Dynamical Systems and Control
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Lecture – 29
Lyapunov Stability - I

Hello friends, welcome to this lecture, in this lecture we will discuss the Lyapunov stability of dynamical system in fact, we have already discussed some concept of stability in previous few lectures also but in this lecture, in all those lectures when we try to discuss the say, stability of linear system then it is a problem of finding the say, sign of real part of the eigenvalues of the linear matrix A but when we try to solve the; when we try to find out the stability of non-linear system, then we consider the corresponding linear system.

And then we try to find out the this similar thing which we have solve which we have done for linear system, so in all those cases, we basically try to find out some solution and then we do all this thing but in case of Lyapunov stability, we need not to find out solutions of the dynamical system in fact, here we try to find out a function which is known as a Lyapunov function having certain properties, so here without solving without finding the solution of dynamical system, we can find out the stability behaviour of the dynamical system.

In fact, one more thing which we can obtain through the Lyapunov stability or with the help of Lyapunov function that here we can find out the region in which your dynamical system is stable, unstable or say asymptotically stable thing, so here the new thing is that we can find out the region in which your solution is stable here, so these things we can obtain with the help of Lyapunov function but the difficult part in case of Lyapunov function is to construct the Lyapunov function.

So, if you are a good enough to find out the Lyapunov functions, then we can achieve all these benefit here, so in fact there is no general method to find out the Lyapunov function for given dynamical system, so all this thing we try to discuss in this and the next lecture so, let us first start the Lyapunov stability of a non-linear system.

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Lyapunov stability of autonomous system

Consider an autonomous system

$$x' = f(x) \qquad (1)$$

where $f \in C[\mathbb{R}^n; \mathbb{R}^n]$.

Assume that f is smooth enough to ensure the existence and uniqueness of the solutions of (1). Let Ω be an open set in $\mathbb{R}^n$ containing the origin.

So, first consider an autonomous system x dash = f of x, here f is a continuous function from Rn to Rn, so it is basically n cross 1 system, x is a n cross 1 vector and assume that f is a smooth enough to ensure the existence and uniqueness of the solutions of 1, so it means that here we are assuming that dynamical system is having solutions and solutions are satisfying the uniqueness conditions also.

So, let omega be an open set in Rn containing the origin here, so now the function; Lyapunov function V which we are talking about, it satisfy certain properties, let us do some basic concept and try to identify those properties, so first let us define certain things so, suppose Vx is a scalar continuous function defined in a omega, so omega is already defined, omega is an open set in Rn which contains the origin.

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Suppose $V(x)$ is a scalar continuous function defined on Ω. Then

1. A scalar function $V(x)$ is said to be positive definite on the set Ω if and only if $V(0) = 0$ and $V(x) > 0, \forall x \neq 0$ and $x \in \Omega$.
2. A scalar function $V(x)$ is said to be positive semi-definite on the set Ω when V has the positive sign throughout Ω, except at certain points where it is zero.
3. A scalar function $V(x)$ is negative definite(negative semi-definite) on the set Ω if and only if $-V(x)$ is positive definite(positive semi-definite) on the set Ω.

$V(x)$ PD $\quad -V(x)$ ND

$V(x)$ PSD $\quad -V(x)$ NSD

So, V is a scalar continuous function defined on omega, then we a scalar function Vx is said to be a positive definite on the set omega, if and only if V of 0 is 0 at, it means that at origin, it is 0 and in all other point, it is positive, so Vx is positive for all x which are nonzero and x belongs to omega, so it means that positive definite means that it can it is taking value at origin and in all other point, it is taking only the positive value.

So, in this case we say that Vx is a positive definite on the set omega, where it satisfy this properties. Now, second is a scalar function Vx is said to be a positive semi-definite on the set omega, when V has the positive sign throughout omega except at certain points where it is 0, so it means that positive semi definite means it is taking only the nonnegative values, so it is taking either the value 0 or the positive value, right.

So, it means that there are certain points where it is 0, in all other points, it is taking only the positive value, so in that case we say that Vx is a positive semi definite on the set omega similarly, we can define the negative definite, so a scalar function Vx, negative definite or negative semi definite on the set omega if and only if - Vx is positive definite, so it means that Vx is negative definite means that - Vx is positive definite.

So, here we simply say Vx is negative definite, means - of Vx is positive definite, similarly Vx is negative semi definite means, - of Vx is positive semi definite here on the set omega, so here Vx is positive definite then - of Vx is negative definite and if Vx is positive semi definite, then - of Vx is negative semi definite here. So, in this way we can define the positive definite, positive semi definite, negative definite and negative semi definite.

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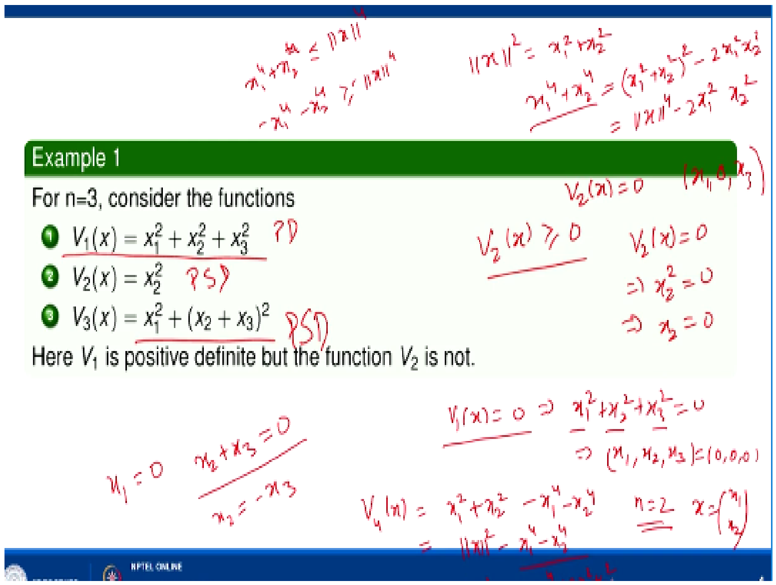

So, let us take some example to identify this, so first let us say for n = 3, now consider some functions, so here first function is V1x = x1 square + x2 square + x3 square, so since it is sum of a squares so, it will always take the value 0 or say, nonzero, positive values, so if you look at when it will take the value 0, so V1 x = 0, implies that x1 square + x2 square + x3 square = 0, so it means that this is possible only when x1, and x2 and x3 are all 0.

So, it means that it is taking value only a 0, only on the; only at the origin and in all other point this is basically, positive value, it is sum of positive values, so it is taking the positive value, so it means that it is taking 0 value only at the origin and in all other point where x is nonzero, it is taking the positive value, so V1x is your positive definite scalar function now, similarly we can define V2x = x2 square.

Now, here again we can say that since it is V2x is given as a square of some x2, so it means that V2x will take only say nonnegative value and if you look at if you put V2x = 0, then this implies that your x2 square = 0, so this implies that x2 = 0 but it gives no condition on the other component x1 and x3, so it means that that V2x will be 0 provided we have all this kind of a structure here.

So, x1, x2 and x1 0 x here, so it means that on x1 x3 plane, where x2 = 0, V2x may take the zero value, so it means that it is taking positive value, nonnegative value throughout the domain and it is taking 0 value other than origin also, so in this case V2x is a positive semi definite function, so here it is PD positive definite, it is positive semi different here, so it is taking only the nonnegative value here.

So and it is a taking value 0 other than origin also, so it is a positive semi definite function, now defined V3x = x1 square + x2 + x3 whole square, again it is given as some of the squares function, so it is basically taking nonnegative value. Now, we need to find out that at what point it is taking the value 0, so here it is taking the value 0 provided that x1 = 0 and x2 + x3 = 0, so here this may gives you the entire plane.

So, we can say that x2 = -x3, so it is, it may take value other than origin, so it means that here it is taking nonnegative value and it may take value zero other than origin also, so in this case your V3 function is again a positive semi definite function. Similarly, we can define one more function say V4x as x1 is square + x2 square – x14 – x24, here let us take n = 2 here, so it means that we have only x as x1 and x2 here, right.

So, if we look at this V4 x square, now it is quite difficult to find out whether it is taking all the positive value or negative value, so here if you look at x1 square + x2 square, I can write this as norm of x whole square – x1 to the power 4 - x2 power 4 here, now here we know that norm of x square is basically x1 square + x2 square, now what about this x1 to the power 4 + x2 power 4, if we add, so this I can write as x1 square + x2 square whole square of this – 2 x1 square and x2 square, right.

So, - of this will be; so I can write this as this is what; this is norm of x to the power 4, -2 of x1 square –x2 square, this –x1 square x2 square, so here these 2 are minus, okay, so this implies this, so here if you look at x1 to power 4 + x2 power 4 is some values which is < x2 power 4 here, so I can write that x1 to the power 4 + x2 to the power 4 is something which is less than so, is > =; sorry, this is x to power 4 here.

So, it means that here we are subtracting some positive value, some nonnegative value to get x1 to the power 4 + x2 to the power 4, so it means that x1 to the power 4 + x2 to the power 4 is < norm of x to the power, so - of x1 to the power 4 – x2 to the power 4 is > = norm of x2 to the power 4 here, so I can write this as this is > =; so here we can say that x1 to the power + x2 to the power 4 is written as norm of x2 to the power 4 – 2x1 square + x2 square.

So, if we use the value x1 to the power 4 + x2 to the power 4 as this, then we can write V4x as norm of x square -; in place of x1 to power 4 – x2 to power 4, we are writing –x2 to power; norm of x2 to power 4 + 2 x1 square x2 square.

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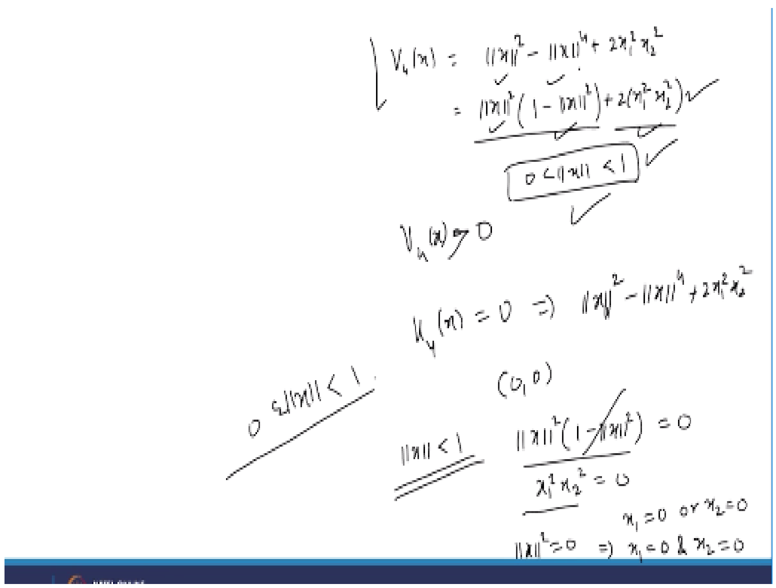

So, we can let us use this here, so V4x we have written as norm of x square - norm of x to power 4 +2 times x1 square x2 square, now we simply take the first 2 term, we can take that norm of x square 1 – norm of x square +2 times x1 square x square, so now if we define a domain like this that if norm of x is < 1 and > 0, then it will take this value is positive, this value will also be positive because norm of x is < 1 and this is also positive here.

So, it means that in this range, your V4x is all the time positive here and if V4x is 0 implies what; that norm of x square - norm of x2 to the power 4 +2 times x1 square x2 square, so here we can simply say that at 0, 0 it is taking value 0, so 0, 0 is certainly the point where V4x is 0 and if we put this condition that norm of x is < 1, then this is the only point where it is vanishing because in that case when norm of x is < 1, then this is positive, this is positive, this is positive.

So, some of these square will be positive, it means that this can be make 0, this will be positive if x1 square x2 square is 0 and this term is also 0, so this is possible, this cannot be; okay, so let me write it here, if norm of x is < 1, then this will be 0 provided norm of x square * 1 - norm of x square = 0 and here x1 square x2 square = 0, simultaneously, this implies that x1 = 0 or x2 = 0 here and this 0 implies this cannot be 0.

So, only possibility is that norm of x square = 0, so this implies that x1 = 0 and x2 = 0 here, so it means that if we restrict our domain as norm of x < 1, then V4x will take the value 0, only at the origin and no other point, so it means that V4x is a positive definite function in a domain, norm of x < 1, right and it is > =0, so in this, so in unit cycle, so in n the interior part of unit cycle, it is a positive definite scalar function, okay.

So, here we have defined this now, more, in general we can define the Lyapunov function as a quadratic form basically, so we will now discuss some concepts related to quadratic form, so let V of x is given as summation i, j = 1 to n, B ij xi xj, here xi and xj are say component of x, so here it is x1 to say xn here, right.

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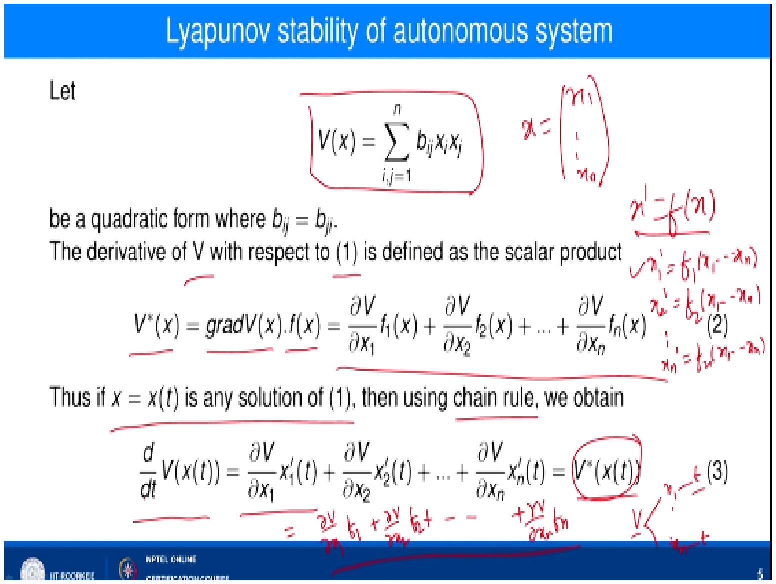

So, with the help of x, we can define Vx as this, i, j = 1 to n bij xi xj where bij = bji and we call this as quadratic form, so now with the help of this quadratic form, let us find out the derivative here, we will see that how this calculation of derivative is quite important, so that derivative of V with respect to 1 means, x dash = f of x is defined as the scalar product, V star x = grad of V of x dot f of x.

So, grad of V of x will be dou V/ dou x1, dou V/ dou x2 and dou V/dou xn and then f of x means your f of x is basically f1x, f2x and so on, so when you take the vector product say dot product, then it is dou V/ dou x1 * f1x+ dou V/ dou x2 f2x + so on dou V/ dou xn fnx here, so V star x we defined as grad Vx dot f of x here and it is given by the expression given into. Now, if x =; xt is any solution of 1, means x dash = f of x.

Then we can; in fact, use same rule to obtain d/dt of V of x t, so here it is given as any x1 to xn, may not be the function of n here, so it may not give function of t, here, so now let us assume that x is a function of t and now find out d/dt of V of xt, so this can be done by choosing the; while using the chain rule, so here V is the function of x and x is a function of t, so to find out d/dt of V of xt, so here it is V is a function of say x1 to say xn.

And each one is a function of t, here, so we to find out dV/dt here, we simply say derivative of V with respect to x1 and derivative of x1 with respect to t, so dou V/ dou x1 and dx1/dt + dou V/

dou x2 dx2/dt and so on, so now if you look at here, if you look at the 1 here, so this I can write it as x1 dash = f1 x1 to xn, x2 dash = f2 x1 to xn and so on xn dash = fn x1 to xn here, so using the expression for x1 dash, I can write this as douV/dou x1, in place of x1 dash, we can write f1.

And dou V/ dou x2 in place of x2 dash we can write f2 and so on, so if you look at this is nothing but dou V/ dou x1 and this is f1 + dou V/ dou x2 is f2 and so on dou V/ dou xn fn, so if we look at this is nothing but this expression given into, so we can write this as V star of x of t, here, so by d/dt, V xt is nothing but V star xt, so it means that this the derivative of V with respect to 1 which is a given as V star x is given by d/dt of V of xt, when x is a function of t, here, right.

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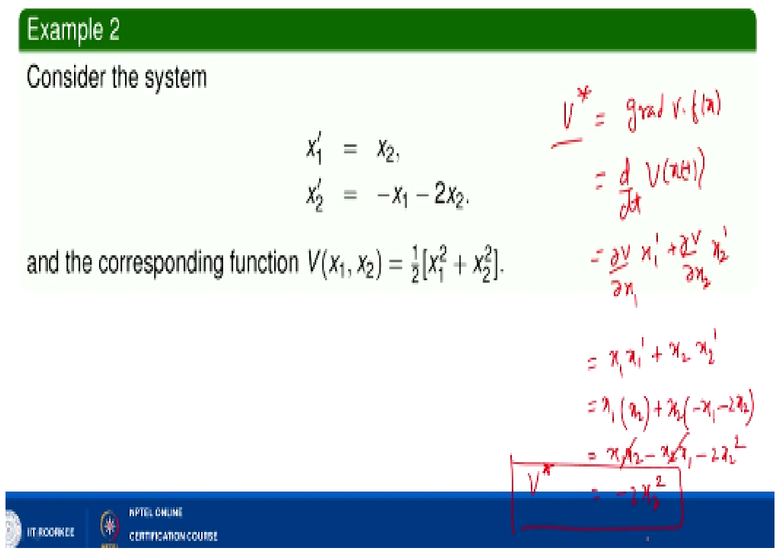

Now, let us consider the system x1 dash = x2 and x2 dash = - x1 – 2x2, now here we try to find out the v star here, so first look at let us say that V x1, x2 is 1 upon 2 x1 square + x2 square and we want to find out V star here, so V star is basically what; V star is nothing but grad of V dot f of x, right or we can say that it is nothing but d/dt of V of x of t, here, right so we will use this thing.

So, if we use this then it is what; it is dou V/ dou x1 and x 1 dash + dou V/ dou x2 x2 dash, so what is dou V/ dou x1; so if you look at here we can find out dou V/ dou xn is nothing but simply x1, it is 2 times x1 and 1/2 is already there, so we can simply say it is nothing but x1, so it is x1 x1 dash +, similarly dou V/ dou x2 it is 2 times x2, 2 is can be cancelled out from this, so it is again x2 and x2 dash here.

Now, we already know what is the expression for x1 dash and x2 dash, so we can write x1 dash is x2 here + x2, x2 dash is –x1 -2 of x2 here, so if you simplify x1, x2 – x2 x1 – 2 of x2 square so this will be cancel out and it is nothing but -2 of x2 square, so V star is given as - of 2 of x2 square here, so here this V star is given as -2 of x2 square, we can easily check that this is nothing but negative semi-definite function.

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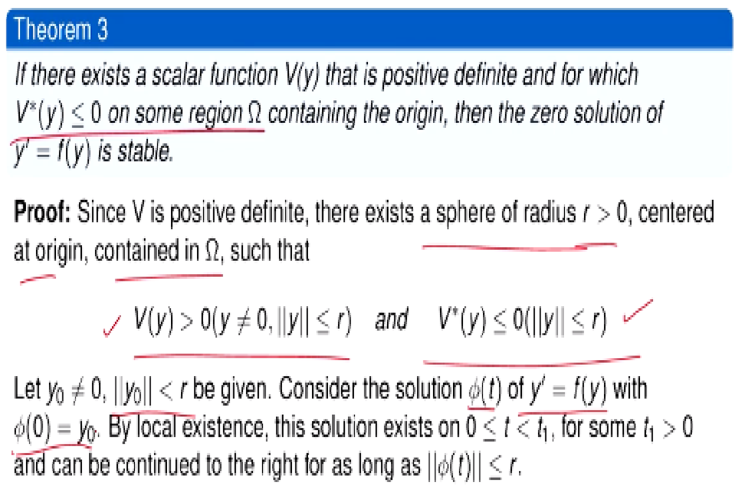

Anyway, so now once we have something related to this function vx, now let us consider the main theorems for finding the stability of given system, so one important theorem is given as theorem number 3 three. So, it says that if they exist a scalar function V of y that is positive definite and for which V star y is < = 0 on some region omega, which contains the origin then the zero solution of y dash = f of y is stable.

So, here we are assuming that zero is a solution of y dash = f of y and we are able to find out a function V of y which is positive definite and V star y is negative semi-definite, in that case origin which is zero solution is a stable solution, so basically it is what; we are able to find out a positive definite function whose derivative is negative semi-definite, then we are saying that y dash = f of y has a zero solution which is a stable solution, okay.

So, since V is positive definite, then they exists a sphere of radius r > 0 centred at origin containing omega and such that that V of y is > 0 y != 0 and norm of y is < r, < r = r and V star y is < = 0 norm of y < = r, so here the first expression is given with the help of positive definiteness of V of y, so since V of y is positive definite, so it can take value 0 only at origin and in all other point, it is taking the positive value only.

So, V of y is positive when y ! = 0 and norm of y is < = r, we are considering everything in the inside the sphere of radius r > 0 and it is already given that V star y is < = 0 on some region omega containing the origin, so this sphere is inside you know, omega, so this also be true that V star y is < = 0 norm of y < r. Now, let us take one initial point say y0 which is nonzero and it is inside your omega, inside your sphere of radius r.

So, it means that norm of y0 is < r be given, now we find out one solution phi t of y dash = f of y with phi of 0 = y0, so let us consider a solution of initial value problem that is y dash = fy with the condition that Phi of 0 = y0, so we; so here I am assuming that f is smooth enough, so it means that we assume that solution exists so, by local existence, this solution exist on some interval say, 0 t < = t1, for some t1 > 0.

So, this is a standard existence theorem that if f is smooth enough, then this initial value problem has a solution in fact, if we take more smoothness on f, then we can have a unique solution here and this can be continued to the right of t, t1 as long as that norm of phi t is < = r, so we keep on applying the same result and we can say that the solution of y dash = fy with phi of 0 = y0, can be continued, can be extended till the right of your t1 also as well as t1 also, only till your norm of phi t is < = r.

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Suppose that $[0, t_1)$ is the largest interval of existence of the solution $\phi(t)$ that can be achieved by continuation. Then either $t_1 = +\infty$ or $0 < t_1 < +\infty$. We will show that for small enough $||y_0||$, second case, i.e., $0 < t_1 < +\infty$ can not arise. Since

$$\frac{d}{dt}V(\phi(t)) = V^*(\phi(t)) \leq 0, \qquad (0 \leq t < t_1)$$

integrating

$$V(\phi(t)) - V(y_0) = \int_0^t V^*(\phi(s))ds \leq 0$$

$y' = f(y), \phi(0) = y_0$

$\phi(t) \neq 0$

Therefore,

$$0 < V(\phi(t)) \leq V(y_0) \qquad (0 \leq t < t_1) \qquad (4)$$

where the inequality follows from the assumption $y_0 \neq 0$.

It means that phi t is inside the sphere of radius r, so suppose that 0 to t1 is the largest interval of existence of the solution phi t, so it means that by doing, by repeated application of existence theorems, suppose this is the interval 0 to t1 is an interval where your solution exists, so phi t that can be achieved by the continuation. Then claim is that either t1 is < infinity; = infinity or t1 is < infinity.

Now, we wanted to show that if norm of y0 is small then and t1 is < infinity, then we can get some contradictions, so it means that for small enough norm of y0, the second case that is t1 is < infinity is not possible, so idea is to show that if we start, if we take a solution inside the sphere or then it will remain all the time in the sphere of radius r only, so it means that when t1 is tending to infinity, then also your solution is will remain inside the sphere of radius r.

So, it means that the possibility that t1 is < infinity may not arise, so t1 has to be = infinity, so to show that this is true, we use this condition that V star phi t is < = 0 and we have already shown that V star phi t is nothing but d/dt of V of phi t here, so through t < = t1, sorry t > = 0 < t1 d/dt of V phi t is < = 0, this is already given that V star phi t is negative, so we simply integrate with respect to t here, so we will get V of phi t – V of y0, from 0 to t V star phi of s ds.

Now, V star phi of s is < = 0, t is positive, so this will be < = 0, so it means that your V of phi t is < = V of y0 which we have achieved from this. Now, we already know that since y0 is nonzero, then phi t cannot be 0, why because if we have y dash = f of y and initial condition is that your phi of 0 = y0, now if y0 is nonzero, then if y0 is 0, then it will have only a unique solution that is zero solution.

But if y0 is nonzero, then zero solution cannot be a solution of this and by existence and uniqueness theorem, if phi t is a solution of this initial value problem then phi t is cannot be equal to a zero solution, so it means that phi t is not a 0, so it means that V of phi t has to be > 0 because V is positive definite, so it means that with the condition that V star phi t is non-positive and V is a negative definite, we have obtained that 0 < V phi t is < = V y0, for all t < t1 here.

So that is the inequality given in equation number 4, so where the inequality follows from the assumptions at y0! =0, so this inequality is followed from the assumption that y0 is nonzero, so it means that 0 is not a solution of this and hence phi t is all the time positive, nonzero.

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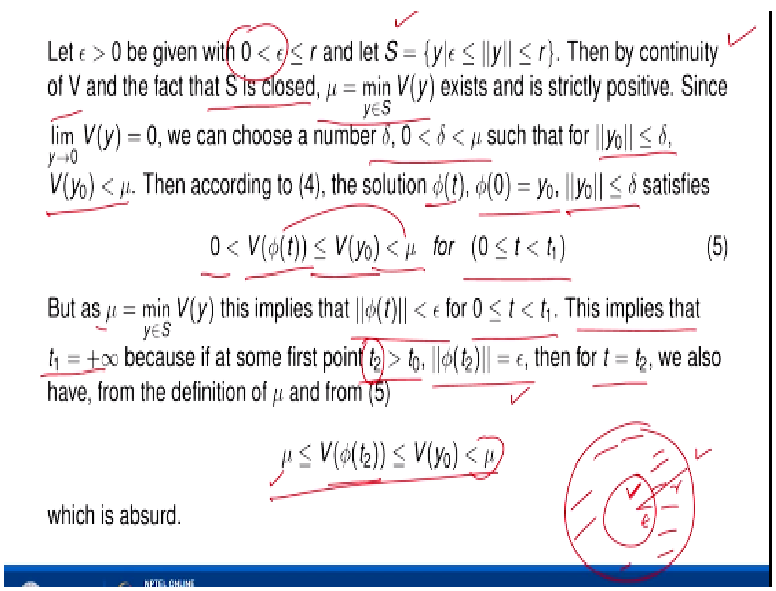

So, let epsilon > 0 be given with 0 < epsilon < = r and let S = all those y such that norm of y is < = r and > = epsilon now, what we try to prove now that if this 4 is true then your solution will remain very small if we take y0, norm of y0 very small, so that is what we want to show that we can, we want to find out for every epsilon, they exist a > 0, they exist a delta > 0 such that norm of this phi of t is < epsilon, whenever norm of y0 is < delta.

So, we want to show this thing, so let us say that corresponding to this epsilon, you define a domain like S, so let S = all those y whose norm is lying between epsilon and r, right and we assume that this then by basically, it is this, so it is epsilon and it is; so this is the reason which is given by this is, right so it means that S is close because it contains the bounded point also, so S is close and V is continuous.

So, we can use the (()) (31:42) theorem and we can say that continuous function define on a closed region will attain the maximum and minimum value inside your close region here, so let us say that Mu which we define as the minimum of V of y were y is running over this S, so it means that Mu exist and it is strictly positive, why so, why because norm of y is > = epsilon and epsilon is positive right.

So, it means that Vy take only positive value in this region S, right and it will not take any negative value, so here we simply say that Vy is strictly positive, it can take only value 0, when your y is 0, okay so since limit y tending to 0, Vy = 0, we can choose the number delta which is < this Mu such that norm of y0 is < = delta here and V of y0 is < Mu here. So, it means that we are choosing norm of y0 < delta and delta is < Mu.

So, it means that this norm of y0 is < Mu here, okay so we simply say that V of y0 is < Mu here, so here since the limit of y tending to 0, V of y = 0, so we can find out a number delta which is positive, such that norm of y is < delta and V of y0 is < the Mu which we have define as minimum of V of y here, then according to 4, the solution phi t with the condition that Phi of 0 = y0 and norm of y0 is < = delta, it satisfy the following equality that V of phi t is > 0.

And it is < = V of y0 and V of y0 is < Mu, so here we assume that norm of y0 is <= delta, where y0 is nothing but phi of 0, so this is true for t < t1 here, but we already know that what is Mu here, Mu is the minimum of V of y, y belongs to S, then this implies that norm of phi t has to be < epsilon for t between 0 to t1 here, why so because this implies that that t1 = + infinity, first of all, since V of phi t is < Mu, you can look at here.

Then your phi t cannot be the element of S here and S is this annulus region, so it means that your phi t is somewhere outside this annulus, so either it will be here or it will be here now, since phi t is tending to 0, then this V of phi t is to tending to 0, so it must be in a region which involved, which contain the origin, so it must be here, so it means that if phi t is here, so it means that norm of phi t must be < epsilon for, t < between; t lying between 0 and t1 here.

So, and this also implies that t1 = infinity because if it is not, then we can find out a sometime t2 > = t0, with the condition that phi of t2 = epsilon, then for t = t2, we also have from the definition of Mu, we can say that since phi of t2 is = epsilon, so it means that V of phi t2 to and since Mu is a minimum of all those; Mu is a minimum of values of Vy over S and phi t2 is belonging to S, so it means that V of phi t2 is > = Mu.

And V of phi t2 by 4, it is < = V of y0 and V of y0 is < Mu, so it means that this condition cannot hold true, so it means that we cannot have any point t2 where norm of phi t2 = epsilon, so it means that norm of phi t2 is < epsilon and this t1 is all the time + infinity, right.

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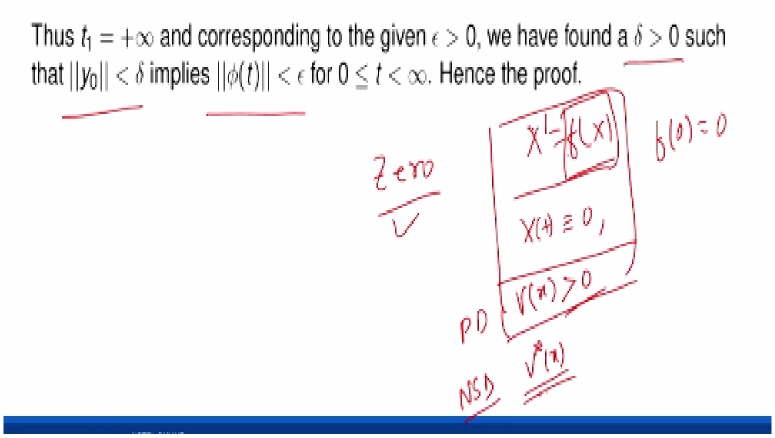

So, it means that thus t1 = + infinity and corresponding to the given epsilon > 0, we have found a delta < 0 such that norm of y0 is < delta implies that norm of phi t is < epsilon for 0 lying between t is < infinity and hence the stability of the solution phi t is proved here, so solution here the stability of zero solution is proved here, so it means that if given a system x dash = f of x where f is smooth enough, then a zero solution of this, it means that xt identically = 0 here.

We are assuming that f of 0 = 0, so 0 solution of this system is stable provided that they exist a scalar continuous function V of x which is positive definite, so V of x is positive definite and v star x is negative semi definite. So, here if you look at the expression of V of x is we have not said that it is somewhere related to x dash = f of x, so here our; how to find out V of x is up to us such that the V star x, it is the derivative of Vx along the system x dash = f of x satisfying this condition that it is negative semi definite.

Then your zero solution is stable solution, so if you look at these theorem, then here the important part is to construct this function V of x and here if you look at we have not obtained the solution of x dash = f of x, what we require is in finding the V star x, the expression of f of x which is already given here, so without finding the solution of this dynamical system, we are able to predict the stability of zero solution here.

And hence sometimes we call this method as Lyapunov direct method for finding the stability of the zero solution. So with this, we conclude our this lecture and we will continue discussing some more result based on the Lyapunov function, so in next lecture, we will discuss some more properties telling about the asymptotically stability of zero solution and unstability of zero solution, in fact equilibrium solutions and how to construct Lyapunov function in some important cases, so with this we end here and thank you very much for listening, thank you.