Dynamical Systems and Control
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Lecture - 32
Introduction to Control Systems - II

Hello viewers. Welcome to this lecture on control systems. In this lecture, we will see some formulation of certain control systems.

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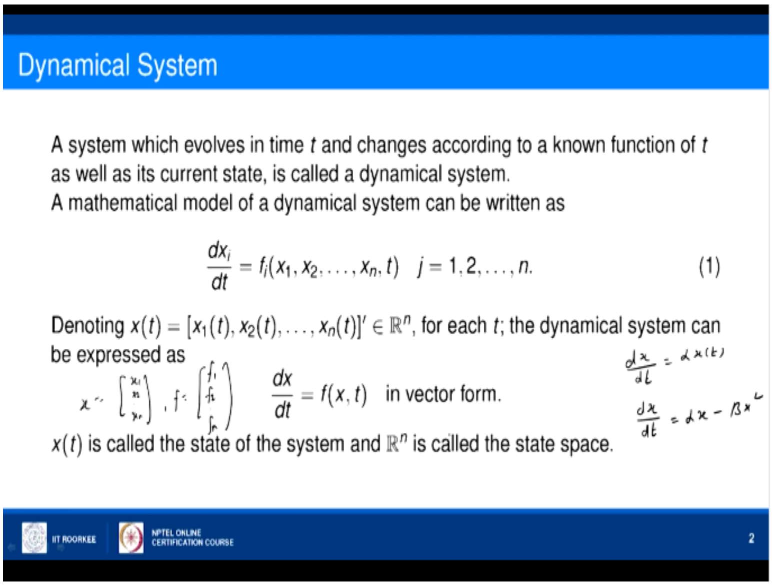

Or in other words some mathematical models of some practical control systems. In previous lectures, you might have studied various dynamical systems. So the definition of a dynamical system is a system which evolves in time and changes according to a known function of t, t is time as well as the current state of the system. So such systems are called dynamical system. So for example earlier we have seen the population dynamics.

So it is of the form say dx/dt is some alpha times x of t. So it is a simple model of a population dynamics where x of t represent the population of a species at a time t and how it changes, the rate of change of the population dx/dt is proportional to the population at time t, so alpha times x of t is the simple mathematical model which have been analyzed or improved model of the population dynamics is it is something like alpha x-beta x square.

So this represent it is called the logistic model of the population growth and so here also the rate of change of the population x is proportional to x itself and x square, a combination of x and x square is given. So it is in detail you have studied earlier about this population growth. So these are called the dynamical systems.

In general, if there are more variables of the state, so let us say x1 of t, x2 of t, xn of t are the n state variables, then in equation 1 we have a model dxi/dt, the rate of change of the state xi is=a function fi of all the states x1, x2, xn and the time. So this n equation together n first-order differential equation together is called a dynamical system and it maybe linear or nonlinear depending on the function fi.

If all the functions fi are linear functions of the variables x1, x2, xn, then it is called a linear system otherwise it is a nonlinear dynamical system. This x1, x2, xn are called the state of the system, etc. So in the vector form, we can write if x is x1, x2, xn vector, then we can write the equation 1 as dx/dt is f of x, t where f is a vector, f1 f2 fn the n functions in the right hand side. So it is called a dynamical system and the state space here is Rn.

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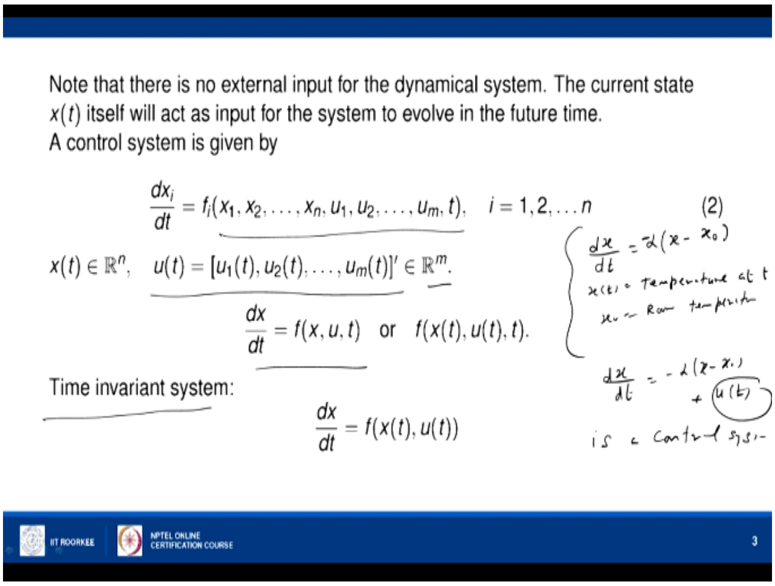

So in this equation there is no external force or external input in the system because the system is evolving on its own. The current state x of t itself acts as the input for the evolution of the x of t for the future time, but if you include an external input in the system which will affect the evolution of the system, then it is called a control system. So here we can see that x1, x2, xn are the state of the system and u1, u2, um functions of t or the control of the system.

So for example if you consider the equation dx/dt is some alpha times x-let say x0. So here x of t is the temperature at time t, any substance which is kept in the room temperature let say x0 is the room temperature and this is alpha times –alpha times. So the rate of change of the temperature of a substance which is kept in the room temperature is alpha times –alpha times x-x0 where x0 is the room temperature and x is the current temperature at time t.

So it will behave in this particular way. It is a simple mathematical model of the rate of change of the temperature of a substance. So here this is simply a dynamical system because it is only a function of x the right hand side but if we include some external heating system for the same problem, so this is say u of t is included, then the external heating will also affect the temperature of the substance. So this is a control term u of t so it becomes a control system in this case.

So the evolution of the temperature as a function of time will be affected by this control term u of t. So this is called a control system. So here we write a control system in the equation 2 dxi/dt is fi of x1, x2, xn and u1, u2, um and t, the n equations i=1 to n and u of t that is u1, u2, um of t, it belongs to this control space Rm here. So this equation can be written as dx/dt is f of x, u, t in the vector form.

And if t is explicitly present in the equation like f of x, u, t then it is called the time varying system. Time invariant system is the functions of f is function of x and u but x and u are function of t but in the equation t is not appearing explicitly. So this is called the time invariant systems. So time invariant system does not mean that the system does not change with respect to time, it changes.

But in the model, the mathematical model t does not appear explicitly, the rate of change of the variable x is not directly affected by t but as a function of x and u it is changing with respect to time, so such system are called time invariant system.

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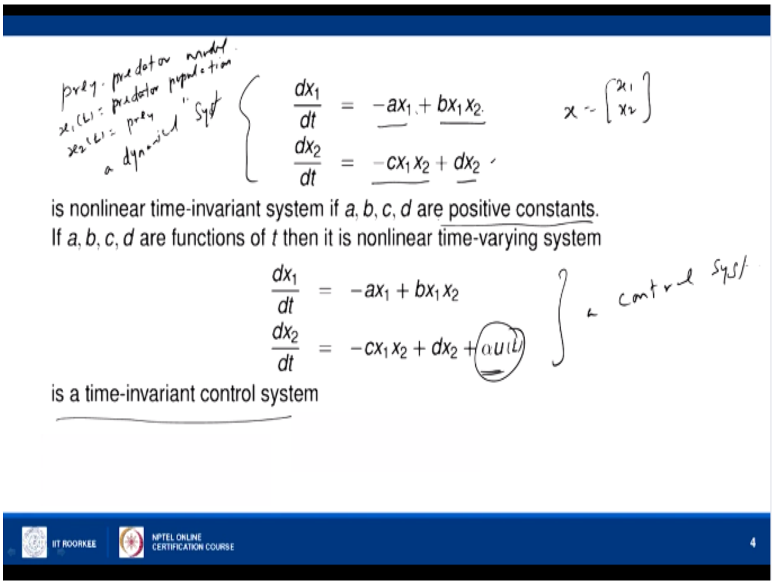

So for example of a time invariant system we see this one dx1/dt=-ax1+bx1 x2, dx2/dt is –cx1x2+dx2; a, b, c, d are positive constants. So this is a system where x means x1, x2. This is the state of the system and the right hand side of the two functions of the state variables. So this is familiar prey-predator model. It is a dynamical system because there is no external control available in this equation.

So here x1 is the predator population, predator population at time t and x2 of t is the prey population. So the rate at which the predator changes the population changes is the – sign represent that in the absence of any prey the population will decrease because the food for the x1 prey so the absence of x2 x1 will decrease but in the present of x2 the population x1 will increase, so the ratio in which it will increase is its own population.

So b times x1 is the ratio in which the population sorry b times x2, more and more availability of x2 will increase the population of x1. So b*x2 is the ratio in which x1 will increase, so that is the + sign and dx2/dt here in the absence of the prey, in the absence of the predator, the x2 population will increase on its own and the presence of the predator the population of x2 will decrease.

So how it will decrease? If x1 is of a bigger size, then c times x1 will be a bigger size. So the rate of decrease of x2 will be proportional to c times x1 of x2. That is the rate of decrease and this is the rate of increasing. So this represents the prey-predator equation. It is a dynamical system. Now if we artificially increase the population of the prey so that the predator population is maintained.

So we can use some methodology for increasing the population, the prey population, x2 population by a term alpha times u of t. So this is some population control method is applied on x2. Then, it becomes a control system. So this can be either x2 can be increased or decreased by the population control. If you want to decrease the population of x1, the rate of change of x2 can be decreased by a control u so that x1 also decreases.

So the controlling the population by a control term, this is a control system and time invariant, this both the systems are time invariant system because there is no t appearing explicitly in the equations.

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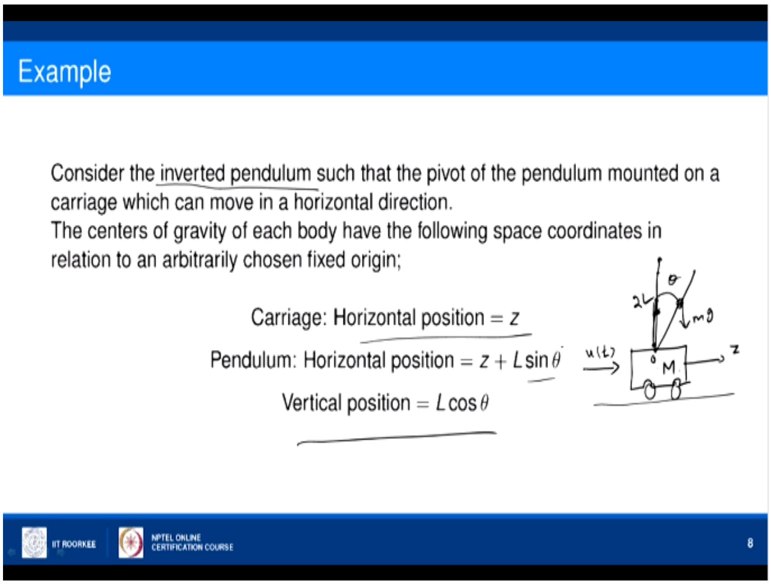

So now we will consider an important problem that is the inverted pendulum. So the inverted pendulum is like this. We have a cart with less there is no friction, it will be moving freely and the inverted pendulum is attached at the point O here. So this is the fixed position so this will move like this. So if it is not stable, it will fall down. So the problem is to keep this inverted pendulum in a vertical position by adding controlling this one.

By controlling u of t is given the control, so that if it falls down in this right side then it will be pushed to the right side so that it will come to the vertical position. If it falls in the left side, then it will be pulled back so that the pendulum will be again kept in the vertical position. So the aim is to just to keep it in the vertical position that is a stable position but it is a highly unstable position, a small disturbance will make the pendulum fall down.

So it should be controlled by this control u of t, so how to formulate the equation, so let us say that the length of this is 2L okay and the center of the length is 2L, the center of mass is there m is the mass of the pendulum and capital M is the mass of the cart on which the pendulum is positioned.

And let us say z is the displacement in the horizontal direction. So we can formulate the displacement in the horizontal position is z and now at a particular position the angle is theta let us say. Then, this L sin theta is so we will be watching the movement of the center of mass. So we will say that L sin theta is the horizontal motion and L cos theta is the vertical motion of the center of mass.

So vertical position at any angle theta is L cos theta and L sin theta is the horizontal movement. Now if you differentiate it two times, the displacement, differentiate the displacement two times, you will get the acceleration in that direction. So we will formulate the equation of motion of this pendulum in the following way. So let us consider the 4 variables.

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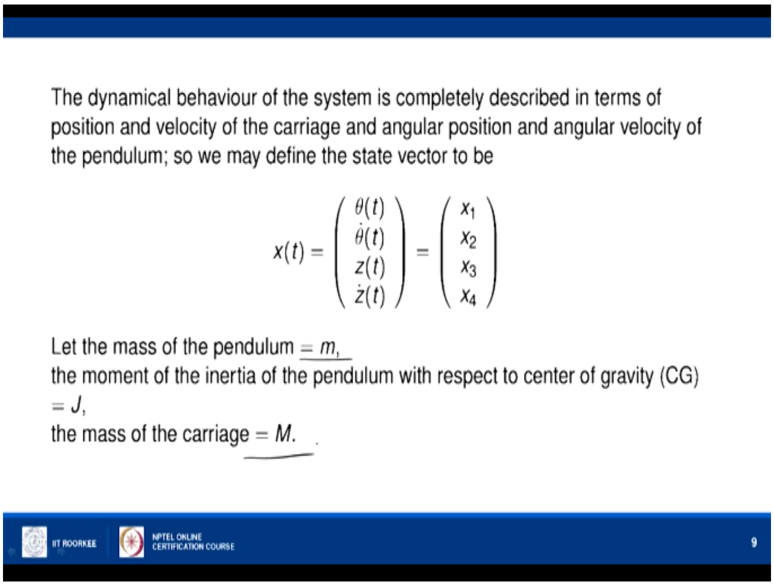

Theta and theta is the angle, theta dot angular velocity, z is the displacement in the horizontal direction, z dot is the velocity in the horizontal. So we call the 4 variables as x1, x2, x3, x4. The mass of the pendulum is small m, mass of the carriage the cart is capital M.

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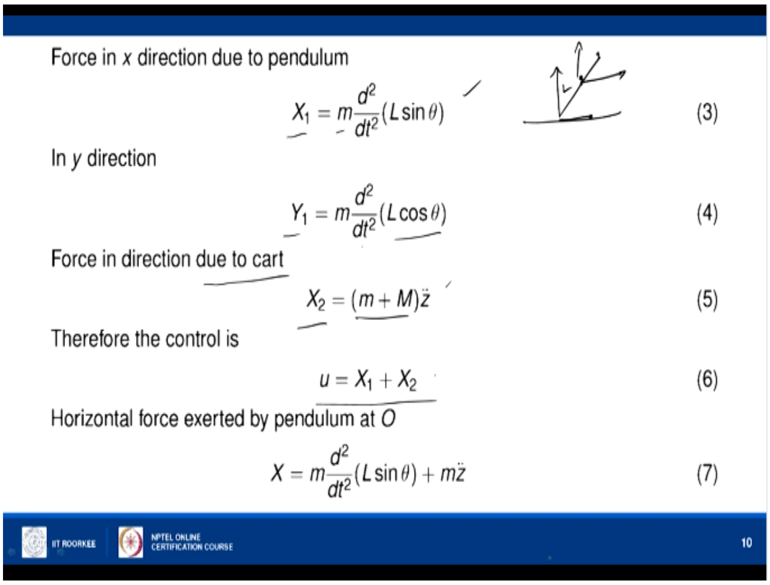

Now you can see that in the x direction, this direction, this is the pendulum and this is the x direction. In that direction, the force in the x direction due to the pendulum is because the L this is L sin theta this position and L cos theta is the movement of the center of mass is L sin theta and if you take the second derivative of L sin theta that gives the acceleration in the x direction.

Then, mass*acceleration will give the force X1 in the x direction of the force in the x direction. Similarly, the Y direction L cos theta is the displacement in the Y direction, so second derivative mass*the second derivative give the Y1. Now the force in the direction due to the cart motion. The cart is moving in the X direction, so z is the displacement, the second derivative will give the acceleration.

The mass of the cart+the pendulum together it gives m+capital M*z double dot is the force due to the cart motion, the carriage motion. So the control to be balanced, it is this two forces, x1 force and the x2 force. So u=x1+x2 is the control action so that is what we have in this previous picture, u of t is the control. So it should balance the two X direction forces only because Y forces the vertical direction. This control u should balance the x direction.

So we have u is=X1+X2. Now the horizontal force is given by this expression due to the pendulum only we will get the X1 part m*d square/dt square of L sin theta+m*z dot because only m force we are measuring here due to the pendulum we have the X direction force which is exerted at the point O because the mass capital M is below that O position. So whatever is the above the force exerted at this point O is given by equation 7.

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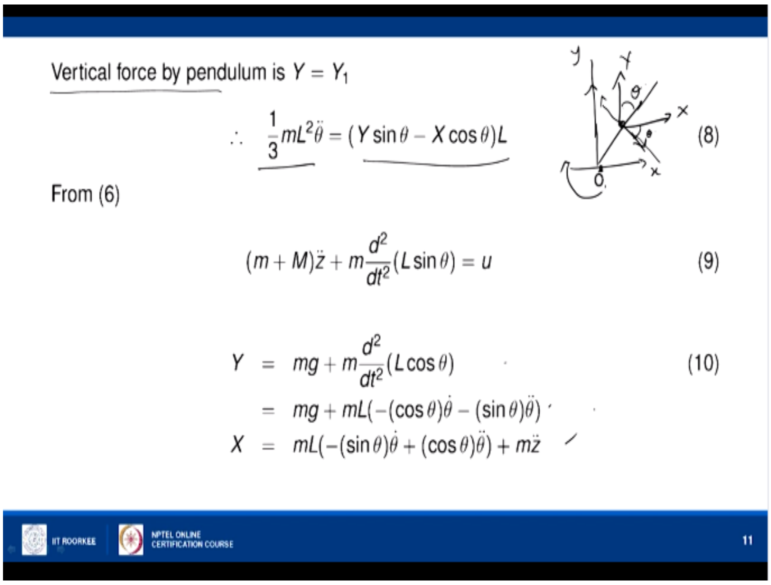

And equation 8 it gives the vertical force by the pendulum is Y=Y1. Now in the equation 8, we have the inertia*the acceleration theta double dot is the torque. So the torque exerted by the pendulum is given by the Y force*sin theta. So we have to measure the torque like this. So this is the pendulum, center of mass here. So the force exerted in the Y direction and force exerted in the X direction, so this is the X force and Y force at the center of mass.

So it will give a torque like this. If you resolve it in the perpendicular direction, so this angle this is theta and so this is 90-theta so X*cos theta will be the force exerted in the vertical direction to the pendulum that is X cos theta and it will give a torque because it is pulling it downwards in this direction. The torque direction will be in this way; this is the vertical. So the torque exerted by the X force is – because this is the Y direction, this is the X direction.

The torque is in the negative direction, so we get a – sign X cos theta is the torque due to the X force. Similarly, Y is this and the vertical direction to this is Y cos theta sorry Y sin theta is this expression and this will exert a positive torque here. So we get a positive sign, Y sin theta and –X cos theta is this one, so this is the theta angle, sorry this is X cos theta and a negative sign and Y sin theta with a positive sin*the length will give the torque exerted at this point O.

So from this equation, the previous equation we get from this equation 6 that is u=X1+X2, X1 is given by 3 and X2 is given by 5, so summing these two we get this equation m+capital M z double dot+this expression=u. Now exactly we will calculate this Y and X, capital Y is given by this expression as mentioned in the previous slide, we substitute this and expanding d square/dt square of this expression will give this one.

And capital X by using the same expression in the previous slide we get the expression as given in the last line can be calculated.

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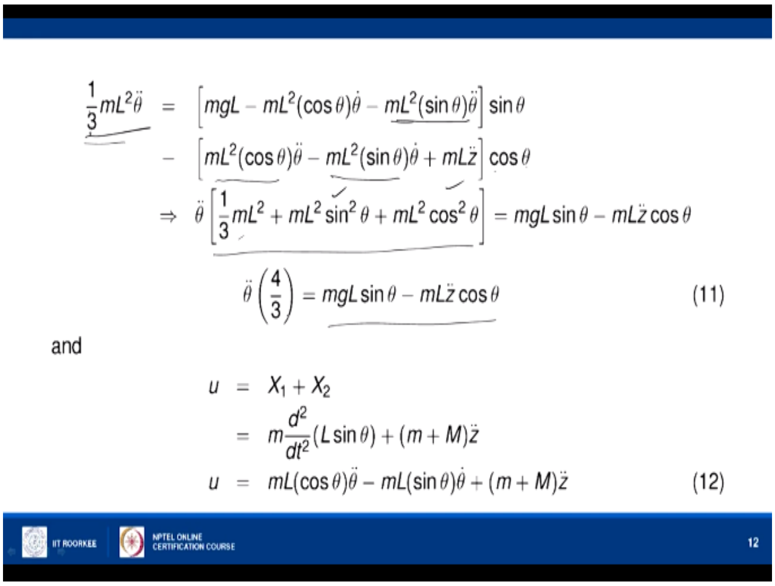

Now the balancing of torque, the torque is the inertia*the angular acceleration and the right hand side from the previous step if you collect the sin theta cos theta coefficient, we will get this expression theta double dot*this term because we take all the theta double dot to the left side, so we will get this expression. First term is 1/3 mL square and from here we get mL square cos theta.

And here we get mL square sin theta sin square theta because we have to multiply from this one okay. Here we get mL square sin square theta with a – sign when it comes to the left side we will get this one and this expression. So the remaining expressions are in the right hand side. So ultimately simplifying this equation, we get theta double dot and this whole bracket will become 4/3 and the right hand side is given as it is. So the equation 11 is the equation of motion.

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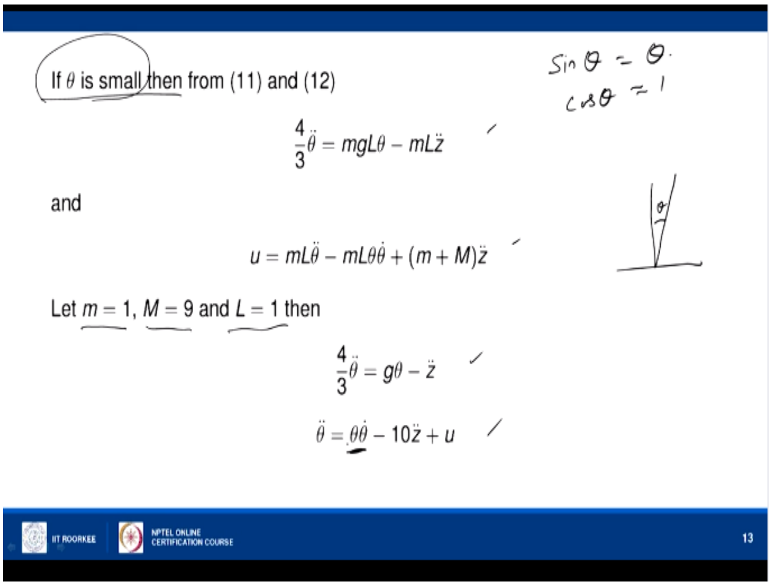

And the control equation is given by this expression okay. Now equation 9 if you take and then simplify that expressions, the second derivative expressions we will get u is given by this one, this expression in terms of theta double dot and z double dot, etc. So ultimately what we get is the equation 4/3 theta double dot=mgL theta-mL z double dot, that is one equation.

Another is u is=mL theta double dot-mL theta*theta dot+m+capital M z double dot. This is due to this expression. So we will replace sin theta approximately=theta and cos theta is approximately=1. So when we substitute these expressions in the previous two equations, equation 11 and 12, so wherever sin theta is there will put theta and cos theta to be 1 and in all the expressions we get these two equations.

Now for a specific case if the mass of the pendulum is 1 unit and mass of the cart is 9 units and the length of the pendulum is 2 units, so the half of the length is 1. Capital L is half length it is 1, then we get the equation to be like this. These two equations by substituting the m, L and capital M values we get this equation. Now when the pendulum is in the vertical position and if it is disturbed slightly, so theta is small.

The assumption theta is small; it means that from the vertical position it is slightly disturbed, the theta value is not deviating too much. Similarly, the velocity it is moving with a very slow velocity because already it is near the vertical position. So velocity as well as the angle theta both are the angular velocity and angle theta both are very small, so we can also neglect this theta*theta dot, both of them are negligible values, so we can neglect it.

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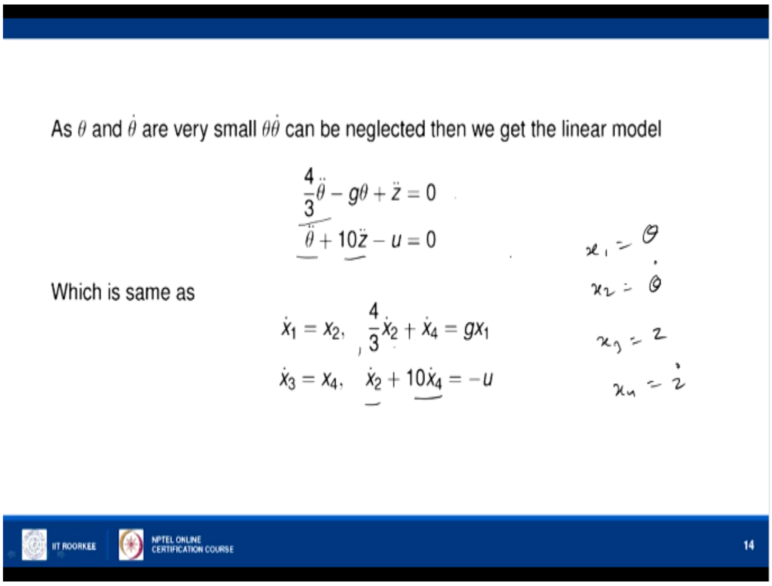

So the equation finally becomes like this 4/3 theta double dot-g theta+z double dot=0 and theta double dot+10 z double dot-u=0, these two equations. So we get the equation in this particular form and now if we write the variables x1 is theta, x2 is theta dot, x3 is z, x4 is z dot. So the relation is x1 dot=x2 and from the equation 4/3 x2 dot this theta double dot-g x1 and z x double dot is this one, so it should be +sign here, sorry 4/3.

So we get this equation=g x1 and x3 dot is x4 and here x4 dot that is 10 x4 dot is this and x2 dot is this one=u.

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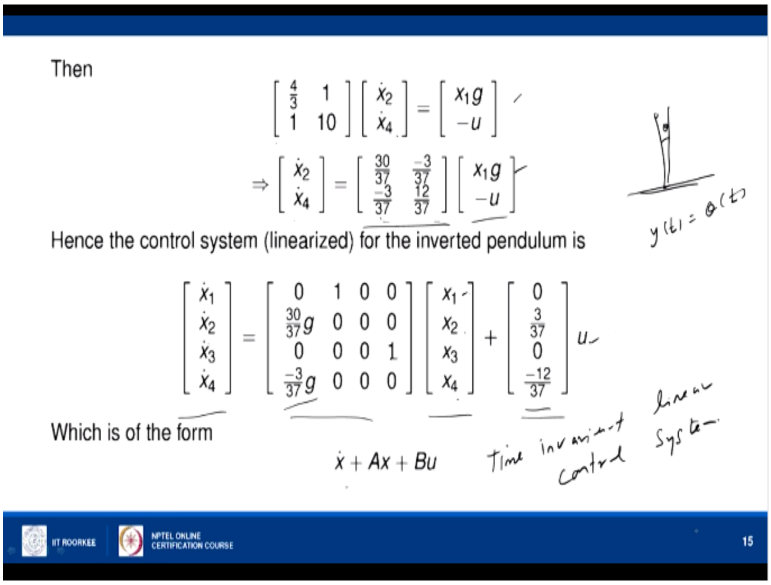

So these two equations we get, from here we can write it in the form of a matrix the expression 4/3 1 1 10, these are the coefficients of the matrix x2 dot x4 dot, so that is the right hand side x1 g and –u from the previous equation. So now we can calculate the values of x2 dot, x4 dot by taking the inverse of the matrix in the left hand side. So the inverse is given by this expression into this.

So directly we can get the values of the x2 dot in terms of x1 and u. Similarly, x4 dot in terms of x1 and u. Now so from here we can easily see that x1 dot is 1*x2 and x2 dot is from this equation. So it is 30/37*37g*x1 and -3/37*-u therefore it is +3/37*u. So that is x2 dot equation and x3 dot is x4, so we have 1*x4, x4 dot is from this equation again so -3/37 g*x1 and 12/37*-u, so this expression is given.

So we can write this equation as x dot the left hand side=A*x+Bu, so it is a standard form of a control system. It is a time invariant linear system, linear control system, this formulation. Similarly, when the pendulum is in motion if the angle can be measured because we are interested in making the pendulum to a vertical position that is stabilizing the pendulum position as a vertical position.

So it is important to know at what angle it is there at each instant of time. So if you are able to measure the angle at each instant of time, then the observation means we are measuring theta of t.

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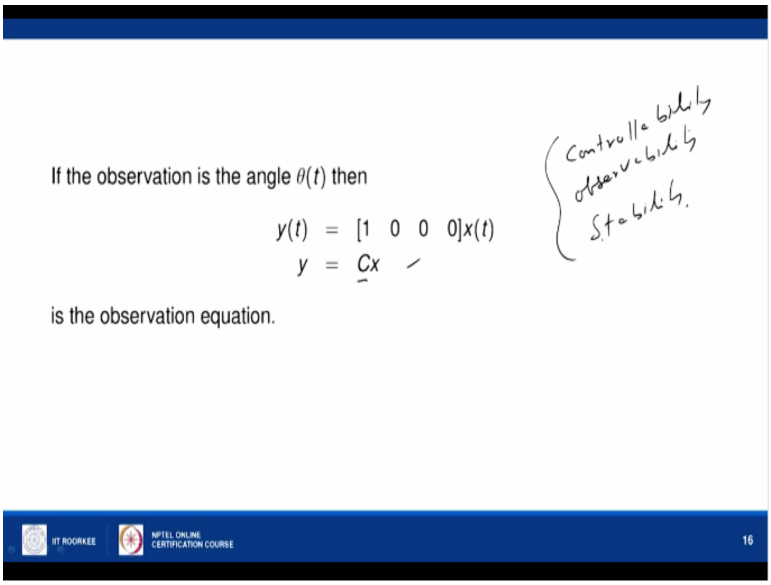

So if the observation of this experiment is the angle theta, then it can be written as 1*theta so y the observation=C matrix is 1 0 0 0*theta dot is z dot we get the observation to be equal to theta. So we have the equation, linear equation and the observation equation like this. So this is the formulation of a simple the control system, the control system of the inverted pendulum okay.

So for this because it is an important problem in the mechanics, various concepts like the controllability, observability and stability of this particular control system is analyzed extensively in the control theory courses. So we can study this as an example in various concepts in the coming lectures on the controllability, observability, etc. Thank you.