Dynamical Systems and Control

Prof. D. N. Pandey
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 03
Existence and Uniqueness Theorem - I

Hello friends, welcome to this lecture, in this lecture we will discuss the existence and uniqueness theorem for dynamical system, so if you; let us start with this.

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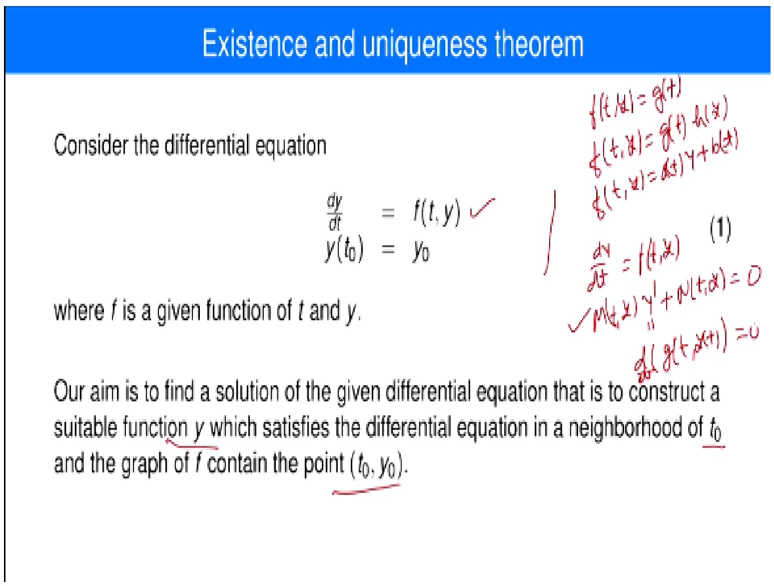

So, consider the differential equation, dy/dt = f ty with initial condition y t0 = y0, here the function f is a given function of t and y, here t is the independent variable and y is a dependent variable on t and on this is a typical example of a dynamical system, so our aim is to find out a solution of the given differential equation that is to construct a suitable function y, we satisfy the differential equation dy/dt = f ty.

In a neighbourhood of the point t0 and the graph of f contained the point t0, y0, so our idea is that how we can solve this initial value problem since the condition given at initial point, so we call this as initial value problem, so our; problem is how to solve this initial value problem and from the basic course of ordinary differential equation or differential equation, we know that under some special cases of this f ty.

For example, if f ty is a kind of a separable form in terms of t and y and or it is just a constant function, it is just a function in terms of t or some other suitable form, like if it is this equation number 1 can be reducible to say linear equation or say it is reducible to some kind of exact form then, we know how to solve this equation number 1 but the problem comes when this f ty is not given in these standard form.

For example, if f ty is some function gt only, then this can be integrated directly and we can find out the solution and if f ty is given as some gt * hy, then we can solve this by using separation variable method and if f of ty is given as like some at y + bt, then it is this dy/dt = at y + bt is nothing but linear equation and we can solve but this are already given and one more thing that if f ty is given in a way such that this dy/dt = f ty can be written as say M ty, say y dash + N ty = 0.

And we can choose M and N in a way such that this can be written as exact differential of this can be written as d/dt of say some g of t yt, then we call this equation as exact differential equation and we know how to solving these simple cases but the problem comes when this f ty is not of these simple form then, forming or finding out the exact solution say, finding out the solution given in terms of elementary function is quite difficult.

And then it is quite difficult to check whether this particular equation is solvable or not and right now we have several other method or we can say numerical methods or say using computers and all that is already available but for that also, we need to know whether this system has a solution or not. So, if our f ty is not deducible, dy/dt = f ty is not deducible in these set forms, then it is quite difficult to proceed.

And we cannot apply these numerical solution method or say approximation solution method to this equation number 1, until and unless we do not know whether this system has a solution or not because it may happen, because they generally use the say, approximation technique and we really do not have any method to check whether this that obtained solution is a solution of this equation or not.

So, these exist in; existence and uniqueness theorem is very, very important for these kind of system, so in this lecture, we will focus on how to say give say, condition on this function f ty such that this system one has a solution and if there is any other condition on f such that the solution, the existence; the solution is unique or not, so that all these questions we try to answer in this lecture here.

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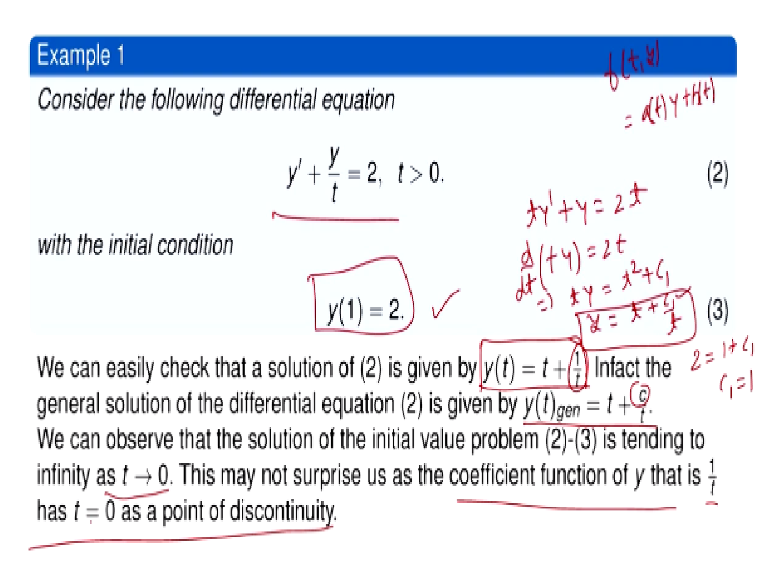

So, first let us start with the simplest form that is linear form, so here we assume that your f ty is given as some at y + bt and try to find out the effect of at and bt in this solution, so let us consider the following differential equation, y dash + y upon t = 2 and t is > 0 and the initial condition is given as y1 = 2 and we try to find out the solution of this and if you look at this can be solved in a very easy manner, you can simply rewrite this as t of y dash + y = 2 of t.

And this can be written as dy/dt of here ty = 2 of t and you can simply integrate, you can write it here ty = t square + some constant C1 and we can write y as say t + some C1 upon t, so this is a general solution and the constant C1 can be obtained using the initial condition, so y1 = 2 is the initial condition given, so with the help of y1 = 2, you can find out, you can fix this constant C1, so we can easily check that the solution of equation number 2 is given yt = t + 1/t.

In fact, the general solution which we have just obtained is given by t + C/t and this C can be obtained by the initial condition y1 = 2 that is I can write it here 2 = 1+ C1 and you can find out

C1 as 1, so it means that the solution is given as yt = t +1 / 2 and we can observe that the solution of the initial value problem 2 to 3 is tending to infinity as t tending to 0 because of this term because of this term as t tending to 0, this solution yt = t +1/t is tending to infinity.

So, this may not surprise us because the coefficient if you look at the equation here, the coefficient of y is 1/t and it also have the same feature that as t tending to 0, this 1/t is also tending to infinity, so it means that the coefficient function of y that is 1/t as a point of discontinuity at point t = 0, so it means that if the differential equation has say, point of discontinuity at some point, it may happen that the solution will also have the point of discontinuity at the same point.

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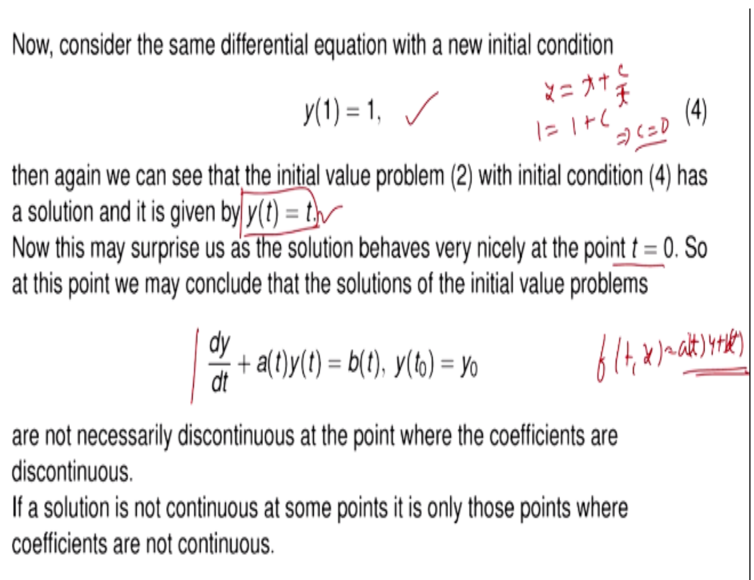

Now, look at another situation when initial condition is now replaced by y1 = 1, earlier it is y1 = 2, now it is replaced by y = 1, then if you look at the general solution is given by t + C upon t and if you use y1 = 1 that is 1 = 1+ C and you can say that this implies that C = 0 and in this case our; since C = 0, so our solution is given by yt = t. Now, here we can observe this thing that though your differential equation has point of discontinuity at point t = 0.

But the solution yt = t is behaving very nicely at point t = 0, so if we combine these two thing then we can say that in the case of linear differential equations that is dy/dt + at yt = bt with the initial condition yt0 = y0 are not necessarily discontinuous means, I am talking about the solution, this is solution of this initial value problem or not necessarily discontinue at the point where the coefficients are discontinuous.

But if a solution is not continuous at some point, it is only those point where coefficient are not continuous, so if you look at this, this is linear differential equation here and here we can say that in if we take the initial condition y1 = 2 then we have a solution like yt = t +1/t and here the point of discontinuity is at t = 0 which is the same point at which the coefficient of y has a discontinuity.

But if you look at the initial condition y1 = 1 and corresponding to that if you look at the solution yt = t, then it has no discontinuity, so it means that if it may happen with that the coefficient or the discontinuity at some point but the solution may not be say, may not be a discontinuity at all but if it has discontinuity, it must be at those point only where the coefficients of differential equations failed to be continuous, so that is the example given here.

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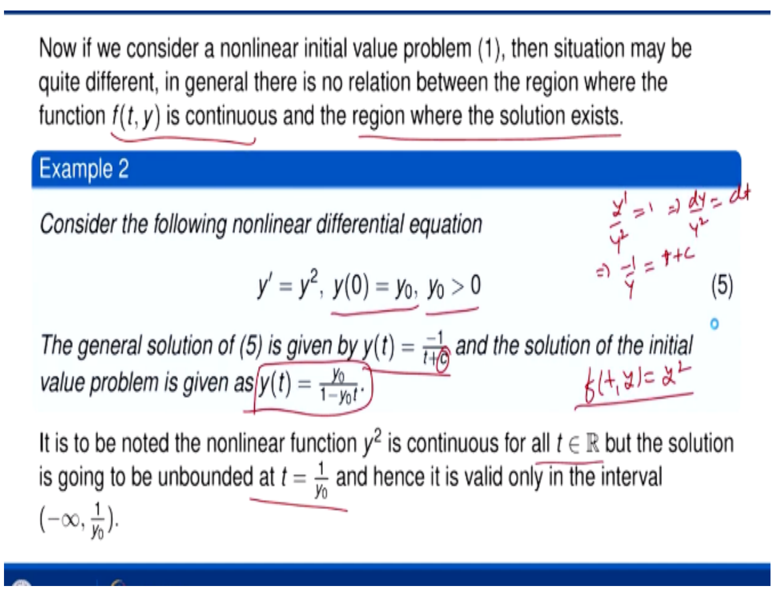

So, now after consider this linear differential equation now, let us consider a non-linear initial value problem, so now if we consider a non-linear initial value problem 1, then situation may be quite different, it means what that in general there is no relation between the region where the function f ty is continuous and the region where the solution exists, so here it may happen that f ty is continuous in the entire region R but the solution may not exist for the entire R.

So for let us consider one simple example, so consider the following non-linear differential equation y dash = y square with the initial condition y0 = y0, where y0 is some positive value then the general solution of 5 is given by yt = -1 upon t + C and this can be easily obtain because if you look at this is nothing but say, separation variable form and you can write it y dash upon y square = say 1, so we can write it dy upon y square = dt.

And we can simply integrate and you can write it –y1 upon y = some t +C and we can write down that yt = -1 upon t + C is the general solution of this equation number 5 and the solution of the initial value problem that is y0 = y0, we can fix this value C and we can write down the solution of this initial value problem as yt = y0 upon 1 – y0 t and here your f ty is what; here f ty is y square.

And if you look at this y square is continuous for all values of t, so it means that here y square is continuous for all t in R but the solution is going to be unbounded at t = 1/1; 1/y0 because if you look at this yt is unbounded provided that this 1 – y t0 is 0, so 1 – y0 t is 0 is as t = 1 upon y0 and since y0 is positive, so it means that for given some positive value of t that is 1/y0, the yt is going to be unbounded.

So, here you are f ty is say continuous for all t but your solution may not exist for the all t and in the previous case when f ty is your this at y + bt and then the coefficient of y, the coefficient at and bt will give you some information about the solution.

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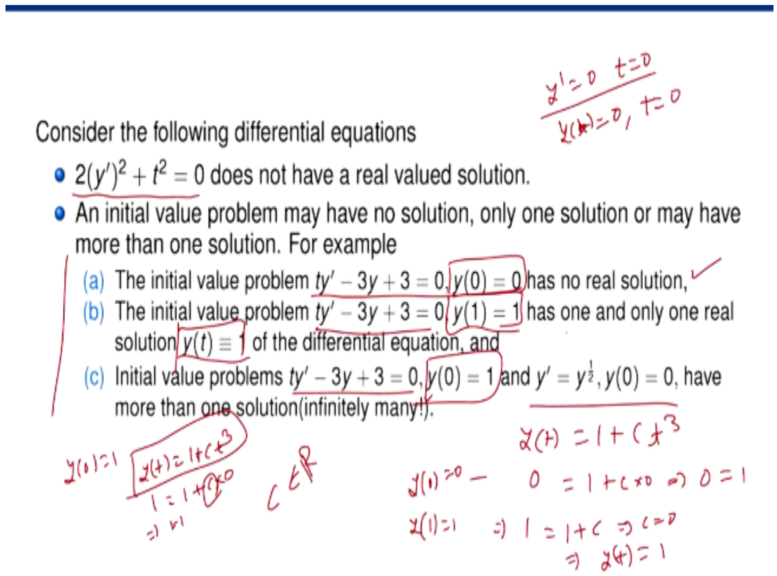

Now, look at some following differential equation, so 2y dash square + t square = 0 and if you try to solve this problem, then this can be solved only when y dash = 0 and t = 0, so it means that here if you solve this will not have any real value solution to satisfy this because this implies that this can be solved only yt = 0 and that is also true only for a particular point that is t = 0, so here we cannot define a solution of this differential equation.

So, here we have a differential equation which may not have any real value solution now, consider the next initial value problem and we can check that depending on the initial value, we may have no solution, one solution or more than one solution for example, if you consider this differential equation ty dash – 3y +3 = 0 and if you look at the all the part a, b and c, equation is same, the only thing is that initial condition not change.

In first case, it is y0 = 1, in second case, it is y1 = 1 and in third case, it is y0 = 1 and we try to check that if the only thing we are changing is the initial condition and we can check that by changing this initial condition, what is effect on the solution, here, so to find out the solution here, I will use this thing.

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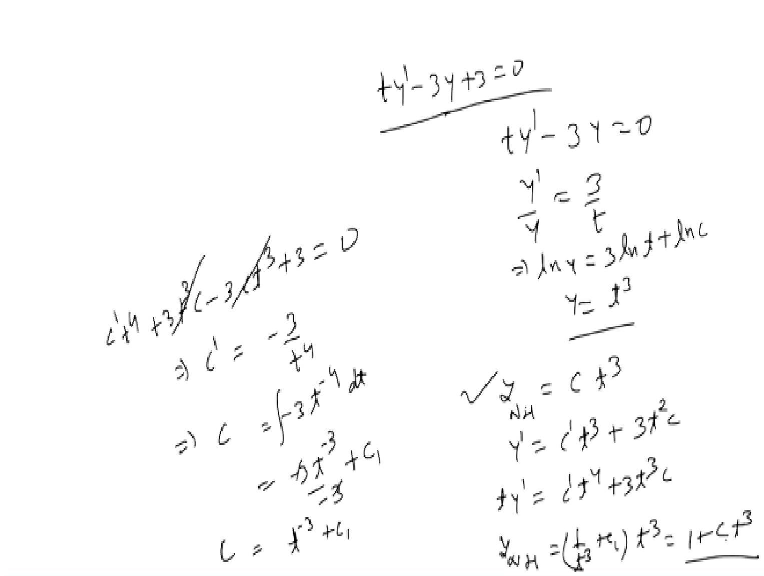

So, here we have ty dash – 3y + 3 =0 and we want to find out the solution here, so here first we find out the solution of the homogeneous problem that is dy dash – 3y = 0, so here y dash upon y = 3/t and you can write ln y = 3 ln t and we can note that y = t cube say solution of this + some constant we can write it and we can put on since you can put some constant also and then since we want only one particular solution I can write it y = t cube.

Now, to find out the solution of this non-homogeneous problem, we can use variation of parameter method and we can write y, non-homogeneous solution is some C * t cube, here C is a parameter here, now to find out this parameter which; for which this yNH is the solution of this non homogeneous problem, we can simply say y dash = C dash t cube + 3 t square C and we can put it here.

So, ty dash is basically C dash t to the power 4 +3 t cube C and we can plug in the value of ty dash and y and we can write it here, C dash t to the power 4 +3t cube C – 3C t cube + 3 = 0 and you can see that these are cancelled out and you can write down C dash = -3 upon t to power 4 and we can integrate and you can get it C as -3 t to the power -4, integration of this, and you can write it this is nothing but -3 t to the power -3 upon -3 + some integration constant, let us say C1.

And you can write it this is as t to the power -3, you can write it C1 here, so so we can find out the general solution like this, general solution is y of NH is given as this you can write it here t cube we multiply and you can write it thus C is 1 upon t cube C1 * t cube and you can write it here as 1 + C1 t cube, so your solution is given by 1 + C t cube, so here we can write down general solution as yt = 1 + C t cube.

So, this is our general solution and now, initial conditions are given so, we can fix our C here, so if you look at the first one that is y0 = 0 and if you want to find out the solution in this case then it is 0 = 1 + C * 0, so we can say that it is given as u = 1 which cannot be true, so it means that if we consider this differential equation along with this initial condition, then we will have no solution.

So, this initial value problem given in a has no real solution but if you look at the initial value problem given in b that is we are considering this condition this is y0 = 0 and let us consider this y1 = 1, then in this case, it is 1 = 1 + C and you can check that C = 0, so here we can say that solution is what; the solution is given as yt = 1 only, so here we have 1 and only 1 real solution that is yt is an = 1.

So, in the case b when initial condition is replaced by y1 = 1 then we have only one solution that is y1; yt = identically = 1. Now, let us change further and we write, we use y0 = 1 here and here the solution is what; yt = 1 + C t cube using the condition that is 1 + 1 and this C * 0, so we can say that this is 1 = 1 and C can; here it has no effect of this C, so it means that whatever C you can take and still it is satisfy this initial value problems.

So, it means that for any value C belongs to R, this yt = 1 + C t cube is the solution of this and similarly, you can handle this situation y dash = y to the power 1/2 y0 = 0 and you can simply solve this problem, this is a simply separation and variable problem and you can see that this has at least 2 solution that is 0 on t square upon 4 and not only 2 solution you can check that it has infinitely many solution, so that I am leaving it to you.

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We have seen that there are initial value problems which have one solution, more than one solution or no solution at all. This leads to ask the following questions:

1. How do we know that the initial value problem (1) has one or more than one solution.
2. If we have a solution of (1), then whether it is unique or not. There may be two, three or infinitely many solutions of (1).
3. Why bother asking the first two questions? After all, what is the use of determining whether (1) has a unique solution if we are not able to find it explicitly?

Here we want to discuss the conditions by which we can answer these questions.

So, it means that if we combine all these thing, we can say that we have seen that there are initial value problems which have one solution more than one solution or no solution at all, so this leads to ask the following questions. Question number 1; how we know that the initial value problem has one or more than one solution? We are not generally interested those problems in which we have no solution at all and basically, you can say that these initial value problems are say some mathematical model of some real say, model.

And in real situation, we are expecting that if we are making any mathematical model, we are expecting that we are having at least one solution, so it means that the problem which has no solution we generally are not much interested. Now, the first question is that how do we know that the initial value problem has one or more than one solution. In the second question is that if we have a solution of one so, somehow we know that it has one solution then, we are interested in whether it is unique or not.

They may be 2, 3 or infinitely many solution of one that we have already seen that in this last case, this ty dash – 3y + 3 = 0, y0 = 1, it has infinitely many solution, so it means that once we know that it has a solution we are interested in knowing that whether it has a unique solution or not. Now, next question; which is kind of important one or why bother asking the first 2 question that whether it has a solution or a unique solution after all, what is the use of determining whether one has a unique solution if you are not able to find it explicit manner.

So, it means that if you do not know how to find out the solution then why we are worried about the first 2 question whether it has a unique solution or more than one solution but the question number 3 with the development of this technology, this question number 3 is now not much useful because with the help of say, softwares, many softwares are already available.

Once we know that solution exist and it has a unique solution then, we can always use some numerical technique to find out the solution which is say which is approximately exact solution up to say, decimals 10 to the power -15 or 16 up to that level. So, we are not much worried about question number 3 here right now, we are very much interested in trying to consider the first 2 question that is whether it has a solution or if it has a solution, it has more than one solution or a unique solution.

So, here we want to discuss the condition by which we can answer these questions.

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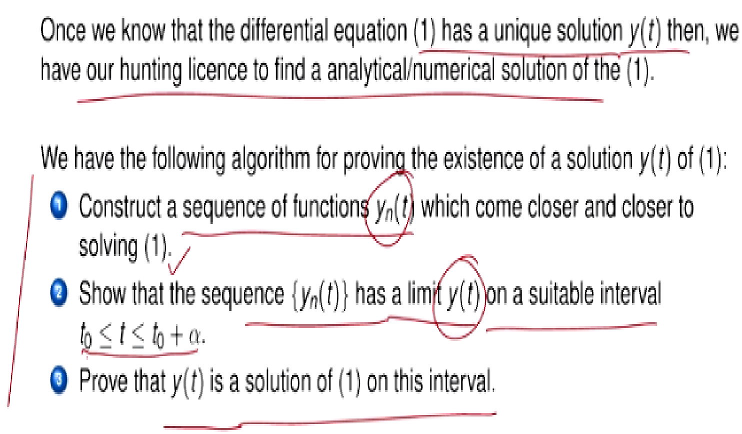

So, once we know that the differential equation one has a unique solution yt, then we have our licence to find out analytical or numerical solution so, first thing is to find out the analytical solution, if you are not able to find out the analytical solution, we try to find out the numerical solutions and right now, we have very much added by say, softwares which are freely available here.

Now, so to find out the first thing that whether we have a unique solution or not, we try to look at this problem in the following way, so we have the following algorithm for proving the existence of a solution yt of 1, what we try to do; we construct a sequence of function y and t which come closer and closer to solving in one and so we have to construct the sequence y and t, we will see that how we can find out this sequence of function which is approximating the exact solution.

Now, showing that the sequence y and t has a limit yt, so once we have a sequence next step is to show that it has a limit and it has a limit, let us call that as yt on a some suitable interval say t0 to t0 + alpha and last step is to prove that the limit which we have already obtained is a solution of one on this particular interval that is t0 to t0 + alpha. So, first thing is constructing a sequence, second thing is to prove that this sequence is converging.

And once it is converging defined that limit as yt and try to prove that yt is a limit of the; yt is a solution of the given problem, so that is how we try to proceed, so let us proceed with this.

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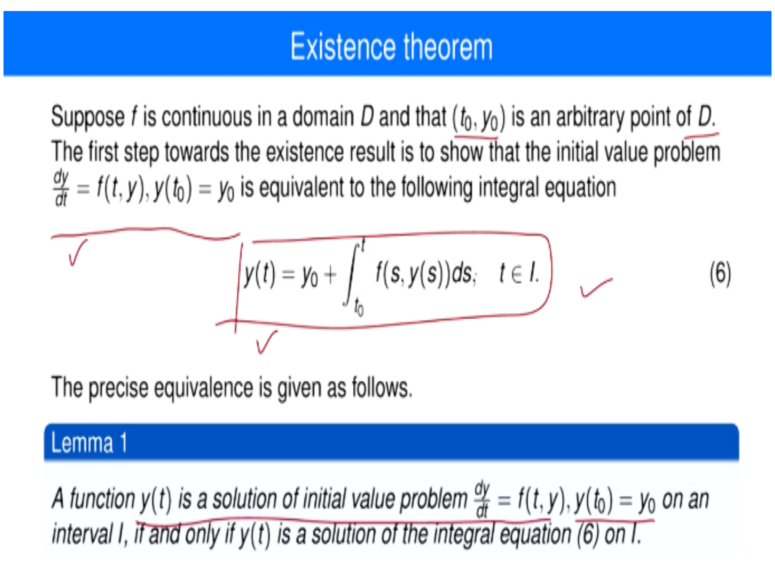

So, suppose f is continuous in a domain D and that t0 y0 is an arbitrary point of D, so we have a domain where this f is continuous in its argument that this t and y and t0 y0 is an arbitrary point of in that domain. The first step toward the existing result is to show that the initial value problem dy/dt = yt y with the initial condition yt0 = y0 is equivalent to this integral equation that is yt = y0 + t to t0 f of s y of s ds, in fact it is t0 to t and we try to show that this initial value problem is equivalent to this integral equation.

And why this is important because using this equation number 6, we try to construct the sequence of function which will converge the solution of this initial value problem, so the precise equivalence is given as follows, number 1; a function yt is a solution of initial value problem dy/dt = f ty yt0 = y0 on an interval I, if and only if yt is a solution of the integral equation 6 here, so it means that here we say that yt is a solution of this initial problem, if yt is a continuous solution of this integral equation number 6 here.

So, here I am assuming that the solution of this initial value problem that is dy /dt = f ty yt0 = y0 is equivalent as a continuous solution of this integral equation number 6 here, so let us prove the equivalence.

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Proof: If y is a solution of $\frac{dy}{dt} = f(t, y)$ on I satisfying $y(t_0) = y_0$, we have

$$y' = f(t, y(t)) \tag{7}$$

and integrating from t_0 to t on I, we obtain

$$y(t) - y(t_0) = \int_{t_0}^{t} f(s, y(s))ds, \quad t \in I. \tag{8}$$

Thus, using the initial condition $y(t_0) = y_0$, we can see that y satisfies (6). Conversely, if $y(t)$ is a continuous solution of (6), then by the continuity of the function $f(t, y(t))$ the right hand side of (6) is differential. Then by the fundamental theorem of calculus we can verify that $y(t)$ satisfies the differential equation

$$y' = f(t, y(t)), \tag{9}$$

and putting $t = t_0$ in (6), we have $y(t_0) = y_0$.

And if y is the solution of dy/dt = f ty on I satisfying yt0 = y0, then we can simply integrate here from t0 to t and we can write yt – yt0 = t0 to t f of s y of s ds, t belongs to I. Now, here we can put t = t0 and we can say that yt is coming out to; yt0 is coming out to be y0 and we can see that if I y is a solution of this initial value problem, then y is a solution of this integral equation that is yt = y0 t0 to t f of s y of s d of s. Now, so this is one way, now prove the other way round that if yt is a continuous solution of integral equation 6 then since yt it is continuous and f is also continuous in its argument.

Then we can say that f is a continuous function of t and we can say that the right hand side that is t0 to t f of s y of s ts is basically a differentiable function, so it means once it is a differentiable function then we can apply the fundamental theorem of calculus and we can write that y dash = sorry; this yt = y0 + this, if we simply differentiate this what will get; y dash t = and here if we differentiate, we will get f of t y of t.

So, it means that if y which is a continuous function is a solution of this equation number 6, then y dash t = ft yt is true, now regarding the initial condition we can put t = t0 and we can have the initial condition that yt0 = y0, so and putting t = t0 in 6, we can say that yt0 = y0, so it means that the solution of this initial value problem is equivalent to a continuous solution of this integral equation number 6.

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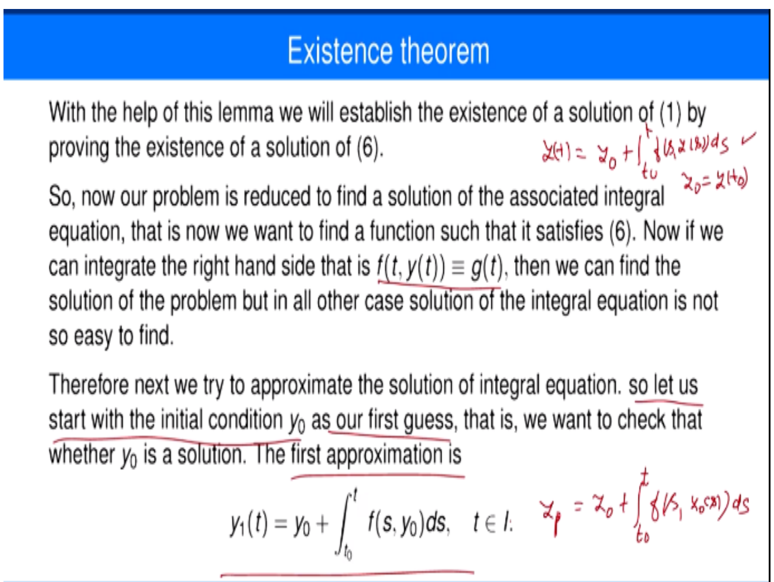

So, once this equivalence is proved, we will try to proceed further and with the help of this lemma, we will establish the existence of a solution of 1 by proving the existence of a solution of integral equation that is equation number 6. So, now our problem is to is reduced to find a solution of the associated integral equation that is now we want to find out a function such that it satisfy the integral equation 6 that is yt = say y0 + t0 to t, f of s, y of s ds.

Here, y0 is basically y of t0, so here once we have this integral equation and we know that if f of s ys is integrable, then we can simply integrate and we can have a solution for example, if we take ft yt as gt then we can simply integrate if gt is integrable but since we are assuming that f is continuous functions, so it means that g is also continuous function and so it can be integrable. So, it means that in a particular case when fs ys is a simple function or integrable function, we can integrate.

But suppose, it is not then we cannot find out the exact solution or say analytical solution of this, then we try to proceed further to find out the approximate solution, so how to find out the approximate solution; so, let us start with the initial condition y0 as our first case. We try to see whether this y0 satisfy this integral equation or not, so it means that if we write look at this y0 + t0 to t f of s, now replace y by y0 and try to find out this value.

If this is coming out to be y0, we simply say that y0 is a solution of this, if it is not then we are in trouble and we call this expression as y1 and we call this y1 as first approximation so, the first approximation we can define as y1t = y0 + t0 to t f of s y0 ds, t belongs to I.

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If $y_1(t) = y_0$, then $y(t) = y_0$ is indeed a solution; if not, then we try $y_1(t)$ as our next guess.

In this way we can define a sequence of approximating solution $y_1(t), y_2(t), \ldots, y_n(t), \ldots$ as follows.

$$y(t_0) = y_0,$$
$$y_{j+1}(t) = y_0 + \int_t^{t_0} f(s, y_j(s))ds, \quad j = 0, 1, 2, 3 \ldots \qquad (10)$$

These functions $y_n(t)$ are called successive approximations, or Picard iterations.

And we have already discussed that if y1 t = y0, then we can say that yt = y0 is indeed a solution and if not we try y1t as our next case and in this way we can find out a sequence of approximating solution y1t, y2t, ynt as follows that yt0 = y0 and yj +1 t = y0 + t0 to t f of s yjs ds, where j starting from 0, 1, 2, 3, and all, so what we have achieved so far is that we have shown that this initial value problem is equivalent to an integral equation.

And with the help of integral equation, we try to find out the sequence of approximating solution and it is given as equation number 10, now these function y and t are called successive approximation or Picard iteration. So, first step is done that is sequence of approximating solution is already given.

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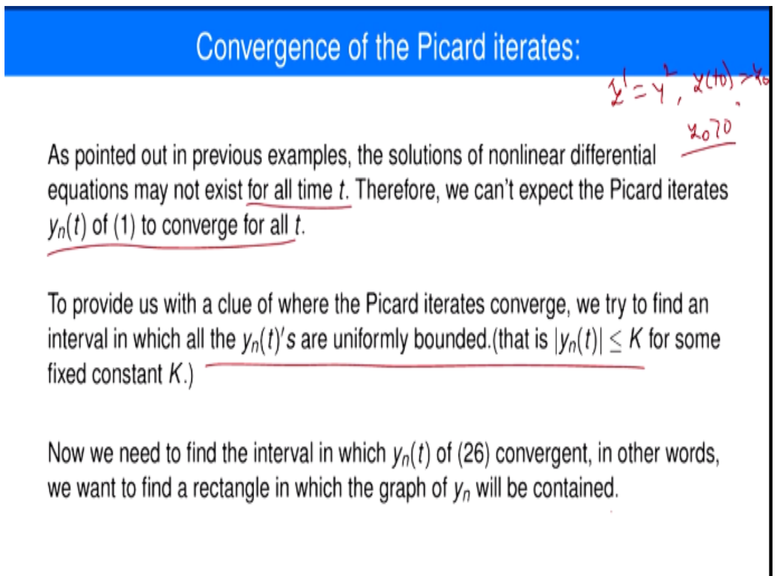

Now, we try to find out whether this converge or not, so once we have sequence of approximating solution, next thing is to discuss is convergence of the Picard iteration, so as pointed out in previous example, the solution of nonlinear differential equation may not exist for all time t, we have discuss this case, y dash = y square with some condition y0, here y0 is; here we have seen that solution may not exist for all time t.

So, it means that we cannot expect that Picard iteration y and t will converge for all time t, so here we try to find out a range in which this sequence will converge, so to provide us with a clue of where the Picard iteration converge, we try to find out an interval in which all y and t's are uniformly bounded that is modulus of y and t is < = K for some fixed constant K that constant K will be determined later.

So, now we need to find out the interval in which this y and t is convergent in other words, we want to find out a rectangle in which the graph of yn t will be contained.

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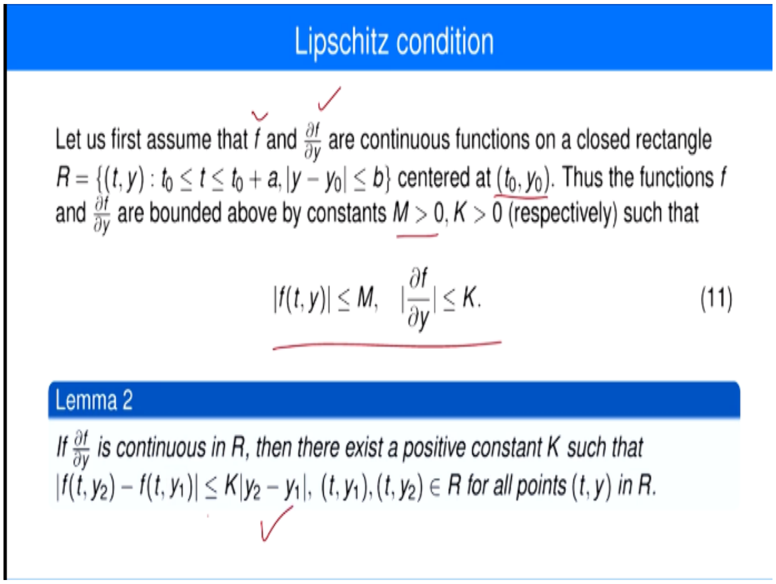

So that we are going to discuss but before that we will consider one more very important condition which is known as Lipschitz condition, so let us first assume that f and dou f/dou y are continuous function on a closed rectangle R, now here this is the rectangle which we are talking about, it is set of all ty, such that t is lying between t0 to t0 + a and y is in a way such that y – y0 is < = b, so it is a closed rectangle and it is centred at t0, y0.

And here we are assuming that this f and dou f/dou y are continuous function on this rectangle; closed rectangle are then we can say that this is f and dou f/dou y are bounded thus the function f and dou f/dou y are bounded about by constant M, and we can write this as modulus of f ty is bonded by M and dou f/ dou y is bounded by; now, we consider the next lemma that is if dou f/dou y is continuous in R, then they exist a positive constant K such that modulus of f ty2 –f ty y1 is < = K times y2 – y1, where ty1 and ty2 belongs to this rectangle R for all points t, y in R.

And we say that if dou f/ dou y is continuous then f satisfy this condition and we know that this condition is quite useful and we call this condition as Lipschitz condition and we say that if dou f/ dou y is continuous then f satisfies the Lipschitz condition that is what we wanted to prove in this lemma number 2.

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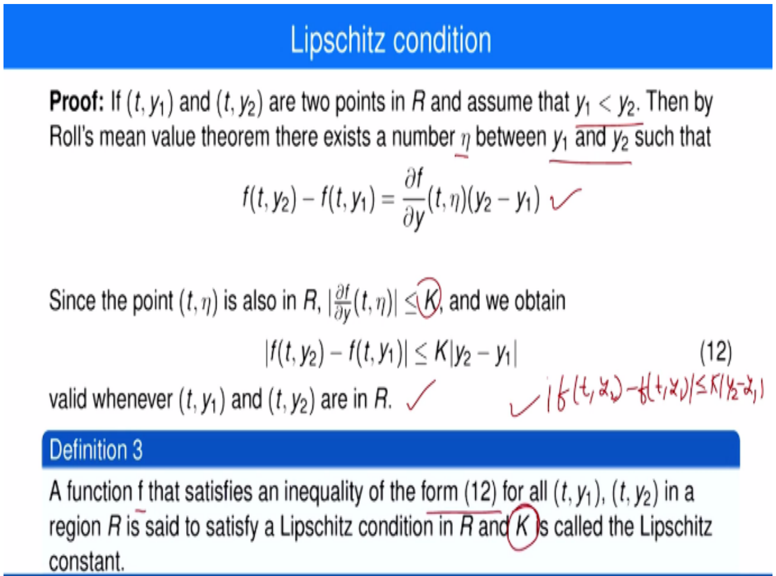

So, proof is not very difficult, let us look at here, if t, y1 and t, y2 are 2 points in R and assume that y1 is < y2 without loss of generality then by Rolle's mean value theorem, we can write down that they exist a eta between y1 and y2 such that fty2 – fty1 is = dou f/ dou y t, eta y2 – y1 and we already know that this t, eta is bounded in R, so it means that dou f/ dou y at this point is bounded by K that we have already assumed here that dou f/ dou y is bounded by this scale.

So, using this we can write this result as modulus of f ty2 – f ty1 is < = K times y2 – y1, valid; whenever this ty1 and ty2 are in R, so it means that if dou f/ dou y is continuous in closed rectangle R then it satisfy this condition and we define this condition as Lipschitz condition, so a function f that satisfy an equality of the form 12 for all t, y and t, y2 in a region R is said to satisfy a Lipschitz condition in R and the constant K is called the Lipschitz constant.

So, it means that if we have this condition that f t y2 – f t y1 is < = K times y2 – y1 modulus, then we say that f satisfy the Lipschitz condition and we will see that if f satisfies the Lipschitz condition then the initial value problem has a unique solution that is what we wanted to prove.

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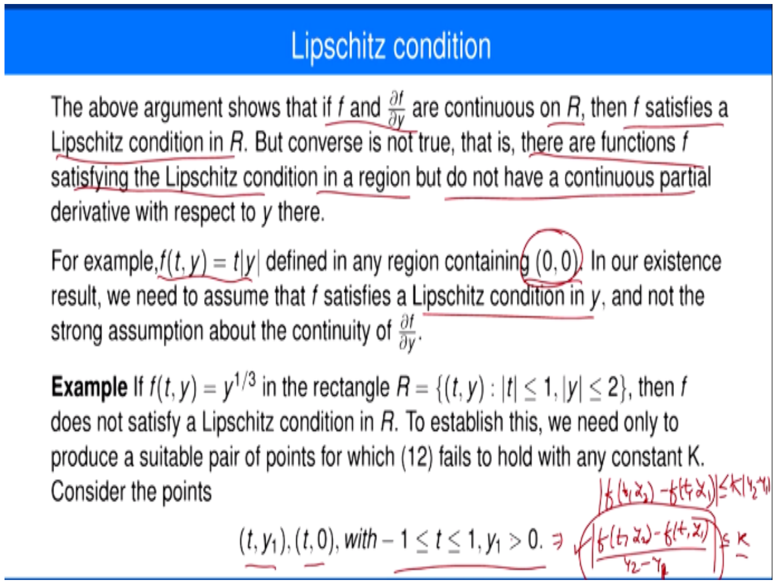

Let us discuss some more point about it, we have already shown in the previous lemma that if f dou f/ dou y are continuous in R, then f satisfies the Lipschitz condition that we have shown, but converse may not be true in fact, they are functions f satisfy the Lipschitz condition but in a particular region but they do not have continuous partial derivative with respect to y for example, if you consider this f ty = t modulus of y, we can check that it satisfy the Lipschitz condition in a region containing 0 0.

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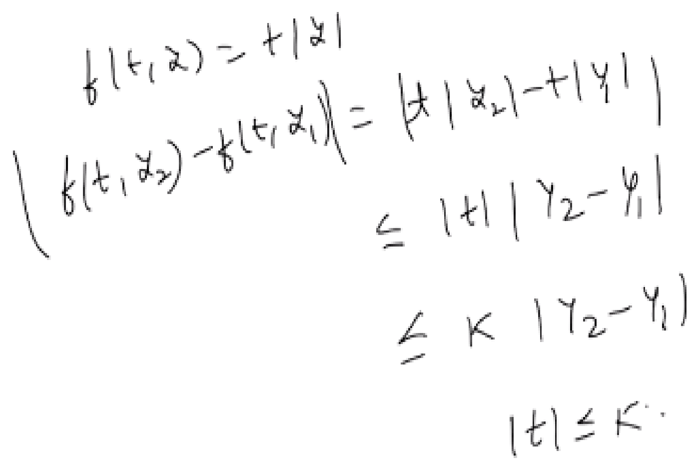

So, this is not very difficult to show we can simply say that is f ty = t modulus of y, so f t y2 – ft y1, you can write it t modulus of y2 – t of y1 and you can check that if you take the modulus here, then you can simply check that it is modulus of t and you can write it this is y2 – y1 here and if t belongs to some bounded domains, you can always write it like this, y2 – y1, if this t; so it means that this function f ty = t modulus of y satisfy a Lipschitz condition in y.

But here we can say that region containing the point 0 0 then this may not satisfy the differentiability criteria that is dou f/ dou y may not exist in this region, let us consider one more example here, this example says that it may happen that a particular function may not satisfy the Lipschitz condition in one region, but it may satisfy the Lipschitz condition in another region, it may happen.

For example, consider this f ty = y2 power 1/3 in the rectangle R, where R is defined as modulus of t < = 1, modulus of y < = 2 then f does not satisfies the Lipschitz condition that we want to check that we can check like this that f of t y2 – f of t y1, we want to show that it is K times y2 – y1 or you can simply say that ft y2 – ft y1 divided by y2 – y1 is < = K, so it means that if this quantity is bounded by some K, we say that f ty satisfy Lipschitz condition.

And to show that it is not satisfying the Lipschitz condition, you want to show that this quantity is on bounded, so let us take a pair t y1 and t, 0 in a rectangle that this t from -1 to 1 and y1 is > 0 here.

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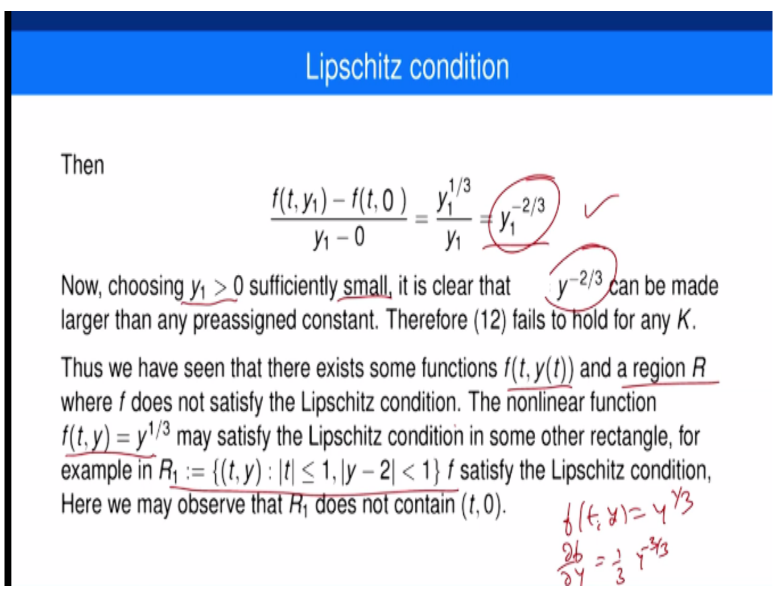

So, if we take this, look at this quantity ft y1 – ft y0 divided by y1- 0 and since y0 is 0, you can simply say that it is nothing but ft y1 that is y1 to the power 1/3 divided by y1 here and if we simplify you can get y1 to the power -2/3. Now, if we choose y1 > 0 sufficiently small close to 0 then this quantity is on bounded, so it means that this y to the power -2/3 cannot be bounded by, it means that it is clear that K = y to the power -2/3 can be made larger than any preassigned constant.

Therefore, this 12 that is f ty = y to the power 1/3 is not satisfying the Lipschitz condition here, so thus we have seen that they exist some function ft yt and it is an R where f does not satisfies Lipschitz condition but if you look at if you change the region here, the non-linear function f ty = y to the power 1/3 may satisfy the Lipschitz condition in some other rectangle that is this that R1, where it is set of all ty such that modulus of t is < = 1 y – 2 < 1.

So, here we have removed the point that is t, 0 here, so that point is missing here because if you look at this is creating problem in the neighbourhood of 0, so it means that in R it may not satisfy the Lipschitz condition but in this R1, it is satisfying the Lipschitz condition, so that you can check in fact, you can check that f ty = y to the power 1/3, this you can calculate dou f/ dou y and it is coming out by 1/3 y to the power -2/3.

And this defined if y is not defined in the neighbourhood of origin, so in this rectangle, this is perfectly valid and you can say that here, it satisfy the Lipschitz condition. So, we have discussed the Lipschitz condition here now, with this let us stop here and we will continue our discussion in next lecture, so what we have done in this lecture; we have defined the equivalence of initial value problem with the integral equation.

And we have define the sequence of approximating solution and we have also discuss the condition imposed on the non-linear function ft y and we say that if f satisfy the Lipschitz condition then the sequence of approximating solution will converge to a limit which is the solution of the required initial value problem that is the content of existing and uniqueness theorem, so that will continue in next lecture.

So, here we will stop and will continue in next lecture, thank you very much for listening, thank you.