Dynamical Systems and Control
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Lecture – 34
Controllability of Nonautonomous Systems

Hello viewers, welcome to the lecture on controllability of time varying linear systems this is the continuation of the last lecture. In the previous lecture, we have seen some results on the controllability of time invariant system that is autonomous system if you recall some results of the previous lecture.

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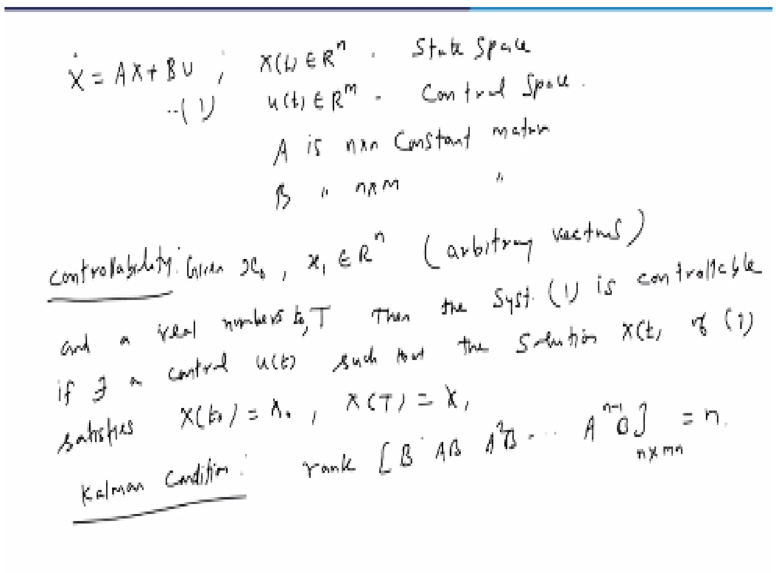

So, we consider the system x dot = Ax + Bu where x of t belongs to Rn and u of t belongs to Rm, so which is called the state space and the control space. So, x of t belongs to Rn for each t as a constant vector so where A is n cross n constant matrix and B is n cross m constant matrix. For this system we have defined the controllability roughly means that given any vector x0 and x1 belongs to Rn.

Arbitrary vectors it is given and a constant time T and a real number T which is the final time of the system then the system is said to be controllable. If you call this system 1 if there exist a control u of t such that the solution x of t of 1 satisfies x of t0 = x0 and x of T = x1. So, here given real numbers t0 and t1 some initial time and final time is given, and 2 arbitrary vectors are given x0 x1.

If you are able to find a control u of t so that the solution has x0 as the initial condition and x1 as the final condition so if it happens then we say that the system is controllable, so we have proved that the system is controllable if and only if rank of the matrix B AB A square B etc. A power n-1 B if it is = n the size of the matrix is n cross m*n. This is for the constant matrices or the autonomous system.

Now if the matrices are time varying then this rank condition may not work even if the rank of let us say A and B are function of t then if you replace it with B of t At * Bt etc. If the rank = n for each value of t then also the system may not be controllable so that we will see that we need a different condition for the controllability of the autonomous system.

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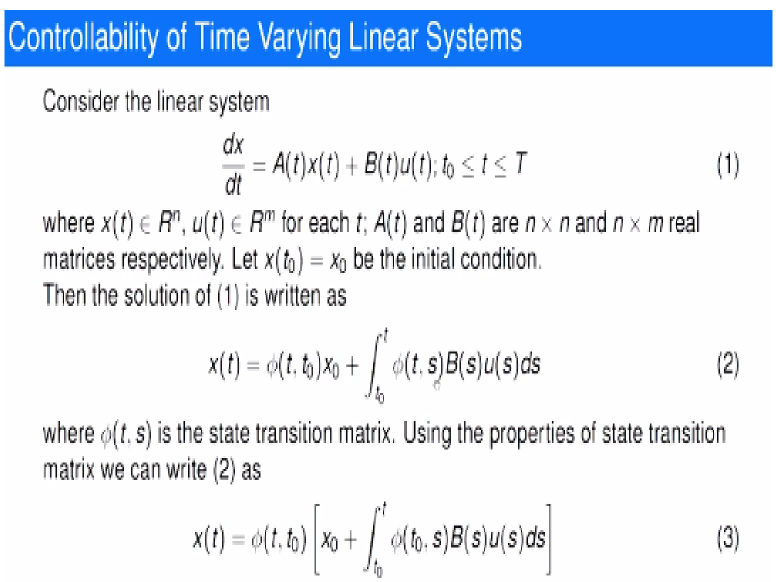

Controllability of the time varying system or the nonautonomous system consider the linear system dx by dt = At xt + But the time intervals t0 to T then it is well known that the solution of the system can be written in the form x of t = Phi of t t0 * x0 + integral t0 to t Phi ts Bs us ds. This is a non homogeneous system of differential equation, so the solution is written in this particular form where Phi of t t0 is the state transition matrix for the time varying matrix A of t.

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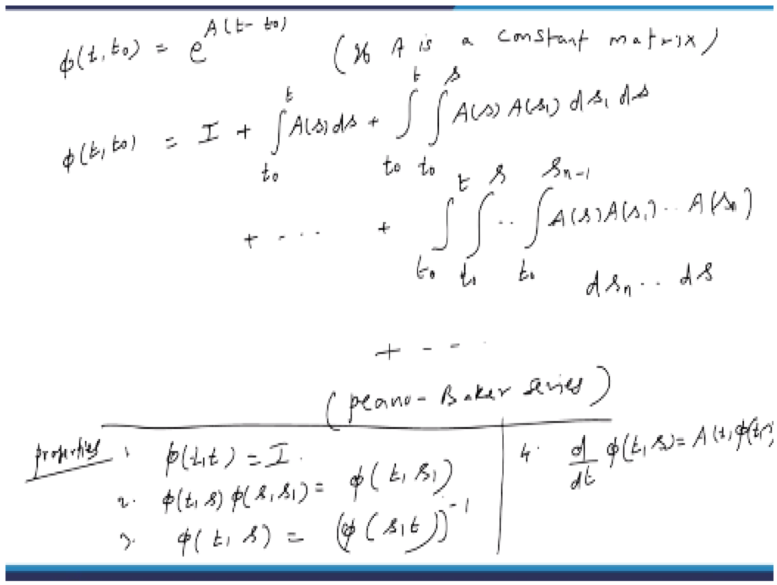

So, if you recall the state transition matrix Phi t t0 it is e to the power A t – t0 if A is a constant matrix so this is known and in case it is a time varying if A is a function of t then the state transition matrix Phi t t0 is written as t0 to t As ds + t0 to t and t0 to s As As1 ds1 ds + etc. So, it is a infinite series we will have the infinite series t0 to t t0 to s etc. t0 s n-1As + etc so this infinite series represent the state transition matrix which is called the Peano Baker series.

Now we can easily see that if A is a constant then it reduces to the value e to the power A t-t0. So, using this state transition matrix we will be able to write the solution of the first equation in the form of the second equation. Now we know certain properties of the state transition matrix so if we again recall the properties of the Phi matrix so if you take the same variable Phi of t comma t is the identity matrix phi of t s and phi of s s1.

If we take variables like this then we get phi of t s1 and phi of t s is Phi of s t inverse and d by dt of Phi of t s that is = A of t * Phi of ts. So, these are the 4 properties of the state transition matrix, so it is clear that the state transition matrix is always invertible matrix. So, using the property we can write equation 2 in the form of equation 3 Phi of t t0 can be taken out and then we can write it as x0 * because Phi t t0 *P phi t0 s will be Phi t s.

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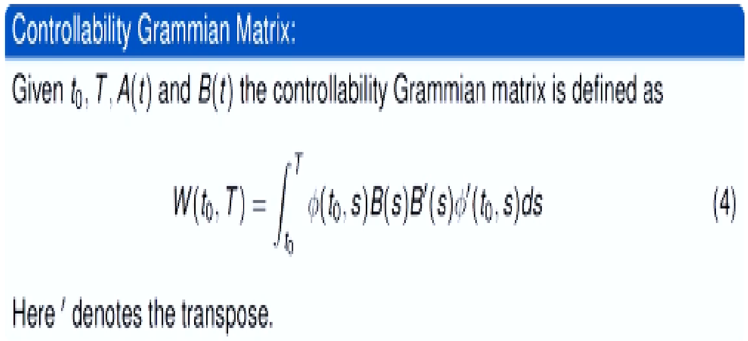

So, we can write it in this particular expression so now we will come to the result that is when the nonautonomous system 1 is controllable so first we define the controllability grammian matrix which is defined as for the given values t0 T and this 2 matrices we can write the expression W t T = integral t0 to T Phi of t0 s B s B transpose s Phi transpose t0 s ds. So, this is a n cross n matrix because Phi is a n cross n matrix B is n cross m.

And all this product will give a n cross n matrix and after integrating each element of the n X n matrix and putting the limits we will get a constant matrix W t0 T that is called the controllability grammian matrix. So we will show that the system 1 is controllable if and only if the controllability grammian matrix is non singular. You can also note that it is a symmetric matrix because Phi * B and Phi B transpose these are multiplied therefore it is a symmetric matrix.

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Theorem :

The system (1) is controllable if and only if the $n \times n$ symmetric controllability Grammian matrix W is nonsingular.

And if it is nonsingular then it is system is controllable. The proof is like this

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Proof: Sufficiency

Let x_0 and x_1 be arbitrary vectors in R^n and let $x(t_0) = x_0$ be the initial condition. Then the solution $x(t)$ at final time T is given by from (3)

$$x(T) = \phi(T, t_0)\left[x_0 + \int_{t_0}^{T} \phi(t_0, s)B(s)u(s)ds\right]$$

$$\therefore \phi(t_0, T)x(T) - x_0 = \int_{t_0}^{T} \phi(t_0, s)B(s)u(s)ds \quad (5)$$

If W is invertible then we can find a control $u(s)$ as

$$u(s) = B'(s)\phi'(t_0, s)W^{-1}(\phi(t_0, T)x_1 - x_0) \quad (6)$$

Which will give $x(T) = x_1$. Hence is (1) controllable.

The initial condition is x t0 = x0 and the solution is written in the form. Earlier slide we have written the solution in the form of 3 now if you substitute t = T we will get the final position of the system. So, x of T is written in the form given here now if you apply the property of phi because phi is invertible. This can be taken to the other side we get Phi t0 T * x of T and take this x0 to the left hand side.

We will get the equation 5 now by observing this term Phi t0 s Bs us ds if you substitute a suitable function u of s vector function. And if it is satisfying the left hand side and right hand side suitably then we can say that this control will steer the system from x0 to x1. So, what we will do is we will just by guess work may be we can say that or by observation we can substitute this particular expression because we have this grammian matrix.

Phi B transpose phi transpose so we will bring that particular form here so u of s can be substituted as B transpose s Phi transpose s if you put that will after doing the integration it will become W. Then we can cancel the W by multiplying with a W inverse so after multiplying here. We will balance it with this vector Phi of t0 T * x1 - x0 if you write it so if you substitute this u of s in this place what we will obtain is finally Phi of t0 T * x1 – x0.

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$$\phi(t_0,T)x(T) - x_0 = \phi(t_0,T)x_1 - x_0$$

$$\Rightarrow x(T) = x_1$$

So, we will get phi of t0 T * x of T – x0 so if you substitute this control B dash transpose Phi transpose to this expression so what we will obtain is so we will get the value Phi t0 T * x1 – x0 So, this automatically will imply that x of T = x1 so which is the required 1 so we can conclude that this control given in equation 6 will take the system from the initial condition x0 to the final condition x1.

So, the system is controllable so we can see that this is not the only possible control which will do this work there may be several other controls available and what we require is at least showing one control for the controllability of the system

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Necessity

Let the system (1) be controllable. To show that W is non singular. If W is singular then there exists a non zero vector α such that $W\alpha = 0$

$$\begin{aligned}
&\Longrightarrow \quad \alpha' W \alpha = 0 \\
&\Longrightarrow \quad \int_{t_0}^{T} \alpha' \phi(t_0, s) B(s) B'(s) \phi'(t_0, s) \alpha \, ds = 0 \\
&\Longrightarrow \quad \int_{t_0}^{T} \|B'(s)\phi'(t_0, s)\alpha\|^2 ds = 0 \\
&\Longrightarrow \quad \alpha' \phi(t_0, s) B(s) = 0 \ \text{ for all } s \qquad (7)
\end{aligned}$$

So, that is the sufficient condition if the W is invertible, we are able to write like this we have shown that if W is invertible the system is controllable. Now we will show that if the system is controllable then the W is nonsingular or the invertible matrix. So, let the system 1 be controllable but the matrix W is singular that is if you assume that it is singular then we will arrive at a contradiction.

So, if a square matrix is singular then we will obtain a non 0 vector alpha so that W alpha = 0 where alpha is in the state space Rn now if you multiply both sides with alpha transpose, we get alpha transpose W alpha = 0 now substituting the expression for W we will get integral t0 to T alpha dash the remaining is the definition of W Phi t0 s Bs B transpose s Phi dash t comma s and then again alpha * ds.

So, this first 3 products and the next 3 products they can be written as the transpose of each other so ultimately, we will get the value is nothing but B transpose Phi transpose t0 s * this expression.

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$$\int_{t_0}^{T} \left(\alpha' \phi(t_0,s) B(s)\right)\left(B'(s)\, \phi'(t_0,s)\, \alpha\right) ds = 0$$

$$= \int_{t_0}^{T} \| B'(s)\, \phi'(t_0,s)\, \alpha \|^2 \, ds = 0$$

$$\Rightarrow B'(s)\, \phi'(t_0,s)\, \alpha = 0$$

$$\alpha' \, \phi(t_0,s)\, B(s) = 0 \text{ for all}$$

So, that is nothing but integral t0 to T and we have this expression as alpha transpose Phi t0 s Bs and then B transpose S Phi transpose t0 s * alpha ds so this expression can be written as this multiplied by this 1.They are transpose of each other so we can write it as norm of B transpose s Phi transpose t0 s * alpha square ds so if this = 0 that is what we have obtained so if the positive function and if the integral of the positive function is 0.

In this interval t0 to T this implies that the positive function which is available inside is itself 0 or its transpose is also = 0 so what we will get is alpha transpose phi t0 s B s = 0 for all s value so that is what we have so this expression is 0.

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As the system is controllable for $x(t_0) = 0$, $x(T) = \phi(t_0, T)\alpha$ we can find a control $u(t)$ such that (from (5))

$$\alpha = \int_{t_0}^{T} \phi(t_0, s) B(s) u(s) ds$$

Multiplying both sides by α' and using (7) we get $\alpha = 0$, a contradiction.

$\therefore$ W is non singular.

So, from here we can Show that the system is not controllable in this case see first we have assumed that the system is controllable, and W is singular, and we arrive at this expression now we will show that it will give a contradiction to the controllability of the system. Now we assume that the initial condition is x t0 = 0 and the final condition is x of T is Phi of T t0 we assume that x of T = phi of T t0 * alpha and x0 = 0.

For this initial condition or this final condition there will be a control available because the system is controllable by assumption so if you substitute this x0 = 0 and x of T = Phi of T t0 * alpha we will get in the left hand side only alpha value and in the right hand side we will get this expression so ultimately, we will get this as the equation alpha = integral t0 to T etc. Now multiplying both sides with alpha dash.

We will get alpha dash * alpha = 0 because of this equation 7 here and hence we will get alpha = 0 which is a contradiction to the fact that this W is singular therefore W should be non singular for the controllability of the system.

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Example

Let $t_0 = 0$ and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then

$$e^{At} = \begin{bmatrix} cost & sint \\ -sint & cost \end{bmatrix} = \phi(t, 0)$$

and

$$W(0, T) = \int_0^T \phi(0, s) B B' \phi'(0, s) ds$$

$$= \begin{bmatrix} \frac{1}{2}(T - \frac{1}{2}sin2T) & \frac{1}{4}(1 - cos2T) \\ \frac{1}{4}(1 - cas2T) & \frac{1}{2}(T + \frac{1}{2}sin2T) \end{bmatrix}$$

So, this result is very useful in computation actually we can make use of this result. That is equation 6 for actual computation of a control which will take the system from the initial condition x0 t to the final condition x1.

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As the system is controllable for $x(t_0) = 0$, $x(T) = \phi(t_0, T)\alpha$ we can find a control $u(t)$ such that (from (5))

$$\alpha = \int_{t_0}^{T} \phi(t_0, s)B(s)u(s)ds$$

Multiplying both sides by α' and using (7) we get $\alpha = 0$, a contradiction.
$\therefore$ W is non singular.

S, for example if you take the matrix A to B 0 to 1 -1 0 and B to B 0 1 it is a constant matrix so this result, we note that it is suitable for both the time varying as well as time invariant system so for the purpose of simplicity of calculation. We take the constant matrices so A is given like this we can easily calculate e to the power At in this space which is = Phi of t 0 and the controllability grammian matrix W 0 T is given by this expression after substituting these values. Phi values and B etc we get the value of the controllability grammian matrix.

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Then a control $u(t)$ which steers the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

from any initial state $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = x_0$ to the final state $\begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} = x_1$ is given by

$$u(t) = B'\phi'(0, t)W^{-1}(\phi(0, T)x_1 - x_0)$$

And then the control is computed come control for this particular system x1dot and x2 dot is given by A* x1 x2+ B * u and if you have the initial any initial condition x0 and any final condition x1 2 vectors. Then the control given by this expression will steer the system from the initial point to the final point. And we can observe that this given system is nothing, but the harmonic oscillator given by this expression.

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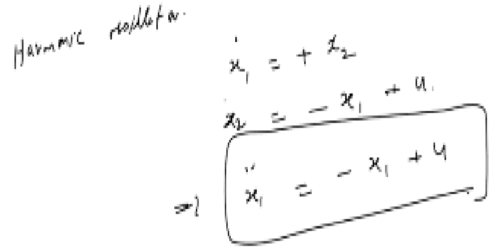

The harmonic oscillator is so we have x1 dot = - that is x2 dot = -x1+u is the given equation in the example so this will imply that if you differentiate once again, we will get the x1 double dot = - x1+u. So, this is the familiar simple harmonic motion equation of simple harmonic and with the external control u of t.

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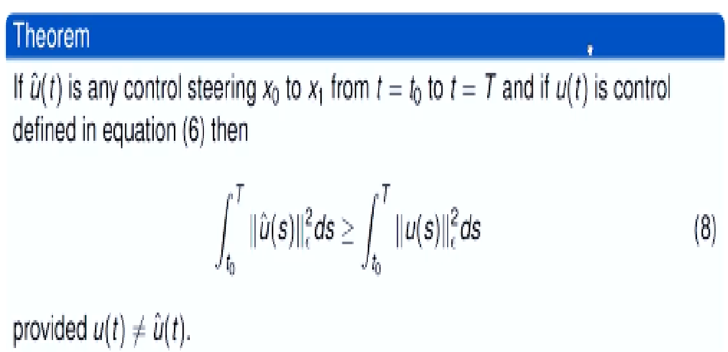

So, this is the expression now we can see that the control which we have seen in the equation 6 this expression.

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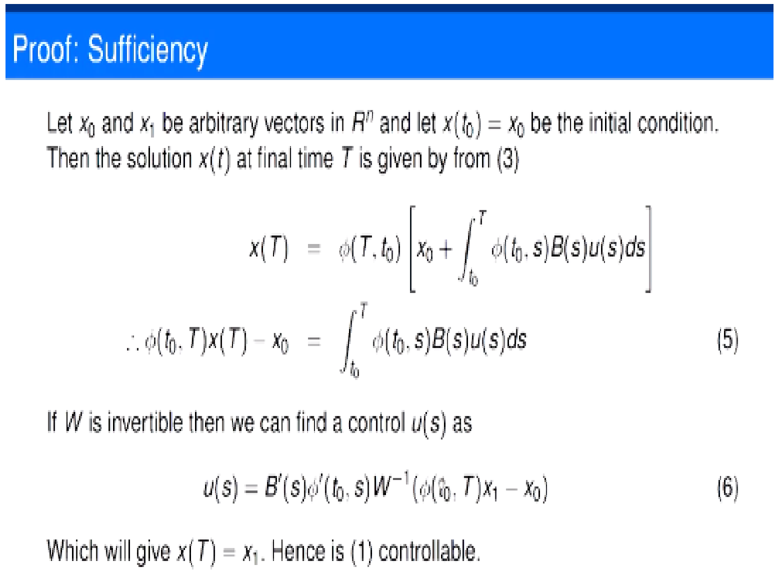

It is the best possible control in the sense that the integral from the time t0 to T of the norm square of this control is always <= any other control which steers the system from the initial condition to final condition.

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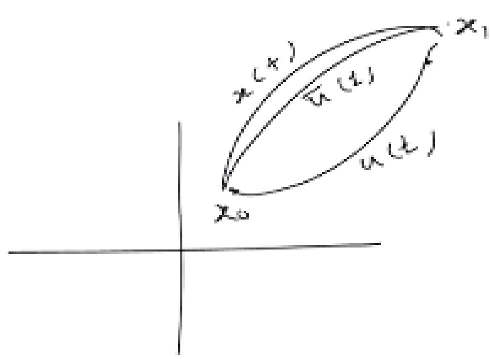

So, we can see that there are for a given initial condition and final condition they may be infinitely many control. In fact there are infinitely many controls for the steering of the system from the initial condition x0 to the final condition x1. So, out of all of this control so if let us say u of t is the control which we have obtained through equation 6 and u bar of t is another control which does the same thing that is starting from x0 to x1.

The solution of the system x of t so then we can show that this u of t given by the equation 6 is the best control and it satisfies this condition. So, in some sense in any control system the energy of the control is very important. The minimization of the expenditure are the minimization of the energy is very important. So, in some sense we can say that this represents the energy of a spent by the control.

And it is the lowest 1 out of all possible controls with steers the system from x0 to x1 so this can be proved by the following steps. Now if you assume that there are 2 controls. So, 1st control it takes the system Phi of Phi of t t0x0+ integral t0 to t Phi of ts Bus ds so us is taking the system this and the other system is taking other control. So, here u is 1 control so the solution will be x of t and u bar is another control.

So, the expression will be different so that is y of t is the solution different solution, but the final position are the same for both of them when we substitute t = T x of T and y of T will be the same expression by the two different controls. So, that implies that when we substitute T in the place of t in all the places and subtract this equation, we will get to 0= these two expressions are the same. And so we will get t0 to T of Phi of T s B of u s – u cap s ds plus only difference is u cap you will; only get this expression. So, that is what is written on the slide here.

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Proof:

Since both u and $\hat{u}$ satisfy (5), we obtain after subtraction

$$0 = \int_{t_0}^{T} \phi(t_0, s)B(s)[\hat{u}(s) - u(s)]ds$$

Multiplication of this equation on the left by $[x_0 - \phi(t_0, T)x_1]'[W^{-1}(t_0, T)]'$ and using (6) we get

$$\int_{t_0}^{T} u'(s)[u(s) - \hat{u}(s)]ds = 0 \qquad (9)$$

Therefore, $$0 \leq \int_{t_0}^{T} (u - \hat{u})'(u - \hat{u})ds = \int_{t_0}^{T} [\|\hat{u}\|_e^2 + \|u\|_e^2 - 2u'\hat{u}]ds$$

$$= \int_{t_0}^{T} [\|\hat{u}\|_e^2 - \|u\|_e^2]ds$$

So we get 0 = integral t0 to t Phi of this * whether it is u of s or u bar s - us so this expression can be obtained now multiply the equation by this expression because if you recall the equation 6 the control specific control which we have. So, if you multiply both sides by this vector x 0 – Phi t0 T * x1 W inverse transpose if you multiply and see compare it with the equation 6. We will get the u transpose s * this expression.

Because this portion x0 – Phi t0 T* x1 multiplied by W inverse if you put in the front of the inverse integral and combine it with Phi t0 s * B s so that will give you u dash that is u transpose of s. And the remaining inside the integral is u cap – u of s because it is 0. You can write it in any order. Now we can get from this equation and from this fact that is integral of u – u bar transpose * u – u bar.

This is nothing but the norm of u – u bar square so it is always positive, so this integral is always positive value. So, 0 <= integral t0 to capital T of norm of u – u cap square ds this expression. And if you open the brackets and if you write like this you will get norm of u cap square+ norm of u square – 2 times u dash * u cap now from the equation 9 we can see that integral of u dash *u which is nothing.

But norm of u square will be = integral u dash * u cap so if you take it to the other side so we substitute in the place of u dash * u cap integral of u dash * u cap you can substitute u dash * u which is nothing but norm of u square so – 2 times norm of u square is obtained here and already there is 1+ of norm of u square so it cancels. We get norm of u cap square - norm of u square inside the integral which is >= 0.

So, this automatically implies that this result the result is this 1 so it shows the control calculated by the equation 6 this is an optimal control in the sense that the integral is the minimum out of all the controls which steers the system from x0 to x1. So, if the system is not controllable is it possible to find a condition or find a control which can steer the system from wherever it can be controlled.

So, if a system is not controllable it does not mean that we cannot reach any position in the state space. The system can reach certain points and it may not be able to reach certain other points so the collection of all the points where the system can reach as the end point. That may be a sub space of the state space. So, if x 0 t is the initial condition and x1 is the final condition and if the system can reach that x1 is it possible to give expression for the control.

The optimal control as we have seen in the equation 6. Is it possible to get control in the non controllable cases also? So, we have seen that the control you are first given by the equation 6 is optimal control in the sense given in the theorem that the integral of the norm of u s square is minimized.

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Theorem:

If, for a given x_1, there exists a constant column $n-$ vector γ such that

$$W(t_0, T)\gamma = \phi(t_0, T)x_1 - x_0 \qquad (10)$$

Then the control

$$u(t) = B'(t)\phi'(t_0, t)\gamma$$

transfers the system (1) from x_0 to x_1.

So, now we will see that if the system is not controllable that is if the controllability grammian is singular in that case is it possible to find a similar control has given in equation 6.So, that it steers the system from initial x 0 to the final condition x1 so if a system is not controllable. It does not mean that from any initial condition we cannot reach any final condition. So, it only means that some final condition can be reached, and some final condition cannot be reached.

So, now we will see that for which initial and final condition 1 can find a control so that the solutions starts from x0 to x1, so this particular theorem gives the result like this.

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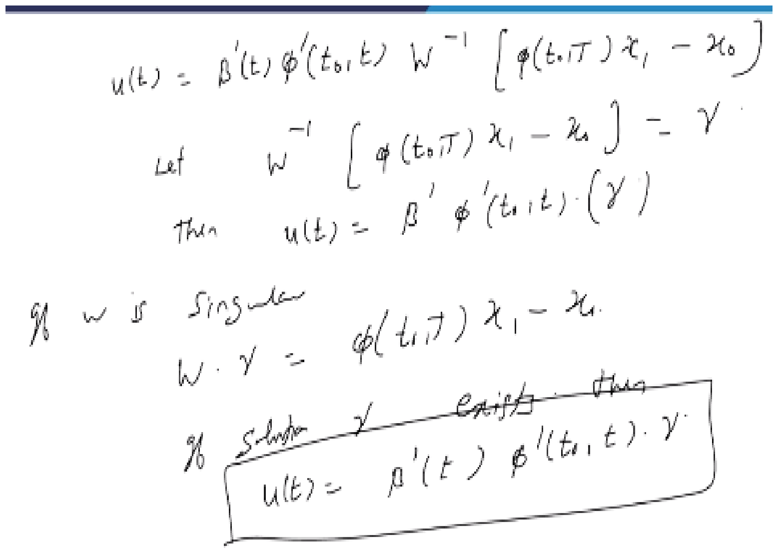

Now If you recall the control given in the equation 6 is u of t = B transpose of t Phi transpose of t 0 to t W inverse * by Phi of t0 to T x1 – x0 so this control takes a system from x0 to x1 provide W inverse exist but if W inverse does not exist so here if you substitute now if you substitute W inverse * by this vector Phi of this x1- x0= gamma then u of t is given by B transpose Phi transpose * gamma.

So, this is equation 6 but if w is singular then the system is not controllable in that case what we can do is instead of finding gamma. And it is not possible because W universe does not exist so if w is singular, we consider the equation W * gamma = Phi of t0 to T x1 – x0 so if the solution exist so if the solution gamma exist for suitable right hand side. Then we can find u of t can be as B transpose t Phi transpose t0 t * gamma.

So, in this case there the system is not controllable then also we can find a control. But here the restriction is only for certain x0 and x1. The solution may exist, so we have this particular theorem so if the equation 10 W of t0 T * gamma = Phi of t0 T x1-x0 if it has a solution gamma then the control given by this expression B dash * Phi dash * gamma it transfers the system from the initial condition x0 to x1, so the proof is quite simple.

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Proof:

The solution of the controlled system (1) with $x(t_0) = x_0$ is

$$x(t) = \phi(t, t_0)\left[x_0 + \int_{t_0}^{t} \phi(t_0, s)B(s)u(s)ds\right]$$

Put $u(t) = B'(t)\phi'(t_0, t)\gamma$ we get

$$x(t) = \phi(t, t_0)\left[x_0 - \int_{t_0}^{t} \phi(t_0, s)B(s)B'(s)\phi'(t_0, s)\gamma ds\right]$$

$$\text{Put } t = T, \quad x(T) = \phi(T, t_0)\left[x_0 - \int_{t_0}^{T} \phi(t_0, s)B(s)B'(s)\phi'(t_0, s)\gamma ds\right]$$

That we write the solution expression let x0 be the initial condition we can write the solution as usual and then in this solution. If you substitute us as given in the previous equation, we will get x of t = to the expression given here replacing u of s by this 1 and by making use of the equation 10.

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using (4), we have

$$x(T) = \phi(T, t_0)[x_0 - W(t_0, T)\gamma]$$

using (10), we get

$$\begin{aligned} x(T) &= \phi(T, t_0)\phi(t_0, T)x_1 \\ &= x_1, \quad \text{as required.} \end{aligned}$$

Since the system is controllable, W is nonsingular and the expression for $u(t)$ in theorem reduces to (6). It can also be shown that the converse of this Theorem is true, namely that only states x_1 for which (10) holds can be reached.

You will get x of T = x1 it is the final condition by making use of the property of the state transition matrix. So, we have seen here the how to evaluate a control even in the case of a system which is not controllable so far suitable initial and final conditions. So, in the next lecture, we will see the definition and some results about the observability of the time varying and time in variant linear systems. Thank you.