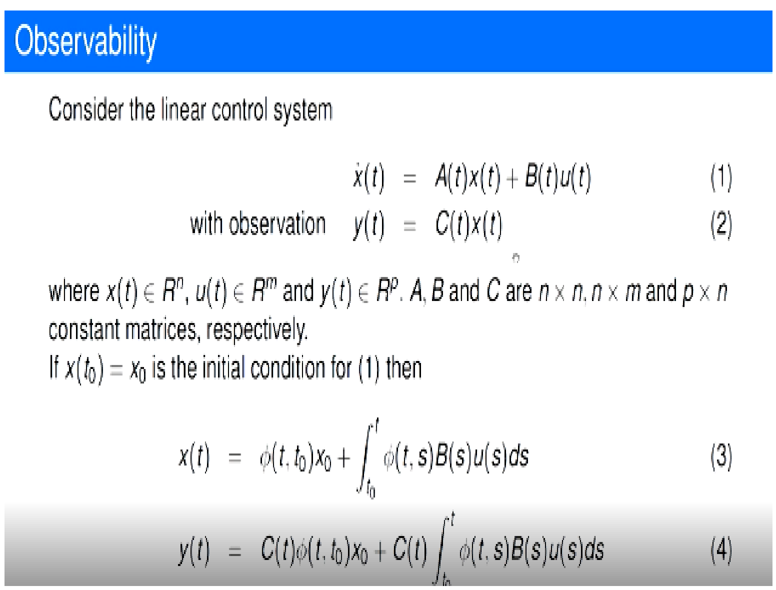
Dynamical Systems and Control
Prof. N. Sukavanam
Department of Mathematics
Indian Institute of Technology- Roorkee

Lecture – 35
Observability - I

Hello viewers welcome to the lecture on observability of linear systems. So, in this lecture we will see the definition of observability and a result a theorem which gives the condition for the observability of the system. So, the result which we will see is suitable for both time varying system and time invariant system. So, consider the linear control system X.=At xt+Bt ut and observation y of t= Ct*xt.

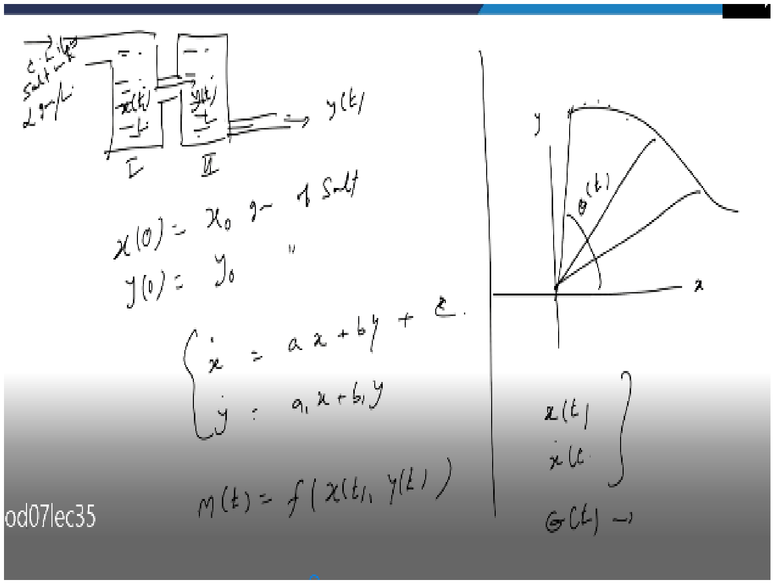
So, in many practical problems we may be knowing the dynamics of the equation which is the first equation the dynamics of the system may be known but it may not be possible to measure all the outputs of the system directly. That is x of t is the output here so if it is not possible to get the x of t value directly but we may be able to get some information in the form of some observations. So, which is a function of the x of t.

(Refer Slide Time: 01:53)



So, in general we will have the system dynamics this is the first equation and it is controlled by a controller you u of t and there will be measurement equation which is given by the second one y of t= Ct xt.

(Refer Slide Time: 02:13)



So, for example in certain cases let us consider so if it is consider a two dimensional space in which an object is moving in the plane. So, if we observe only the angle from the x axis of the; at every instant of time if you measure the angle theta as a function of time not actually the position and velocity of the object with is moving. So, the output of the system that is x of t consists of two values that is the position x of t and the velocity x. of t.

But the measurement is only the theta of t as a function of t. so is it available give the actual value of xt and x. of t by making this observation is the question here. In every system for example if you consider two tanks let us say tank 1 and tank 2. So, which consist of some liquid let us say water with salt and there is an inflow of water with salt water and it is from tank 1 there is an inflow to the tank 2 and tank 2 there is an outflow of the mixed water.

So, in this case if we assume that x of t represent the amount of salt in tank 1 and y of t represent the amount of salt in tank 2 and the capacity let us assume that L is the capacity of both the tanks. So, as the mixed water flows here for example let us say sum C litres of salt water it is flowing inside which consist of some amount of salt. Let us say a fixed amount of salt alpha grams per litre this is mixed in this water and it is mixing in the first tank.

And the first tank is already having for example let us say at time t=0 it has x0 grams of salt is available and y of 0 it contains y0 gram of salt at time t=0 and when the flow starts the inflow and outflow starts the amount of salt keep on changing within the two tanks. Now the equation of these dynamics can be written in the form of the equation x.=sum ax+by+some constant.

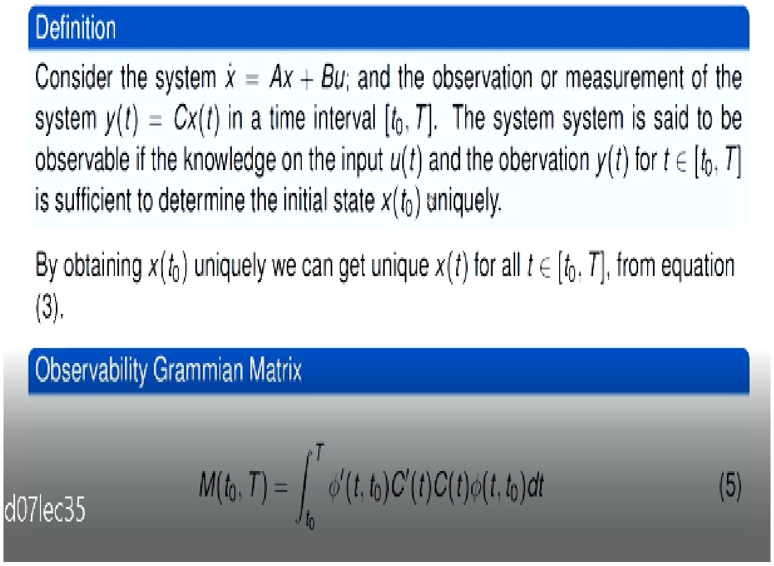
And y. can be written as a1x+b1y and depending on the inflow and outflow now if the measurement let us say m of t that is a function of this x of t and y of t that the measurement which where we are taking the measurement but that maybe we can measure either let us say x of t in the outflow because there is only one option here the flow is coming out of the second tank. So, y of t can be measured all the time.

And by measuring that can we get the information about the state of the system x of t and y of t. So, here the state of the system is xt yt and the measurement is only y of t. So, will it give the information about x of t? That is the question is this system observable? Similarly here the position and velocity is the actual requirement but we are measuring the angle theta at each instant of time.

So, this is the observation which will be the function of the position and velocity and is it a good observation. By observing this will we be able to get the position and velocity of the object at each instant of time? So these are the questions in the observability problem so when we have a system 1 and its observation 2 is it observable in the sense that through this observation will we be able to get x of t for all the time?.

So, now we can write the solution of the equation in this particular form provided the initial condition is given we can write the solution in the standard form. Where phi is the state transition matrix then y of t in terms of x of t so we multiply this first equation 3 with C of t we get the equation 4.

(Refer Slide Time: 08:29)

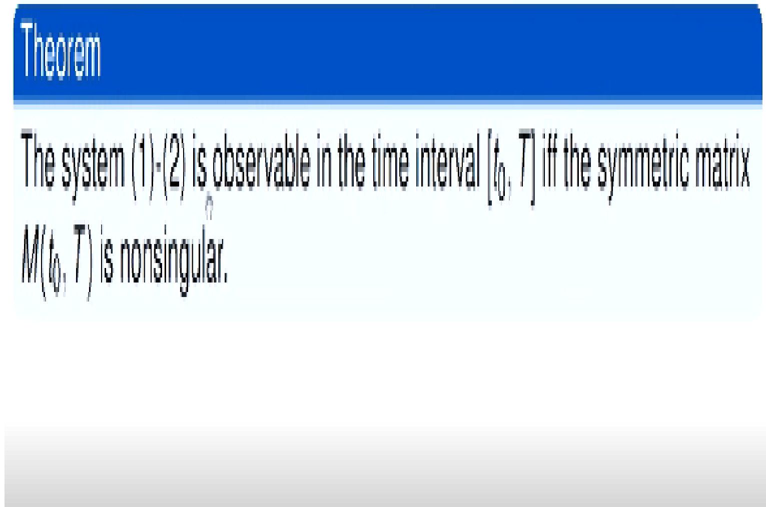


So, the definition of observability is the following the system equation is given x dot =Ax+Bu and the observation is given y of t = C times x of t and these are considered in the time interval t0 to capital T. Then the system is set to be observable if the knowledge of the input u of t and the observation y of t in the time interval t0 to capital T is sufficient to observe the initial state x t0 uniquely.

So, if you are able to get the initial state uniquely then the system is set to be observable why we put emphasis on the initial state only. Because if we know the initial state uniquely then from equation 3 we can get all the states in a unique manner. Because the entire solution is given by the equation 3 so the definition includes only the initial state but it indirectly implies that by this observation y of t we should be able to get the x of t uniquely.

Then system is said to be observable so here as in the case of controllability Grammian here we have the observability Grammian matrix defined by M of t0, T= integral t0 to capital T and phi dash t t0 C dash t C of t phi of t t0 dt. So, this is a matrix n*n matrix again.

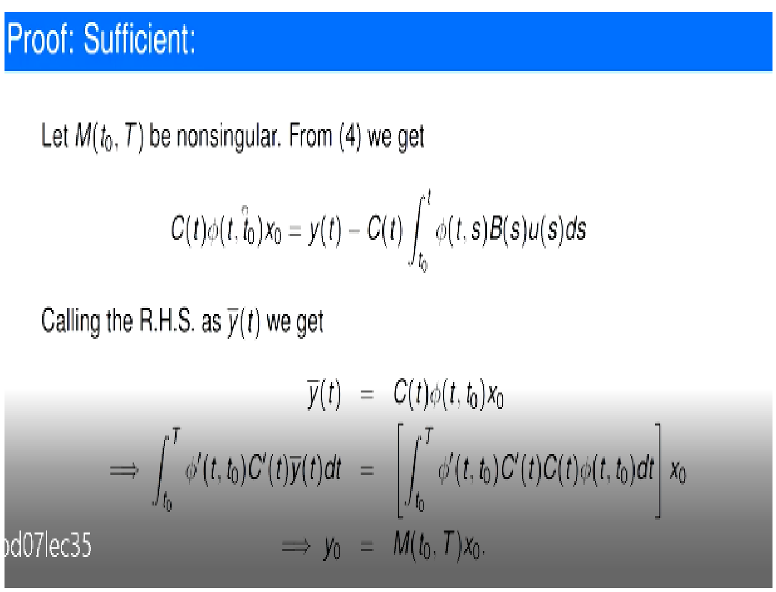
(Refer Slide Time: 10:15)



So, the system 1 and 2 is said to be observable in the interval t0 to T if and only if the symmetric matrix is non-singular. So, it is similar to the controllability theorem here the observability grammarian should be non-singular later we will see the relation between the controllability and observability. A system can be controllable but it may not be observable vice versa and there is a relation for the given system and the dual system.

So, that we will see some kind of relation between the controllability and observability of the system and this dual system in the coming lecture.

(Refer Slide Time: 10:58)

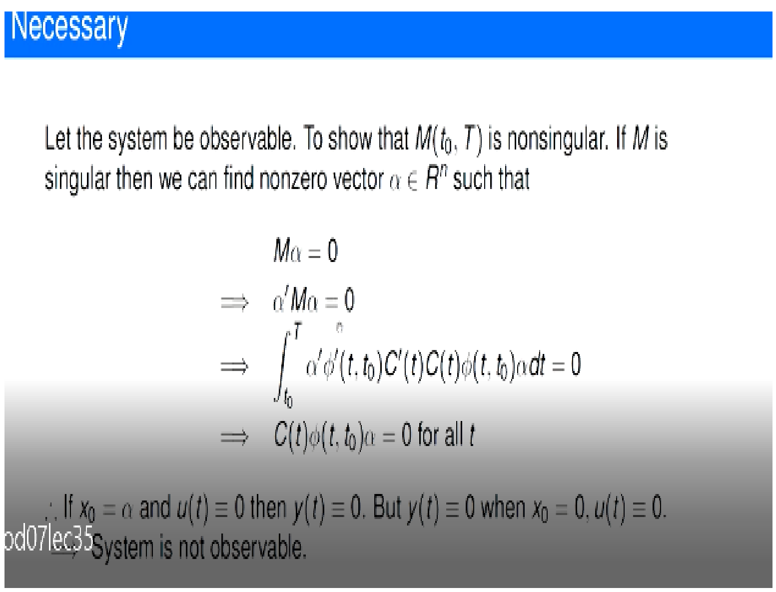


So, the proof of this is if this is if the matrix M is non-singular then we have to prove that the system is observable so from the equation 4 the observation y of t is written in this form y of t that is C of t *this etc. So, the arrangement is given from this equation we have taken this second term to the left hand side so we got the equation like this. Now calling this right hand side to be y bar because we stated that we know all the entries in the right hand side here.

Y of t is observation u of t is the control so these are known to us therefore the entire expression in the right hand side is given. We call it as y bar of t which is =Ct phi t t0*x0. Now simply multiply both sides by phi dash* C dash and integrating from t0 to capital T we will get the control observability Grammian matrix multiplied by x0. So, if you call this left hand side it is a constant vector as y0 and this Grammian matrix multiplied by x0 is this one.

Now it is obvious that if m is say non-singular the inverse exist therefore x0 can be obtained uniquely as M inverse*y0. So, that is there it shows that if M is non singular then the initial condition x0 can be obtained in a unique way therefore the entire dynamics of the system can be obtained uniquely. So, the system is observable.

(Refer Slide Time: 13:04)



Now the necessary condition if the system is observable then we have to show that the matrix M is non-singular. So, we will prove it by contradiction if M is singular then we can find a vector alpha in Rn so that M alpha=0 and multiplying both sides by alpha dash we get this substituting the expression of M we get alpha dash phi dash C dash then C phi alpha dt=0 the integral. So, this from this if you combine the first 3.

And the last 3 terms which are the transpose to each other that implies that the integral t0 to T norm of this expression C*phi of t t0*alpha norm square=0. So, if because the positive entry has the integral =to 0 it implies that the integrant itself should be 0 for all t. So, ultimately we get C of t*phi of t t0 alpha =0 for all values of t. So, now if we assume that the initial condition is alpha where alpha is the non 0 vector which we have seen earlier.

And the control u of t is identically = 0 then we can get y of t is identically = 0 because here if you see the equation 4 in this if you substitute x0= alpha and u of s=0 then we get y of t=C times phi of t t0*alpha. So, but C of t*phi t t0*alpha is =0 for all t that has been proved just now. So, it implies that y of t is identically = 0. So, this is one particular observation y of t is identically = 0 even if the initial condition is 0.

And you have these identically =0 that is in the equation 4 you substitute x0=0 and u of s is 0 then also we will get y of t is identically = 0 for all t. So, this gives two possibilities for the initial condition one initial condition is alpha. Another initial condition is 0 for the same observation y of t=0. So, the initial condition is not identified uniquely therefore the system is not observable. So, this is possible because the assumption that M is singular matrix that is why it happens. So, if M is non-singular then the system is observable system.

(Refer Slide Time: 16:21)

If $M(t_0, T)$ is nonsingular then x_0 is determined uniquely $\implies$ obsevability.

So, to summarize this one we have seen the definition of observability and a theorem which tell us about the observability condition for the autonomous and non-autonomous systems. So, here the same result can be applied for non-autonomous system also. Where phi of ts is replaced by e to the power a*t-s and all the expressions are the same for the time invariant system. So, in the next lecture we will see some examples of controllable and observable systems.

And some computation of the control some results about the feedback contro1 of the linear systems. Thank you.