Dynamical Systems and Control
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Lecture – 28
Limit Cycle

Hello friends, welcome to this lecture and if you recall in previous lecture, we have discussed one example stating that the behaviour of the non-linear system is maybe very different from the behaviour of the linear system in fact, we have pointed out that in nonlinear system, it may happen that the orbit of a close curve; orbit of a solution which is a close curve may spirally approach by the solutions; the orbit of the other solutions.

And that orbit is important in the sense that it will correspond to an orbit of a periodic solution and we had discussed the Poincare-Bendixsen theorem to find out the existence of the orbit which is corresponding to periodic solution. So, in previous lecture, we had discussed the Poincare-Bendixsen theorem which will give you the existence of periodic; nontrivial periodic solution.

But we have also noted down that it gives just the existence, it will not discuss which curve is corresponding to the curve of periodic solution, so in this lecture we will try to focus on the curve which will the orbit, which will correspond to the orbit of periodic solution, so here we will start this lecture by defining the limits cycle and with the help of limit cycle, we try to find out that this will correspond to the orbit of the periodic solution, okay.

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Definition 2

A closed curve C is called a limit cycle of

$$\frac{dx}{dt} = f(x, y),$$
$$\frac{dy}{dt} = g(x, y), \qquad (28)$$

if orbits of (28) spiral into it as $t \to \infty$ or $t \to -\infty$. A Limit cycle is called a stable limit cycle if all the orbits of (28) sufficiently close to this must spiral into it otherwise it is called an unstable limit cycle

So, let us decide the; define the limits cycle, okay, now let us take this following definition that the close curve C is called a limit cycle of dx/dt = f xy dy/dt = g xy, if orbits of this spirals into it as t tending to infinity, so it means that the orbit corresponding to non-periodic solution is called as limit cycle, so it means that a limit cycle is called stable limit cycle, if all the orbits of 28 sufficiently close to this at any given time t must spiral into it.

Otherwise it is called an unstable limits cycle, so it means that first of all what is limit cycle; limit cycle is the orbit; closed orbit on to which your solutions are spiralling onto which, okay, so limit cycle is this, now limit cycle may be of different type; stable limit cycle and unstable limit cycle, when it is called a stable limit cycle; if we take a orbit which is at some time it is near to that orbit must be is spiral on to this orbit; close orbit.

Then your limit cycle is known as stable limit cycle otherwise, it is called an unstable limit cycle, now next example is find all limit cycle of the system x dash = x – xu – xy square and y dash = y – y cube – y of x square.

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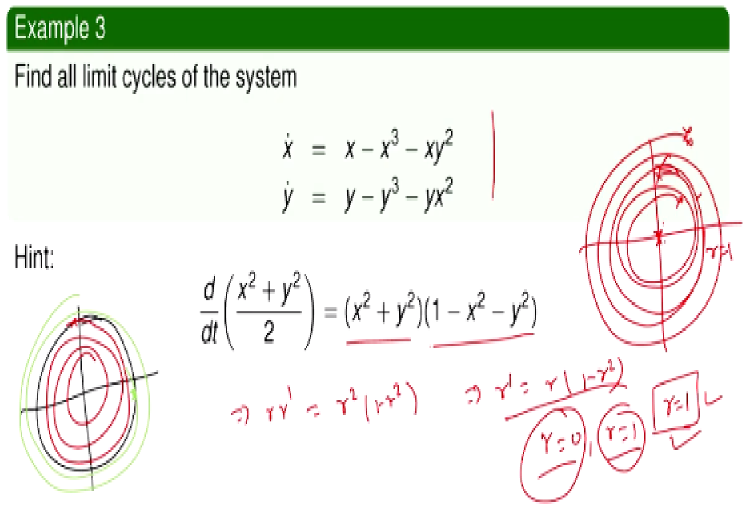

So, here we can find out like as I pointed out that try to find out d/dt of x square + y square upon to; this is nothing but r of r dash, so if you look at this nothing but r square * 1 – r square, so it means that it is r square 1 – r square, so we can say that r dash = r * 1 – r square, right. Now, here we want to find out the limit cycle, so limit cycle is basically if you look at the this will critical point is r = 0 and r = 1 here, r = -1 is; so it means that here we have 2 critical point of this, 0 and this one.

Now, I am saying that this r =1 is the limit cycle, right and we can check that it is a stable limit cycle, so we can say that if we take as we pointed out here, if we take any point here then it will say move away and it will try to reach on this curve r = 1 and if we take any solution outside, then dr/dt will be decreasing and it will try to come on to this, so we simply say that here r = 1 is a close orbit, is a orbit, right.

So, we simply say that there are 2 orbit r = 0 and r =; in fact, it is a critical point of the system r dash = r * 1 – 1, so r = 1 is one of the orbit; closed orbit now, we want to show that this r = 1 orbit is nothing but a limit cycle and that we can show that if we take any solution inside your region r, let me look at here, you can use this thing, so here we have and now, if we take solution which is here, then it will try to move away and try to be on the limits cycle.

Once it is there, it will always remain there and similarly, if we look at some other solution or say, which is outside this then it will always try to be on the say r = 1 and as soon as it is on r = 1, it will always be on the orbit r = 1 here. So, now which it means that your r = 1 is your orbit and all the other solutions are spiralling onto r = 1 and hence by definition r = 1 is a limit cycle and now, if you want to look at the stable and unstable limit cycle, then we can simply say that if you look at this solution somewhere here.

And it is near to r = 1, then we can see that it is always tending towards r = 1 and similarly, if we take any other solution which is outside unit cycle it will also coming to r = 1, so it means that in this case r = 1 is a stable limit cycle right, so it means that it is the limit cycle and it is a stable limit cycle similarly, if you look at r = 0, it is also a trivial limit cycle and we can check that any solution which is near to origin it is going out of the; means, going away from the origin.

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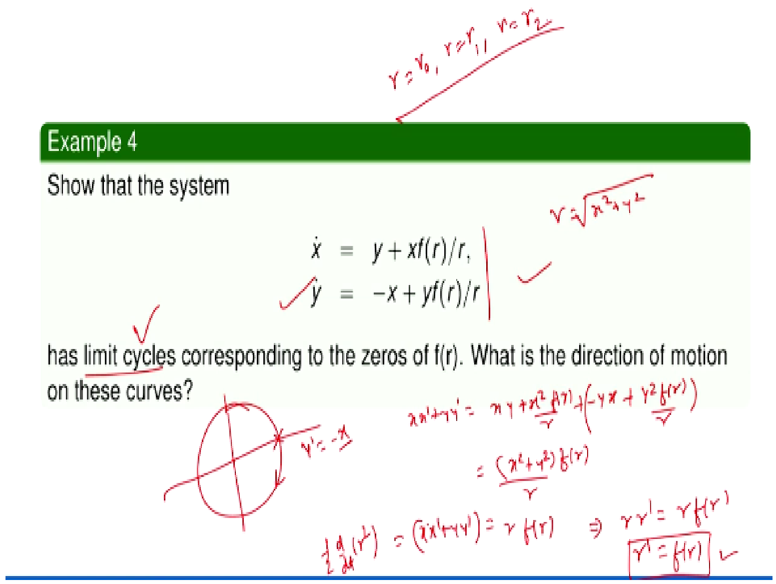

So, we can say that zero is a trivial limit cycle which is unstable limit cycle. Now, next example is a generalisation of the previous example showed that the system x dash = y + x + fr upon r, y dash = -x + y fr upon r, where r = x square + y square under root has limit cycle corresponding to the zeros of, f of r and what is the direction of motion of these curves that we wanted to find out so, if we look at how we look at so, xx dash + yy dash, if you want to find out.

Then it is xy + x c f of r upon r + - y of x + y square f or r upon r and if you look at this is nothing but x square + y square f of r upon r, so it is r * f of r, so xx dash + yy dash = r f of r, now this I can write as 1/2 of d/dt of r square, right, so now I can rewrite this as rr dash = r f of r, so it is r dash = f of r. Now, zeros of, f of r will give you the equilibrium solutions of fix or critical points of r dash, right.

This r dash = f of r, so it means that g rows of this will give you say closed curves, right and now we can check that these close curves are nothing but limit cycles and the direction will be given by this thing, so it means that suppose we have an orbit like this now, we can check that at this point, your x is positive and y dash is; sorry, y is = 0 and x is positive, so the direction will give you by this that y dash = -x + y = 0, so that part is gone.

So, y dash = -f of x, so here x is positive, so y dash is negative, so it means that y is has a tendency to decrease, so it means that the direction is clockwise, right, so it means that the zeros of f or r gives you a close curves which we can show that it is limit cycle and the direction of your movement on the orbits are downwards means, clockwise, so that is what it is given here, okay here.

Now, we say that suppose we have not proven that limit cycles, these are limit cycle, so for that let us say that f of r has to wrote say say r0 and r1, so it means that it has not an r1 which is on one is bigger than r0, so we can use wavy curve method and we can write this thing, right and with the help of this, you can check that which one is a stable limit cycle and which one is the unstable limit cycle here.

So, we have seen here that it is r dash = f of r here and we can also check the direction of these orbits here, so the zeros of, f of r will give you the critical point of r dash, it means that r = say, r1; r0, r = r1, r = r2, if these are zeros r1, where r0, r1, r2 are zeros of, f of r, then this will give you the critical point of r dash, it means that these will give you the orbits of the solutions of this nonlinear system.

And we can see that the moment of your solution on these orbits are given as clockwise that we have pointed out from this, now our claim is that these orbit which are the zeros of, f of r, are basically the limit cycles and correspond to the periodic solution, so solution corresponding to the periodic solutions. So, now let us say that these are orbits or not r1 and r2 are your limits cycles.

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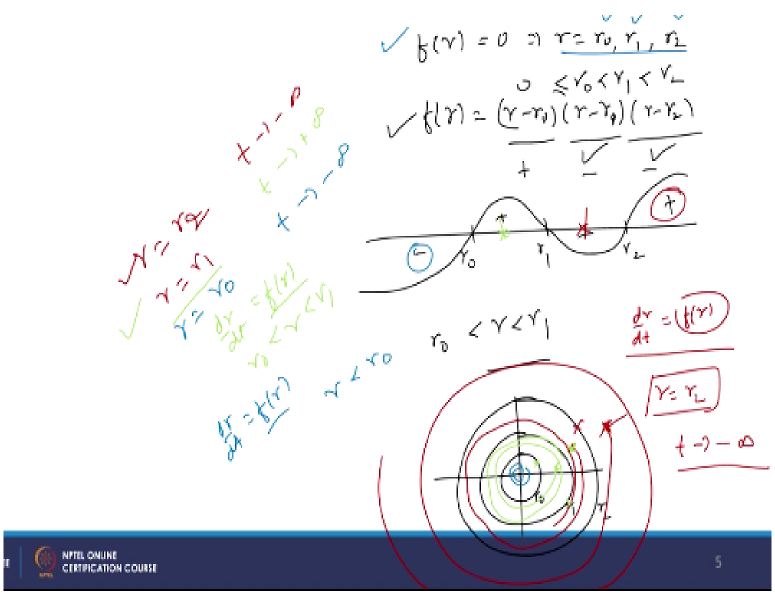

So for that let us look at here that f of r = 0 for r = r0, r1 and r2, so let us arrange in an increasing orders so, r0 is < r1 and r and it is something like that so r0 is certainly positive here, in fact it is > r = 0 here, so let us have these things, so r0, r1 and r2, so why wavy curve method, I can write it f of r = r – r0, r – r1 * r – r2 and we can have wavy curve method like this, it is minus plus, minus plus, it means what; that if r is bigger than re, then what is the sign of, f of r?

So, if r is bigger than r2, then this will be positive, r is bigger than r2 and this will again be positive and all are positive, so f of r is positive here, a so if f of r = r – r0 r – r1 r – r2, then we can use wavy curve method to find out the sign of, f of r as follows, so if it +, so when r2 is; r is bigger than r2 and this - fr is lying between r1 and r2 and it is + again, when r0 is lying between; r is lying between r0 and r1 and negative r is < r0 that you can verify.

Let us say that take r as somewhere between r0 and r1 here, then this r – r1 is here it is, negative right, r – r1 is negative here and r – r2 is also negative, so it is negative and r – r0 is positive, so it will be positive here, so f of r will be positive here. Now, with the help of this we try to check the that these are limit cycle, so this is corresponding to r0, this corresponding to r1, this is corresponding to r2 here.

Now, we want to check whether this will be limit cycle or not, so first if you look at this r = r2 here, so it means that it is one of the orbit, first of all, it is clear. Now, if I take any point r here, right, so it means that let us start with a solution which is somewhere here, so your starting is here somewhere, then if you look at dr/dt will be what; dr/dt = f of r here that we have pointed out here that dr/dt = f of r here.

Now, as r is bigger than r2, then r is bigger than r2, means this is positive, so it means that f of r for r bigger than r2 is positive, so it means that dr/dt is say positive, so it means that it is keep on increasing, so means that any solution which is start from this, it will try to move away from r = r2, so it means it is moving like this right, so it means that it is moving away from r = r2. Now, if I look at any solution which start in this region bigger than r1 and < r2.

And here it means that r is lying between r1 and r2, it means r is lying somewhere here, then your dr/dt = f of r and for that r, your dr/dt will be negative because f of r will be negative, if r is lying between r1 and r2, so it means that for that your dr/dt is decreasing, so it means that it is, it has a tendency to decrease the radius as t tending to infinity. So, it means that if it is start from this, it will try to decrease the radius.

So, it means that the r = r2 any solution which is near to r = r2 has a tendency that as t tending to infinity, your solution will move away from r = r2 or I can say that as t tending to – infinity, all the solutions which are near to r = r2 has a tendency to spiral on r = r2, please mind here that t = - infinity, so it means that this r = r2, which is a orbit of the given nonlinear system has a tendency that all the solution which is lying, starting near the orbit r= r2 has a tendency that as t tending to – infinity, all the solutions are approaching are r = r2.

So, it means that r = r2 is by definition it is a limit cycle and unstable limit cycle right, so we have shown that r = r2, all the solution which is start near r = r2 are approaching r = r2 as t tending to – infinity, so r = r2 is a limit cycle which is unstable. Now, look at r = r1 here, so for r = r1, we can look at this that if your solution start between which is bigger than r1, then we have seen that it is trying towards r = r1 here.

Now, if I look at the solution which starting somewhere between r0 and r1, then solution for r lying between r0 and r1, your f of r is positive, so it means that dr/dt = f of r, where if r is lying between r0 and r1, f of r is positive, it means that radius is keep on increasing, so it means that any solution which start in this annulus region will have a tendency that it will moving towards r = r1 here.

So, it means that if I take this solution which is lying between r1 and r2, then it will converge to this solution at r = r1 and if we take any solution inner to r1, then it is moving towards r1 itself, so it means that corresponding to r = r1, if we take any solution in the neighbourhood of r = r1, then as t tending to + infinity, all the solution will try to merge on the orbit r = r1, so it means that by definition the r = r1 is a close orbit onto which all other solution which are nearby to r = r1 will merge, try to merge on r = r1 as t tending to infinity.

So, it means that by definition r = r1, is a limit cycle which is a stable limit cycle and this will correspond to a solution which is periodic in nature, so r = r1 is a; the orbit of a periodic solution. Similarly, if you look at r0 here, we have already seen that if we take any solution between r0 and r1, it is moving towards r1 not r0, so if we take any solution here which is between 0 and r0, then here we have already seen that f of r is negative.

So, it means that dr/dt = f of r and if r is < r0, then fr are negative, so it means that it is trying to reduce the distance from the origin, so it means that here solution which is inner to your r0 will try away from r0, try to be away from r = r0, so it means that corresponding to r = r0, if we take any other solution which is in the neighbourhood of r = r0 will try to merge to r = r0 as t tending to – infinity.

So, in this case r = r0 correspond to a closed orbit, means it is corresponding to a periodic solution and all other solutions are merging to r = r0 as t tending to – infinity, so by definition this r = r0 is also a limit cycle which is unstable limit cycle. So, what we have shown here that if f of r = 0 for r = r0, r1 and r2, then the orbit corresponding to r0, r1 and r2 are limit cycles which may be stable or unstable depending on the situation here, right.

So, it means that the zeros of, f of r, all limit cycle and it will correspond; these limit cycles are correspond to the solution which are periodic in nature, so that is what we have concluded this that in this problem x dash = y + x fr/r, y dash = -x + y fr upon r has limit cycle corresponding to the zeros of, f of r and the direction of motion on these curves are clockwise that we have already pointed out.

So, this conclude the example 4 here, now next is the Bendixsen's nonexistence theorem, if you look at here in previous example, what we have shown here, here we have exactly shown here the existence of say limit cycle, now in many of the problem it may not be so nice here.

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Bendixsen's nonexistence theorem

Theorem Let D be a domain in x-y plane. Consider the autonomous system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y) \qquad (29)$$

where f and g have continuous first order partial derivatives in D. Then the system (29) has no periodic solution in the domain D if $f_x + g_y$ has the same sign throughout D.

So, here we have one negative theorem in the sense that this will negate the existence of limit cycle because for limit cycle, we must have 2 thing; first thing is that we must have a closed curve and the second thing is that all the other solution which in lying near to your closed curve must spiral onto the closed curve as t tending to + infinity or – infinity, then we say that the closed curve is a limit cycle and a stable and unstable will depend on the condition that your curves are approaching to the closed curve as t tending to + infinity or – infinity.

And many a times, it may also happen that it may happen that one-sided is approaching and a other side it is say moving away, so it means that that kind of thing we can say that it is half stable or half unstable but it is not called as stable or unstable, okay. So that kind of situation may also happen, so now let us look at this important theorem, which negate the possibility of limit cycle.

So, basically it will negate the possibility of closed orbit, so it says that let D be a domain in xy plane and consider the autonomous system x dash = f xy and y dash = g xy, where f and g have continuous first order partial derivative in that domain, then the system 29 has no periodic solution in the domain D, if this fx + gy has the same sign, it means that it is not changing the sign.

Then it will not have any periodic solution in that particular domain D, it means that if there is no periodic solution then there is no possibility of limit cycle, is that okay, so basically it is a negation of the Poincare-Bendixsen theorem, negation in the sense that it is giving you that they exist no limit cycle of the linear system not negation of the linear, Poincare-Bendixsen theorem, okay. So, let us have a proof of this.

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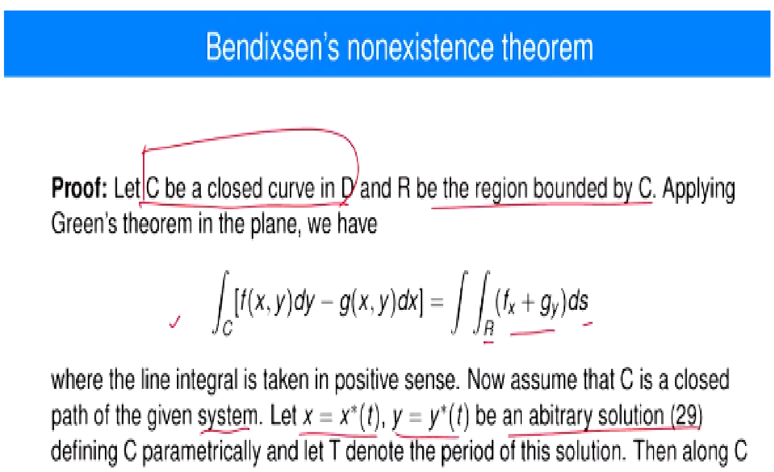

So, let us say that let C be a closed curve in D and R be the region bounded by C, so this is important that C be a closed curve in D, now applying Green's theorem of the plane, we can have integral of; line integral of fx ydy – g xy dx = double integral over R fx + gy d of s, where s is the surface integral, so basically it is a famous result say, connecting the surface integral and line integral.

And here the line integral is taken in the positive sense, anti-clockwise, now assume that C is a closed path of the given system and let x = x star and y = y star be an arbitrary solution defining C parametrically, so it means that let us say that x star and y star defined this closed curve and let us say that this C does not contain any limit point, critical point, so it means that this C correspond to a periodic solution given as x = x star and y = y star.

Now, we say that if fx + gy will not change sign in that domain D, then this is not possible, so let t denote the period of the solution.

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$$\frac{dx^*(t)}{dt} = f(x^*(t), y^*(t))$$
$$\frac{dy^*(t)}{dt} = g(x^*(t), y^*(t))$$

and we have

$$\int_C [f(x,y)dy - g(x,y)dx] = \int_0^T [f(x^*(t), y^*(t))\frac{dy^*}{dt} - g(x^*(t), y^*(t))\frac{dx^*}{dt}]dt = 0.$$

Thus

$$\int\int_R (f_x + g_y)ds = 0$$

which is true iff the last integrand changes sign, which is a contradiction. Thus (29) possesses no closed path in D.

And then along C, dx star dt = fx star y star and dy star; dy/dt of y star = gx star y star t and we can write down this line integral, C fx dy – g xy dx as we simply write dt here and this is dt here and t, so we convert this lined parameterised this C in terms of t, and we can write this line integral as 0 to t fx star t y star t dy star/ dt – gx star t y star t dx star/dt, dt here, right. Now, if you look at what is this dy star dt; it is g here.

So, it is nothing f * g – g * f, so this has to be = 0 here, so it means that the right hand side which is given by this has to be 0, so double integral fx + gy ds= 0, so this surface integral will be 0 provided that the integrand must change its sign but if fx + gy is not change their sign, then this integral cannot be equal to 0, so it means that if fx + gy will keep the same sign, then your, then this may not happen.

So, it means that this is true only when we are assuming that C is a closed curve, so it means that if C is closed curve and fx + gy is having the same sign, then this cannot be true right, so it means that the contradiction we are getting is only because we are assuming that C is a closed curve say bonding the domain R here, right, so because if C is a closed curve, then it has to be 0 but fx + gy is having same sign then it cannot be 0.

So, it means that the assumption that C is a closed curve is not true, so it means that C cannot be closed curve and hence this C is not correspond to a the periodic solution, so C is not a orbit of a periodic solution, so it means that they exist no closed path in domain in D here, so it means that if fx + gy will have the same sign then D cannot have any closed curve right and hence no periodic solution because if it has a closed curve which will not contain any critical point then it must correspond to a periodic solution.

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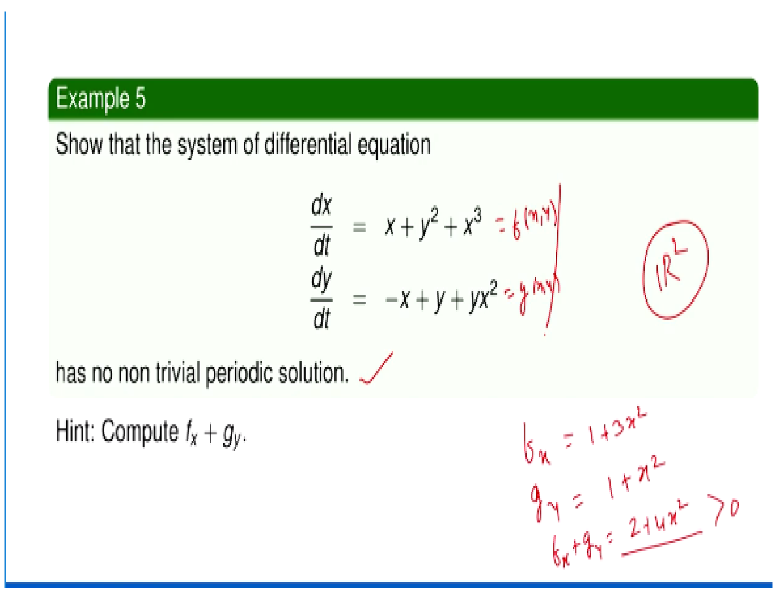

So, let us have one example based on this, so that the system of differential equation ds/dt = x + y square + x cube and dy/dt = -x + y + yx square has no non-trivial periodic solution, so as we have pointed out this is nothing but f of x, y and it is having g of x, y, so look at f of x, f of x is 1 + 3x square, when we tight f of x, we are keeping y as a say, constant, so similarly you can find out g of y; of y is 1 + x square.

And if you look at fx + gy, it is nothing but 2 + 4x square and since x, this will always remain positive, so it means that and this is positive for all xy in R2, so it means that for in R2, it will not have any closed curve, so it means that any of the solution will not form any close orbit and hence this system will not have any non- trivial periodic solution here. Now, let us consider one more example, so if you look at this is always true all values of xy.

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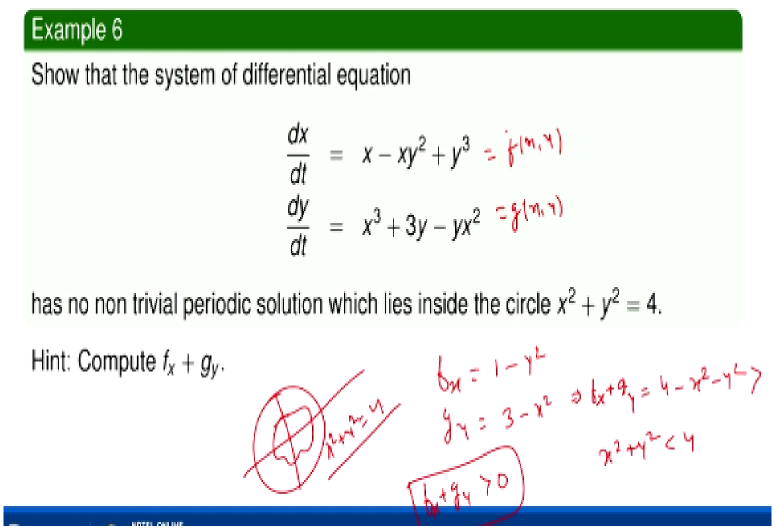

It means that this will not have any periodic solution in a entire R2, now let us look at the next example, showed that the system of differential equation dx/dt = x – xy square + y cube and dy/dt = x cube + 3y – yx square has no non-trivial periodic solution which lie inside the circle x square + y square = 4, so as we have pointed out, this is your f of x, y, and this is your g x, y here, so we can find out f of x which is nothing but 1 – y square.

And g of y =3 – x square and you can see that f of x + g of y = 4 – x square – y square, so it means that if x square + y square is < 4, then it is; then f of x + g of y is positive, it will not change any sign here, so it means that in that domain this x square + y square = 4 in a; in that region, we do not have any closed orbit here, it means that in this domain x square + y square < = 4, we do not have any closed orbit.

And hence we do not have any non-trivial periodic solution which lie inside this cycle, so with this we conclude our lecture here, so what we have shown here that if we do not have any closed orbit then we do not have say limit cycle and we do not have any nontrivial periodic solution, so for existence of periodic solution and hence for limit cycle, we must have a close orbit, so Bendixsen nonexistence theorem will simply gives you that if fx + gy remains the same sign in given domain.

Then in that domain, we do not have any closed curve and hence we do not have any possibility of having a closed orbit and hence we do not have a nontrivial periodic solution as well as they exist no possibility of limit cycle, so with this we conclude and we will continue our study in next lecture, thank you very much for listening us, thank you.