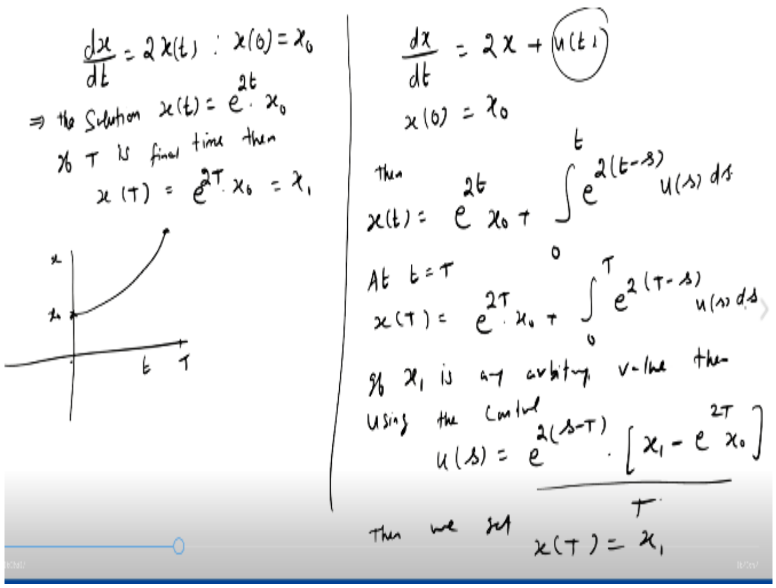
Dynamical Systems and Control
Prof. N. Sukavanam
Department of Mathematics
Indian Institute of Technology- Roorkee

Lecture – 33
Controllability of Autonomous Systems

Dear students, welcome to this lecture on the controllability of linear time invariant control systems. So, in this lecture we will see the definition of controllability and a necessary and sufficient condition for the controllability of a linear autonomous dynamical system or control system. So, controllability deals with the concept of controllability deals with the ability of a dynamical system to reach any arbitrary final state.

From any arbitrary initials state during finite a time interval using a control function. So, this concept is called the controllability of the control system. So, to understand the controllability concept let us consider a simple example.

(Refer Slide Time: 01:29)



So, let us consider the dynamical system dx/dt =2x and initial condition x of 0 is x0, t is the time variable and the x of 0 is the initial condition given. Then we can easily see that the solution x of t or the state of the system at time t it is given by e to the power 2t*x0. So, here we can see that if t is the final time okay any finite value T then the final state x of T is e to the power 2t*x0.

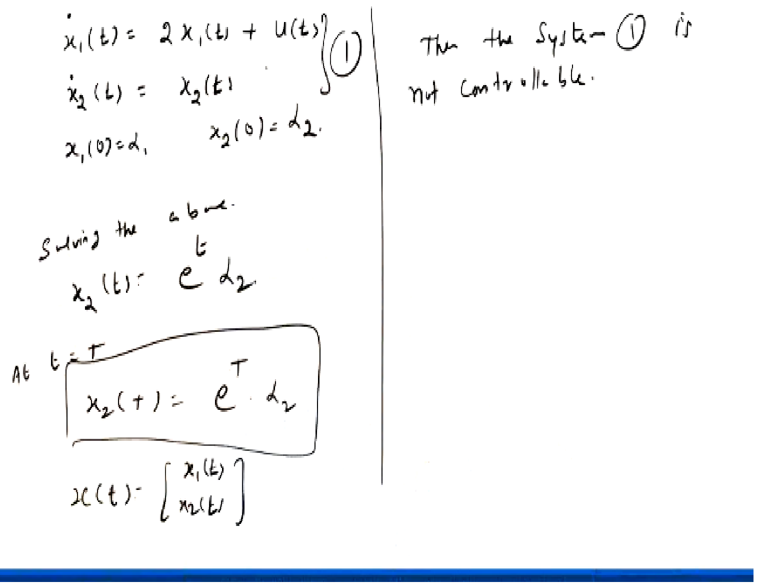
So, what we can see here is this is a t axis and this is x of t. So, at t=0 the value is x0 and when t= T the function reaches the state x1 let us call it as x1. So, you can see that only this particular x of T can be reached by the system if you start with the initial condition x0. Because their existing unique solution for this and so no other final state can be obtained but in this case if you consider a control system with the same dynamical system.

If you include the control term u of t with the initial state x0 then we can see that the solution x of t is given by e to the power 2t*x0+ 0 to t e to the power 2 t-s u of s ds. Now at the final time instant T we will get the final state x of T is given by this expression e to the power 2* T -s us ds it gives the final state of the system. Now we are free to choose the control function u of t.

So, by selecting a suitable u of t function we can get any arbitrary value for x of T. So, if x1 any arbitrary vector here x1 is a real value so any arbitrary value okay then we can get then using the control okay. Let us select the control function u of s as e to the power 2 times s-T if you substitute in the place of u of s multiplied by the function x1 - e to the power 2 T*x0/T So, we can see that if you substitute this u of s function in the inside the integral.

We will get then we get x of T=x1 so it is the simplest way of selecting a control so that the final condition is arbitrary value x1. So, this is the concept of controllability if you are able to find a control function so that the final position x of t is any arbitrary given condition. Then the system is called controllable so we will see the definition of the controllability.

(Refer Slide Time: 06:41)

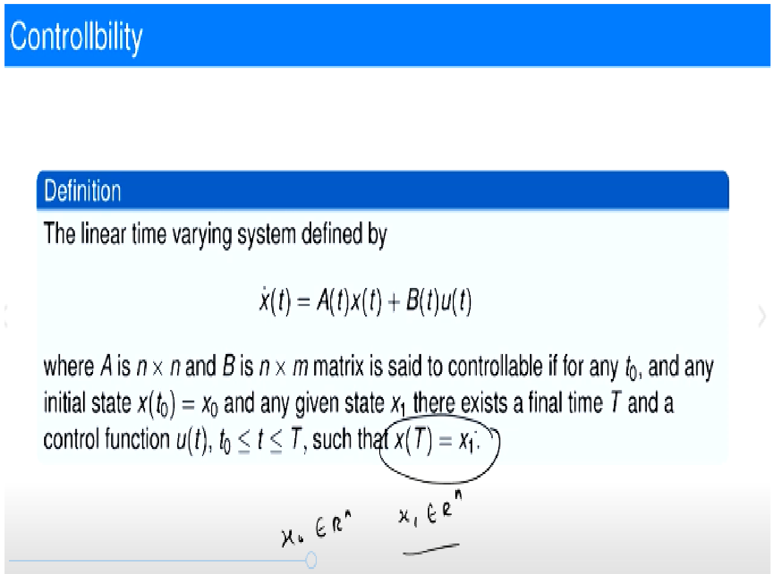


But before that let us consider this example again x1 of t derivative that is x1 dot =say 2 x1 of t + a control function and another variable x2 dot t is let us say x2 of t and some initial condition x1 at time t =0 is a value say alpha 1 and x2 at 0 is alpha 2 so in this case we can see that in whichever ways we selected the control function u of t the second variable x2 of t can never be controlled as we wish.

Because if you solve this equation we will get solving the above equation we will get to x2 of t is e to the power t*alpha 2. So, any final time we take okay we take x2 of T is a constant single value except this value we cannot get any other value for x2 of T. So, the solution of the system or the state variable x of t which is the vector here x1 of t x2 of t so, this cannot be controlled as we wish.

If you assign arbitrary final position for x1t and x2 t we will not be able to reach that position because x2 of T is fixed value only one value can be reached always. So, this system is not controllable so if you call the system to be 1 then the system 1 is not controllable because whatever control we substitute we will never reach any arbitrary final condition. We will be able to read so only if fix to final condition for the variable x2. So, the definition of controllability is given here.

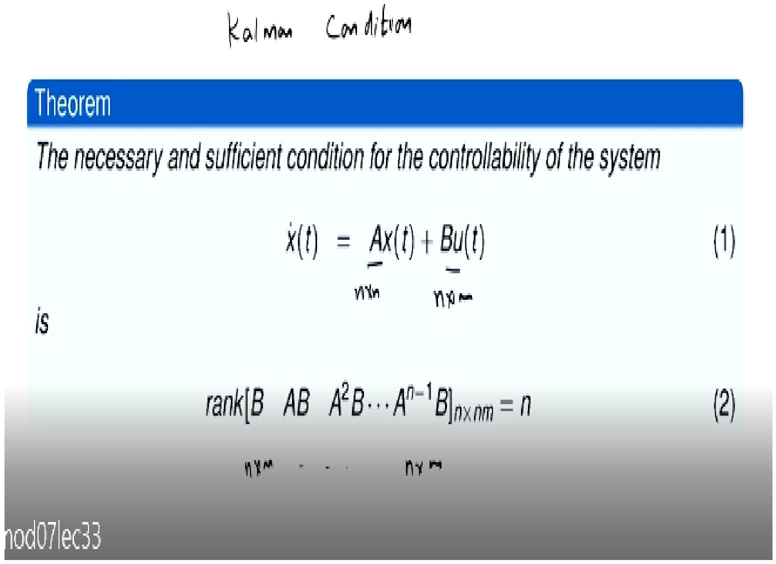
(Refer Slide Time: 09:18)



So, the linear time varying system general system we take x dot of t= A of t x of t+ B of t u of t where A of t is a n*n an B of t is a n*m matrix. So, this system is said to be controllable if for any initial time t0 and any initial state x suffix 0 and given arbitrary state x1 in Rn space x0 belongs to Rn then there exist their final time T and a control function u of t such that x of T is x1. So, that means we will be able to find a control function u of t and a final time T.

So, that the final position of the solution is x1 which is the arbitrary value in the state space. So, if this happens then we say that this system is controllable and if you are not able to find a control function like this for any arbitrary x1 we will not be able to find a control. Then we call it as uncontrollable system but it does not mean that we cannot reach any final condition. We will be able to reach some final condition x of T. But all the points in the space may not be able to will not be able to reach by the system.

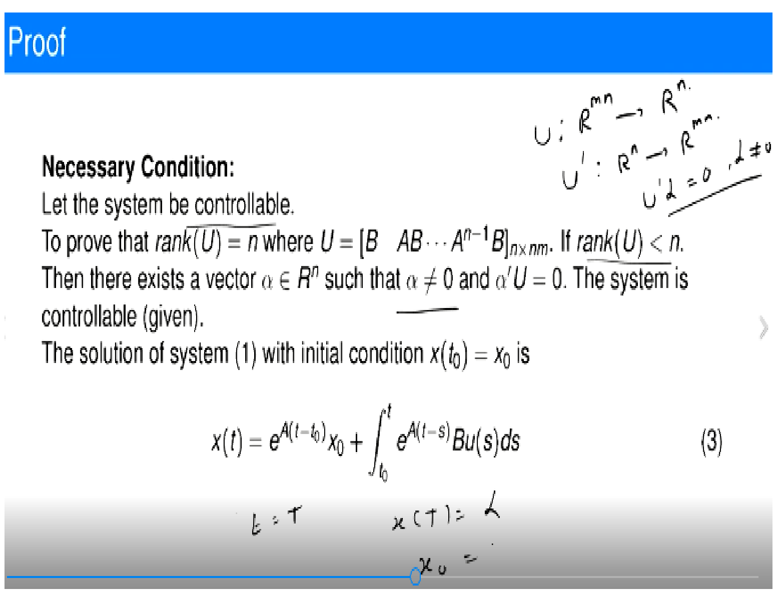
(Refer Slide Time: 11:12)



So, the condition for controllability of the autonomous system that is the matrix A and B are constant matrices. So is given by the rank condition the necessary and sufficient condition for controllability of the system 1 is the rank of the matrix B AB A square B etc A power n-1 B is =n. Here A is the n*n matrix and B is n*m. So, we have n*m matrix n times. So, the size of the matrix given in the bracket is n*m.

So, the maximum rank for this matrix is n the smallest the minimum number out of this two. So, maximum rank can be n but the system will be controllable if and only if the rank is=n. So, that is called the Kalman theorem Kalman condition for the controllability of the system 1.

(Refer Slide Time: 12:33)



So, let us prove it the necessary condition if the system is controllable then we have to prove that the rank of the matrix U=n where U is given by B AB A square B A power n-1 B. So, this is the statement of the theorem now if the system is controllable by the rank is <n then we will arrive at a contradiction in the following way. So, if the rank of U is <n then there exist a vector alpha in Rn the state space and alpha is non-zero such that the transpose alpha dash*U=0.

So, U is a mapping from R mn to Rn so we can show that similarly the transpose U dash is from Rn to R mn so you have the rank of U is <n the rank of U dash is also < than n. Both are from the same rank we will get a vector U dash alpha =0 and alpha naught =0 if rank is <n this will happen if you take the transpose you will get alpha dash U= for some non-zero vector alpha. So, this system is controllable is given here the solution of the equation 1 is given by this.

Where e power A t-t0 is the state transition matrix and the solution at the final point T. So, if you substitute in the equation in the equation 3 the solution if you put t = T the final time and x of t =alpha and x0 =0 because the system is controllable given so for any initial condition and any final condition we will be able to find a control U because the system is controllable. So, now we substitute the arbitrary value alpha for the final condition and the 0 condition for initial.

(Refer Slide Time: 15:06)

Let $x(T) = \alpha$ and $x_0 = x(t_0) = 0$.

$$\Rightarrow \alpha = \int_{t_0}^{T} e^{A(T-s)} B u(s) ds \qquad (4)$$

(we can find a control satisfying the above).

Using Caley-Hamilton theorem we can write

$$e^{A(T-s)} = a_0(s)I + a_1(s)A + \cdots + a_{n-1}(s)A^{n-1} \qquad (5)$$

mod07lec33

So, from equation 3 we will get this alpha =integral t0 to T e to the power AT-s Bu s ds so directly we can check so and this system is controllable. So, we will be able to find a control satisfying the equation 4. Now using Caley Hamilton theorem, the any function of the matrix here it is the exponential A T-s it can be written as a polynomial expression of a polynomial in the matrix A. So, this expression is due to Caley Hamilton theorem so if you substitute in the place of e to the power AT-s.

(Refer Slide Time: 15:49)

substituting (5) in (4) we get

$$\alpha = \int_{t_0}^{T} e^{A(T-s)}Bu(s)ds = \int_{t_0}^{T} (a_0(s)I + a_1(s)A + \cdots + a_{n-1}(s)A^{n-1})Bu(s)ds$$

$$\Rightarrow \alpha = \int_{t_0}^{T} (a_0(s)B + a_1(s)AB + \cdots + a_{n-1}(s)A^{n-1}B)u(s)ds$$

$$= \int_{t_0}^{T} \left(Ba_0(s)u(s) + ABa_1(s)u(s) + \cdots + A^{n-1}Ba_{n-1}(s)u(s)\right) ds$$

$$B\int_{t_0}^{T} \underbrace{a_0(s)u(s)}_{m\times 1}ds + AB\int_{t_0}^{T} \underbrace{a_1(s)u(s)}_{m\times 1}ds + \cdots$$

$$B\begin{bmatrix} v_1^0 \\ v_2^0 \end{bmatrix} + AB\begin{bmatrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_m^1 \end{bmatrix}$$

That polynomial expression we get alpha =integral t0 of T in this expression now we can bring this B matrix inside the bracket a0*B a1 *AB etc and this a0 a1 an-1 all these are scalar functions. So, it can be written in anywhere they are commutative so we can write the first term as B*a0 S*us this u of s also can be brought inside and then second term is AB and a1 of s can be written here multiplied by u of s.

So, we get this similarly A power n-1 B*an-1 of s*us everything is integrated with ds. So, this expression can be written like this the first a term can be written as because B is a constant matrix it can be taken out we will get t0 to T a0 s us ds. Similarly the second term is AB integral a1s us ds +etc. So, here we can see that u is a m*1 matrix. So, when we do the integration t0 to T of this expression we will get a vector here that is B is the matrix.

After doing the integration and putting the limit we will get a constant vector so we will call it as v10 v20 etc vm0 and +AB. Again we will get a constant vector after doing the integration that is v11 v21 vm1 so all these vectors are in Rm space. m vector.

(Refer Slide Time: 18:20)

$$= B\begin{bmatrix} v_1^0 \\ v_2^0 \\ \cdot \\ \cdot \\ \cdot \\ v_m^0 \end{bmatrix} + AB\begin{bmatrix} v_1^1 \\ v_2^1 \\ \cdot \\ \cdot \\ \cdot \\ v_m^1 \end{bmatrix} + \cdots + A^{n-1}B\begin{bmatrix} v_1^{n-1} \\ v_2^{n-1} \\ \cdot \\ \cdot \\ \cdot \\ v_m^{n-1} \end{bmatrix}$$

So, we will get the expression like this the 0 indicates that it is due to the coefficient indicating the coefficient a0 if the coefficient is a1 of s integral we call it says v11 v21 etc. So, we get an expression of this type.

(Refer Slide Time: 18:45)

$$\alpha = [B \quad AB \quad A^2B \cdots A^{n-1}B]_{n\times nm}\begin{bmatrix} v_1^0 \\ \cdot \\ \cdot \\ v_m^0 \\ v_1^1 \\ \cdot \\ v_m^1 \\ \cdot \\ \cdot \\ \cdot \\ v_1^{n-1} \\ \cdot \\ \cdot \\ v_m^{n-1} \end{bmatrix}_{nm\times 1}$$

So, this can be further written as this one B AB A square B etc A power so the first vector B* the first vector + AB*the second vector etc. So, the same previous expression can be written in this particular form. So, it is nothing but the matrix U which we considered and this is a Vector and U is a mapping from R mn to Rn space. So, alpha belongs to Rn here and that is =this expression U* we can call it as a matrix v the vector v. So, U*v =alpha so this is the equation we are getting.

(Refer Slide Time: 19:41)

$$\alpha = U\left[\;\right]_{nm\times 1}$$

$$\Rightarrow \alpha'\alpha = \alpha' U\left[\;\right]_{nm\times 1}$$

Since $\alpha' U = 0$ we get

$$\alpha'\alpha = \|\alpha\|^2 = 0$$

$$\Rightarrow \alpha = 0 \quad \text{a contradiction}$$

$$\Rightarrow U \text{ has rank } n$$

mod07lec33

Alpha =U*the v vector now if you multiply both sides by alpha dash we will get the alpha dash*alpha is alpha dash U* this. But other assumption is U is singular therefore the vector alpha dash exist so that alpha dash U =0. So, this term becomes 0 so what we get is alpha dash*alpha which is norm of alpha square that is =0 this because of this one. So, this implies alpha is 0 its a contradiction so this implies that U has rank n.

(Refer Slide Time: 20:40)

Sufficient Condition:

Let $rank U = n$. To prove that the system is controllable.

Let x_0 and x_1 be arbitrary vectors in R^n. Then from (3) we get

$$\Rightarrow x_1 = e^{A(T-t_0)}x_0 + \int_{t_0}^{T} e^{A(T-s)}Bu(s)ds$$

So, now we will prove the sufficient condition so let us assume that rank of U is n and we want to prove that the system is controllable. So, let x0 and x1 be arbitrary vectors in Rn. So, if the system is controllable we will be able to find the control function so that the initial condition is the x0 and final condition is x1. So, our assumption is the rank of U is n so when we write the solution of the system and put x of T that is when we put t= T in the solution.

And writing x1 as final condition x0 as initial condition we will get the equation like this x1 =e power A T-t0 *x0+integral e power A T-s Bu s ds. Again making use of the same technique which we used earlier using Caley Hamilton theorem e to the power A T-s can be replaced by the polynomial expression.

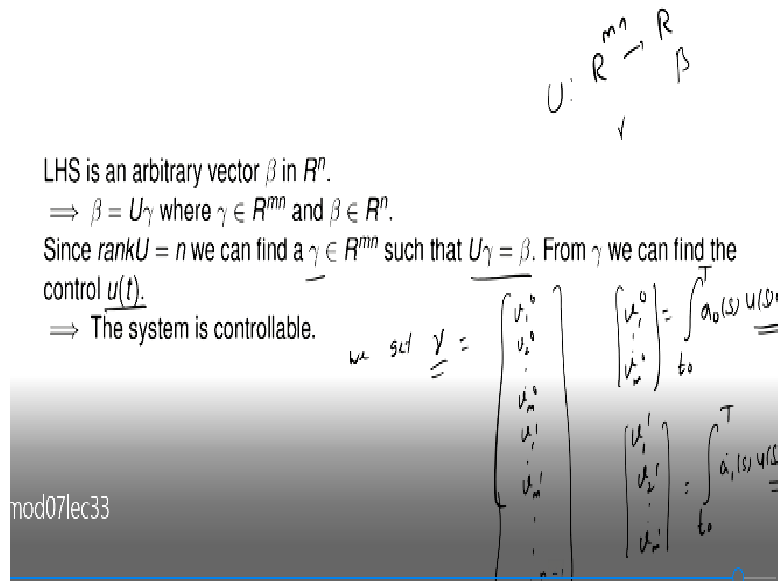
(Refer Slide Time: 21:58)

$$\Rightarrow \underline{x_1} - e^{A(T-t_0)}\underline{x_0} = [B \quad AB \quad A^2B \cdots A^{n-1}B]_{n\times nm} \begin{bmatrix} v_1^0 \\ \cdot \\ \cdot \\ v_m^0 \\ v_1^1 \\ \cdot \\ v_m^1 \\ \cdot \\ \cdot \\ \cdot \\ v_1^{n-1} \\ \cdot \\ \cdot \\ v_m^{n-1} \end{bmatrix}_{nm\times 1}$$

LHS is arbitrary vector in R^n

And then okay making use of the integration, we will get a similar expression the same expression we will get this way only thing is in this case we do not know what is the vector v1 etc this vector v is unknown and x0 and x1 are two given arbitrary vectors so the left hand side is arbitrary vector in Rn. Because x0 and x1 are arbitrary vectors so we will get the equation this form.

(Refer Slide Time: 22:40)



If you write to the left hand side vector as beta and the vector given in the right hand side is gamma here so we get an equation beta=U*gamma and the rank of U is =n has given in the sufficient condition. So, we will get a solution for this expression so U gamma =beta if rank of U is n there exist a gamma which satisfies this equation. So, the system is controllable because we will be able to get a gamma function.

We get the gamma value which is given by this vector v10 vm0 etc and then v11 vm1 so lastly we will get vm so we get the vector gamma it is not a unique solution there will be infinitely many such solution. Because the size of the matrix U is U is a mapping from Rmn to Rn. So, for every given vector beta in Rn you will get many solution vector gamma belongs to Rmn so that U gamma=beta because of the size of the matrix.

So, now each value for example if you take v10 it is nothing but integral t0 to T a0 of s us ds is it not v10 to vm0. So, this vector is given by this expression similarly v11 v21 vm1 so this vector is given by t0 to T a1s us ds so etc. So, then we have the gamma vector with us we can find the function u of s satisfying these conditions because of the relation given by this gamma and the u function.

So, if you are able to get the gamma vector we will be able to get the control function u of t using this relation. So, the system is controllable that is for any given initial condition and any given final condition we are able to find a control u of t for this control system. So, this system is controllable here. So, in this lecture we have seen the necessary and sufficient condition for the controllability of an autonomous system.

So, in the next lecture, we will see the necessary and sufficient condition for the controllability of non-autonomous system that is time varying system. Thank you.