Dynamical Systems and Control
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Lecture – 27
Poincare-Bendixson Theorem

Hello friends, welcome to this lecture, so in this lecture we will focus our study on no non-linear system, before this, we have discussed the phase portrait for the linear system and the types of critical point for linear and weakly nonlinear system and here we considered the kind of a general non-linear system here.

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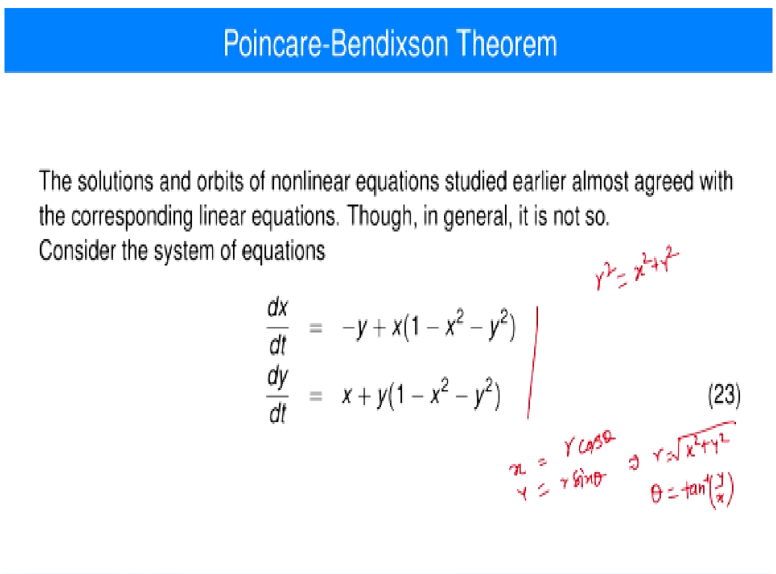

So, here in this case, we have the Poincare-Bendixson theorem which will discuss of the behaviour of the critical point of 0, 0, so the solution and orbits of non-linear equation studied earlier almost agreed with the corresponding linear equations so but in general, it may not be true, so first we will start this discussion with the help of one example and then we summarise the finding of the example as a theorem which is given as Poincare-Bendixson theorem.

So, first let us consider an example and then we try to state what we mean by Poincare-Bendixson theorem, so first let us take one example dx/dt = -y + x * 1 – x square – y square and dy/dt = x + y 1 - x square – y square, so here if we try to find out the behaviour of the critical point, then of course 0, 0 is a critical point here and we can find out the orbit also but let us simplify this equation by introducing the polar coordinate system.

And if we introduce a polar coordinate system, so it means that x = r cos theta and y = r sin theta or I can simply write that r = x square + y square, square root of this and theta = tan inverse y/x, so this is the say, change of coordinate from xy to polar to Cartesian and Cartesian 2 polar here.

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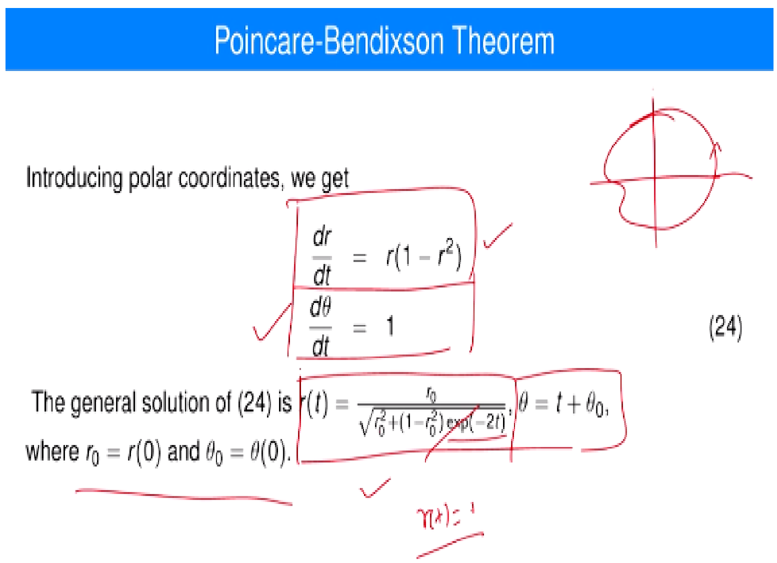

So, introducing polar coordinate system, we claim that your equations given in 23 may be written as dr/dt = r * 1 – r square and d theta/dt = 1, so let us spend some time on this, so here we need to find out dr/dt, so if you look at r square = x square + y square, let me do it here.

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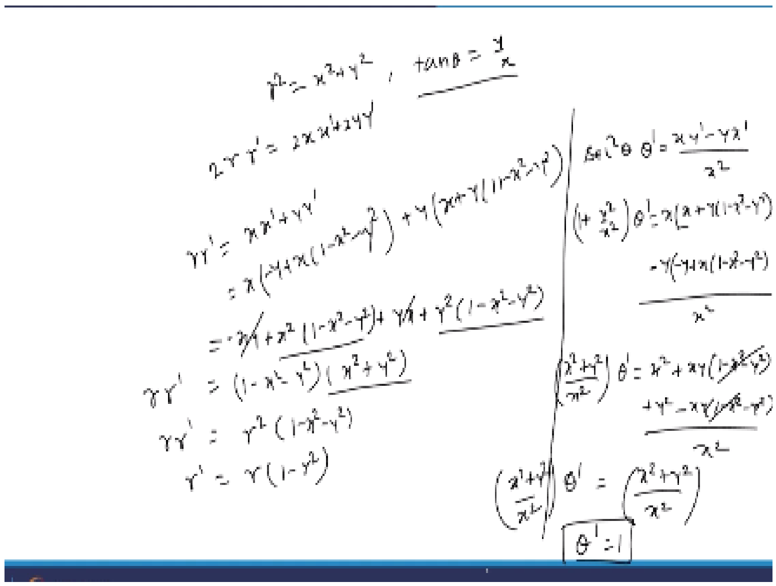

So, it is r square = x square + y square and tan theta = y/x here, so if we look at 2r r dash = 2x x dash + 2y y dash, here dash represent the derivative with respect to t, here, so 2 will be cancelled out, so it is r * r dash = x x dash + y y dash here, now x dash we have already find out, x dash = -y + x, so it is –y + x 1 – x square – y square + y y dash will be what; it is x + y, so it is x + y 1 – x square – y square here.

So, we can simply write this as – xy + x square 1 - x square - y square + yx + y square 1 – x square – y square and this will be cancel out and what is left is; x square * this + y * this, so we can take out this 1 - x square – y square common and what is left here is x square + y square, so it is nothing but r of r dash, now if you look at this is nothing but r square, so we can simply say r r dash = r square 1 – x square – y square.

So, we can simply write r dash = r * 1 – r square that is what we have claimed here that dr/dt = r * 1 – r square, now for the d theta/dt, let us consider this also, so d theta/ dt, if you differentiate this tan theta = y/x, it is sec square theta, theta dash = here, we simply say it is x square x y dash - y of x dash, so sec square theta is 1 + tan square theta, so we can say 1 +; say y square upon x square theta dash = xy dash is basically what; it is this.

So, x + y 1 - x square – y square – y of x dash that is – of y + x 1 – x square – y square divided by x square, so we can simply say that here if we simplify it is x square + y square divided by x square theta dash =; if you simply say that it is x square + xy 1 - x square – y square+ y square - xy 1 – xy 1 - x square – y square divided by x square, so these 2 will cancel out and what you get is theta dash = x square + y square upon x square.

And if you divide it by this, then it is same, x square + y square upon x square, so here we can simply divide and we can simply say theta dash = 1 here, so that is what we have written here that when introducing polar coordinate, we can rewrite the equation number 23 as follows, dr/dt = r * 1 – r square and d theta / dt 1 here, so this d theta/ dt just give you the direction because here we can simply say that behaviour maybe this orbit will be something like that.

And this d theta/ dt will just give you the behaviour of the orientation that orientation is clockwise or anti clockwise, so this d theta/ dt is just give you the direction of the say, orientation of the orbit, the solution is moving clockwise or anticlockwise for example, here d theta/ dt = 1 which is a positive thing, so it means that it is moving in positive direction. Now, what is your positive direction?

It is anticlockwise direction, so it means that your that solution will move in anticlockwise dash here, now first let us find out the general solution of equation number 24, now the region here that we can solve this equation number 24 in a very easy manner that dr/dt = r * 1 – r square, it is quite inseparable form but if you look at the equation number 23, it is quite difficult to solve in Cartesian quantum.

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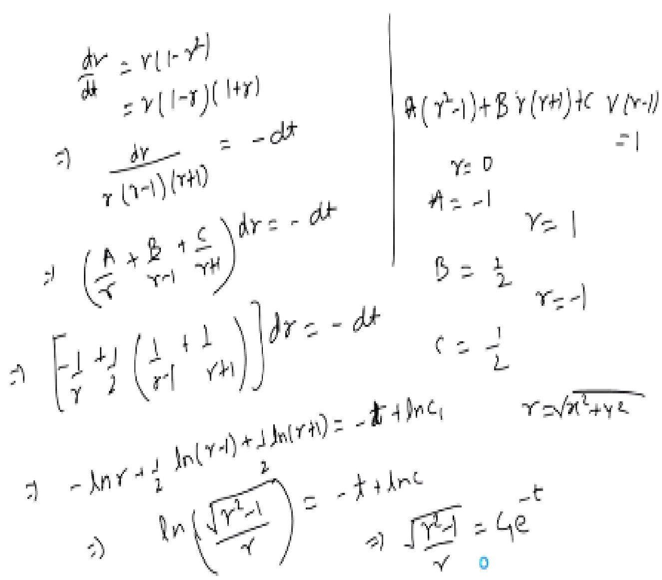

Now, let us solve this equation in polar coordinate, so we have dr/dt = r * 1- r square, so we can write this as r * 1 - r * 1 + r here, so we can simply say that it is dr divided by r r -1 r +1 = - of dt here, so we just multiply by that, now we can use here the partial fraction and we can simply say that it is A/r + B/ r -1 + C/ r + 1 dr = - of dt, we need to find out the values of A, B and C, so I can simply give you the values.

But if you want to verify, you can simply say these 2 are same, so it means that A times r square - 1 + B times r * r + 1+ C times r * r -1 = 1 and this is true for all values of r, so in particular by giving different, different values of r, you can find out the values of A, B, C this is nothing but the method which we have generally used for in integration kind of thing, so here we can put r = 0 and we can get the value of A equal to -1.

And we can put r = 1 here and we can get the value of B as 1/2 similarly, r = -1 we can get the value C as 1/2 so, we can rewrite this as -1 upon r +1 upon 2 of 1 upon r -1+1 upon r + 1, you can verify that it is actually coming to the earlier one, so - of d of dt, so we can simply this –ln of r, now here r is basically x square + y square under root, here we are taking that since r represent the distance, it is never negative, so r is a positive quantity, okay.

So, -ln of r + 1 upon 2 ln of r – 1 and + 1 upon 2 ln of r + = - of dt, - of t, here + some integration constant, let us call this as c, so here I can simplify, I can simply write ln of r square -1 under root divided by r = -t + ln c, so we can simply write this as r square -1 upon r =; this e to the power – t, you write it, now this c I have already used, so let us use some other notation, let us say c1.

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$$\frac{r^2-1}{r^2} = c_1^2 e^{-2t}$$

$$1-\frac{1}{r^2} = c_1^2 e^{-2t}$$

$$\frac{1}{r^2} = 1 - c_1^2 e^{-2t}$$

$$r = 1/\sqrt{1-c_1^2 e^{-2t}}$$

$$r = \frac{1}{\sqrt{1 - \frac{r_0^2-1}{r_0^2} e^{-2t}}}$$

$$r = \frac{r_0}{\sqrt{r_0^2 - (r_0^2-1) e^{-2t}}}$$

$$r(0) = r_0$$

$$c_1^2 = \frac{r_0^2-1}{r_0^2}$$

So, under root r square -1 upon r = c1 e to the power – t, now let us square it out and we have r square -1 upon r square = c1 square e to the power -2t, so we can simplify it is 1 -1 upon r square = c1 square e to the power -2t, so 1/r square you can find out as 1 – c1 square e to the power -2t, so r square is nothing but under root, sorry 1 upon under root 1 - c1 square e to the power -2t, now we had to fix the value of c1.

So, if I assume that r of 0 = say r0, then we can find out the value of c1 here itself, so here your c1 square is basically as what r0 square -1 upon r0 square, so that is this value, so here if you put the value r, sorry, this is r here, so here it is r, so this is r, so r = 1 upon under root 1 - c1 square is this, so r0 square -1 upon r0 square e to the power -2t here, so this we can rewrite this as r0 square divided by r0 square – r0 square -1 e to the power -2t here that is your r, okay.

So, that is your c1 square value, is that okay, so that is what we have written here that the general solution is given by rt = r0 divided by r0 square + 1 – r0 square, exponential of -2t. If you look at here, this is what we have written, minus if you take out, then it is 1 – r square e to the power -2t here and corresponding to d theta/ dt = 1, so it is nothing but theta = t + c, now c you can find out by putting theta 0 = theta 0.

So, we can write down this as theta = t + theta0, so the general solution of this 24 in fact, particular solution satisfying the condition r0 = r0 and theta 0 = theta 0 is given by rt = r0 under root r0 square + 1 – r0 square exponential of – 2t and theta = t + theta 0.

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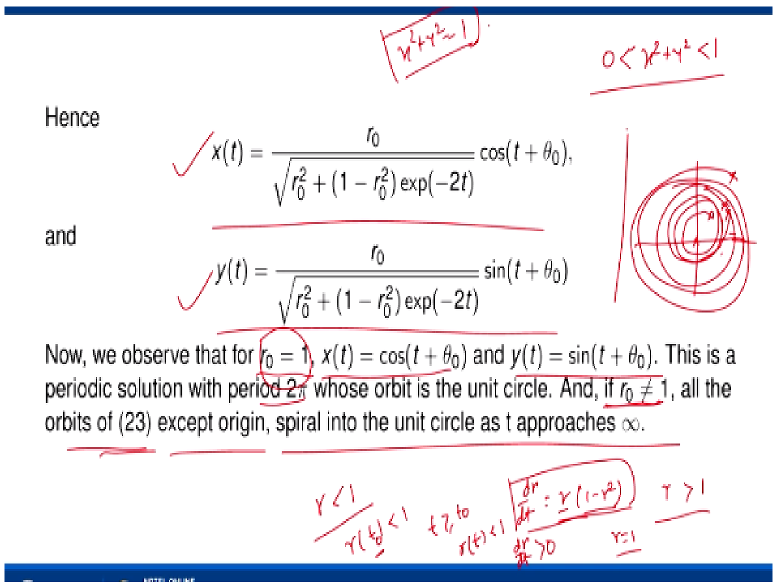

So, this is the say, solution so, it means that now, we can write down the solution xt as r0 divided by r0 square +1 – r0 square exponential of -2t cos of t + theta 0 and yt as r0 upon under root r0 square + 1 – r0 square e to the power -2t sin of t + theta 0, so here the solution of the equation number 23 differential equation number 23, though it is very difficult here how to solve this equation.

But here with the help of polar coordinate system, we are able to find out the solution given as xt and yt here, now once we have a solution, then it means that we can draw the orbit also and then we conclude something. Now, here we simply say that if we take r0 = 1 that initial condition, if we take the initial point lying on the circle x square + y square = 1, r0 = 1 means that x0 square + y0 square = 1.

So, in that case your xt = cos of t + theta 0 and yt = sin of t + theta 0, so it means that if the initial point lie on the circle; unit circle then the solution will always lie on the circle itself, so it means that your circle is like this, so if your initial point is someway here, then future point also it will remain here because x square + y square we can always get as y here, so it means that one of the orbit is given as the unit circle here, right.

Now, if we take r0 =; != 1 means either r0 is < 1 or r0t is bigger than 1, then what is the behaviour of the orbit, so we say that if r0 != 1, all the orbits of 23 except origin, spiral into unit circle as t approaches infinity that we wanted to know. So, if I let us start with say r0 bigger than this, right, so if r0 is bigger than 1, then what we will have? If you look at x of t, here, x of t is is given by this or let us more clearly it will be given by this here.

So, let me look at here on t = r0 upon under root r0 square + 1 – r0 exponential of - of 2t that is the behaviour of rt and at this equation number 24 give you dr/dt = r * 1 – r square here, so this will give you, let me look at here, this dr/dt = r * 1 - r square will give you the change of radius vector with the as t tending to infinity. So, if r is say > 1, then r square is certainly bigger than 1, so this quantity dr/dt is negative right.

So it means that your radius will be a decreasing function as t increases, so as t increases, your radius will try to come down to say, decrease now, up to what it will decrease? So, it means that here it is this thing, so it means that here it will try to decrease and ultimately, it will try to come to this unit circle x square + y square = 1. Because as for r is > 1, it will; dr/dt is always decreasing but as r = 1, it means as it is touching the orbit x square + y square = 1, it means that r = 1, then dr/dt = 0, so it means that then radius vector will not change.

So it means that as your this solution which is starting from say somewhere outside the circle, then it will try to as t tending to infinity, it will try to come to near to your x square + y square = 1. And as soon as it touches the unit circle, then dr/dt = 0, it means that radius will keep on be constant, so it is; it will not change, so it means that if you take any point outside and solution passing through this then it will spirally approach to the circle x square + y square = 1.

Similarly, if I look at the value r < 1, right then your dr/dt is now positive because this 1 – r square is positive and r is also positive. So, dr/dt is now positive, so it means that radius vector is increasing function of time t, so it means that if we take any solutions someway here, then your various will keep on increasing, right and if you look at xt and yt, these are say, periodic functions, so I can look at like this, so xt = this rt cos of t + theta 0. So, if I look at t + some 2pi, then x of t + 2 pi will be what; it will be, let me look at here.

So, here your radius is increasing and it is a periodically; periodic function, so it means that it will spirally increase, so it means that it will go like this or this and ultimately it will move away from origin and at as soon as it will; so now if we take say r < 1 or say r of t is < 1, so it means that if at any time, let us say t0, r of t0 is < 1, then at this particular point dr/dt will be what; this is < 1 but positive, it is positive here.

So, dr/dt at t0 is positive, so it means that for t > t0, your radius vector will increase, right so it is and for any time where r of t is < 1, then dr/dt is always positive, so it means that it keep on increasing for all values of t for which r of t is < 1 till it touches the unit circle x square + y square = 1, where your r will become 1, so at as soon as your r will become 1, then dr/dt = 0, so it means that once this solutions orbit is approaching towards the circle x square + y square = 1.

And as soon as it touches the circle of radius 1, then your radius vector will be a constant so, dr/dt = 0, so it means that radius vector will not change further, so it means that it will always be on the unit circle, so it means that the solution which started somewhere inside the unit circle will ultimately approach of spirally towards the unit circle r = 1 here, spirally because your xt and yt are periodic function of time t with increasing radius.

If you look at xt and yt, these are periodic function that is very clear from this and you can see that the radius means x square + y square is keep on increasing.

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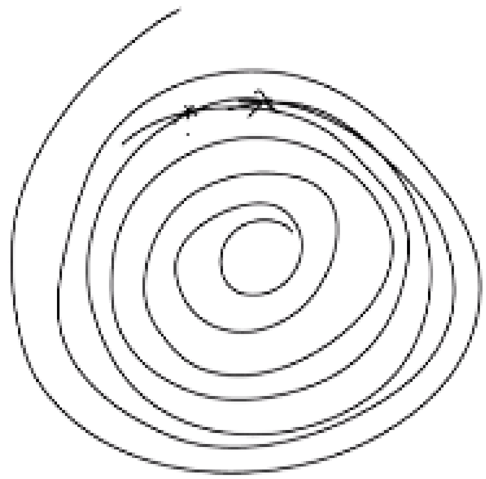

So, here let me write it here this thing that this is a unit circle here, so if I take any solution here, then it will move away something like this and as soon as it will be here, it will, dr/dt will be a constant and it will remove; it will always be on this unit circle on it, similarly if we take any point outside, then your radius vector keep on decreasing and it will come out to be the unit circle here, right.

So, here as soon as is touches the unit circle, your dr/dt is a constant and solution for future time it will always remain on the unit cycle, so this is very unique feature in the non-linear system here, so here what we have observed here that the solution xt and yt is a periodic solution with period 2pi and sorry, this period, for r0 = 1, the solution is periodic solution with period 2 pi and all other solution for which r0 is != 1, all the orbits of 23 except origin spiral into the unit circle as t approaches infinity that we have just pointed out.

And that you can look at from this also, if you look at here, rt = r0 upon r0 square + 1 – r0 square exponential of -2t, so as t tending to infinity, this term will tend to 0, so it means that rt will be what; r0 upon under root r0 square, it means that it is rt = 1 only, so the limiting behaviour as t tending to infinity rt will reach to rt = 1 only, so that is clear from this also.

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The system of equation (23) shows that the orbits of a nonlinear system of equations may spiral into a simple closed curve. Which is not possible for linear systems.

Also, without solving the nonlinear system of equations explicitly, it is often possible to prove that orbits of a nonlinear system of equations spiral into a closed curve.

Now, the system of equation 23 shows that the orbit of a nonlinear systems of equation may spiral into a simple closed curve, so circle x square + y square = 1 is a simple closed curve right, which will represent the our one orbit and all others orbits say, ultimately reaching or converging towards this unit circle and it is not possible for linear system, in linear system, this kind of things will never happen.

So, there we have already seen that none of the orbits are intersecting each other, okay and hence we in linear system, this one circle, one orbit is not spiralling into the other orbit, so this is a unique feature in a nonlinear system, also without solving the nonlinear system of equation explicitly it is often possible to prove that the orbit of a nonlinear system of equation is spiral into a closed curve.

Now, here whatever we have concluded in this example, this conclusion we obtained from the equation and solution of the differential equation basically, we have solved the differential equation and we are able to conclude whatever we are saying but is there any possible that without solving the differential equation, can we conclude something like this. Now, please if you look at here the close curve is corresponding to a periodic solution.

So, it means that we can consider the conclusion whatever we have concluded is to find out a periodic solution that is their a periodic solution possible for a given system of linear equation, so we can look at the this observation in a different manner and to find out a nontrivial periodic solution of the corresponding nonlinear system. Now, here we try to state Poincare-Bendixson theorem to find out the existence of these kind of a phenomena without solving the nonlinear system here.

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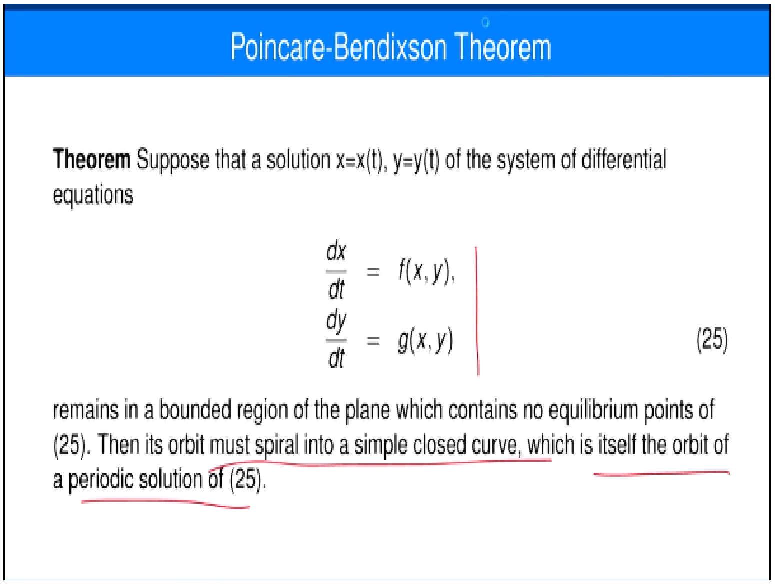

So, here the () theorem which says that suppose that a solution x = xt y = yt of the system of differential equation, dx/dt = f xy dy/dt = g xy remains in a bounded region of the plane which contains no equilibrium points of the 25, then its orbit must spiral into a simple closed curve, now that simple closed curve is itself the orbit of a periodic solution of 25, so that will give you the existence of periodic solution as well as the extra phenomena that orbits of these solutions must spiral into a simple closed curve, okay which correspond to a periodic solution of 25.

So, by this we can also ensure that if this thing is happening that orbit must spiral to a simple closed curve and that simple closed curve is corresponding to a periodic solution, if you look at the previous example here, your solution is remains bounded in a region bounded by x square + y square is < 1 and > 0, so it means that your solution bounded in a region given as annulus of x square + y square bigger than 0 and < 1.

Then the solution orbit which is starting inside that must say spiral into a solution that is x square + y square = 1, the orbit of these xt and yt which is corresponding to a solution of periodic solution with this orbit x square + y square = 1 is corresponding to a periodic solution that we have just given here.

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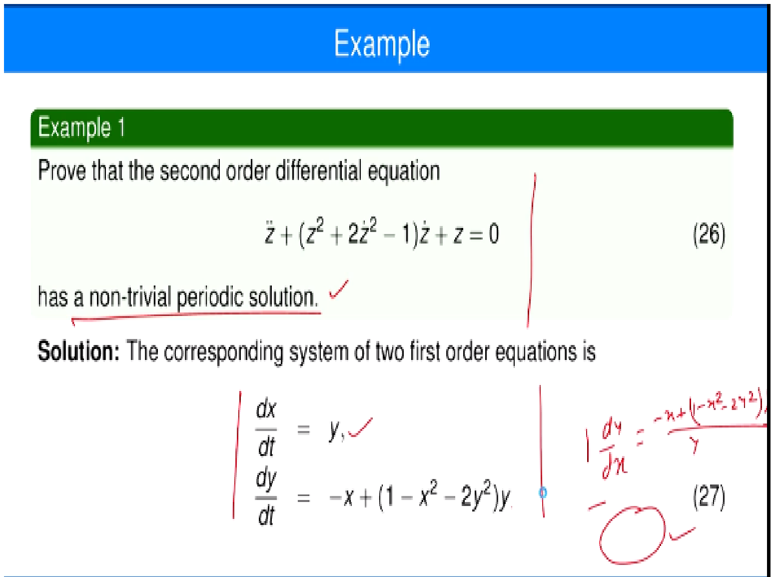

Now, let us take a; I am not giving the proof of this Poincare-Bendixson theorem which is little bit involving rather than we are discussing some very good example based on this Poincare-Bendixson theorem, so one example is that prove that the second order differential equation, z double dash + z square + 2z square – 2z dash square - 1 z dash + z = 0 has a nontrivial periodic solution. So, first of all, look at this, this is a nonlinear differential equation and the corresponding system of 2 first order equation is dx/dt = y and dy/dt = -x + 1 –x square – 2y square upon * y.

Now, here regarding this, the nontrivial periodic solution we have also discuss this problem earlier also, there what we have this thing, we will just convert into system of 2 first order equation like this and then we try to find out the orbit here, dy/dx = this thing -x+1- x square – 2y square y upon y here, right and if this orbit is coming out to be a closed curve and it will not contain any say equilibrium thing.

Then this will correspond to a; this orbit is correspond to a nontrivial periodic solution but it is provided that this dy/dx = this, it is a system of first order differential equation, nonlinear first order differential equation is solvable, if solvable means solvable in explicit manner, if you are not able to find out the equation of orbit, we cannot check whether it is a closed orbit or a just an orbit.

So, here it is quite difficult when this is a general nonlinear system, so here in this kind of problem, we can use Poincare-Bendixson theorem to find out the existence of nonlinear nontrivial periodic solution here. So, now we try to find out a reason now, we want to apply the Poincare-Bendixson theorem and for which we had to find out a region in which your solution will remain bounded.

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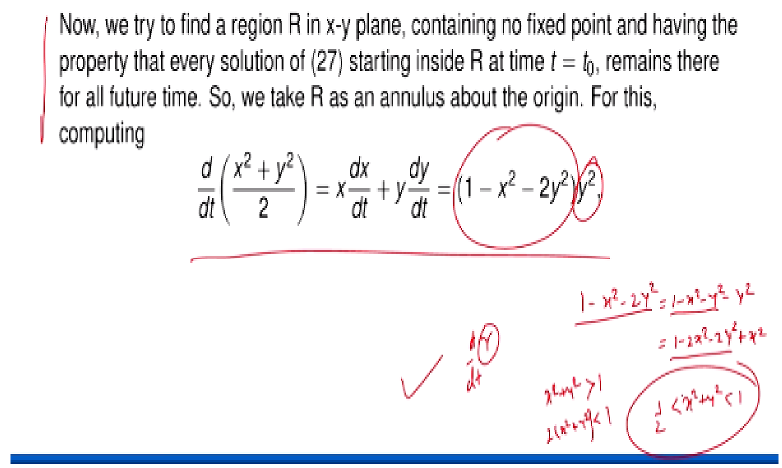

So, let us find out the region R are in xy plane containing no fix point and having the property that every solution of 27 starting inside that region at some time say t= t0 will always remain therefore, all the time, so we take R as annulus region about the origin, so far that we simply look at this d/dt of x square + y square upon 2 = x dx/dt + y dy/dt = 1 – x square – 2y square * y square, so this you can find out, x dash is given by y and y dash is given by this.

You can simplify and you can get it, so to find out the region R, where you; solution will always remain there, it is always advisable to find out dr/dt, so by which we can simply say that by looking at this R, we can conclude something that in region R, your solution will remain there or not, so this is a common practice to find out dr/dt or if it is not possible, then you have to do something to conclude that in your region R, your solution will always there.

If it is starting inside the annulus, in that reason will always remain there having no critical point, so region must be free from critical point, means fix point and solution will remain therefor there, then we can apply Poincare-Bendixson theorem to say that they the solution will spiral in a closed say, orbit and that closed; this theorem does not tell you anything about the closed orbit that which one is your closed orbit, it just simply tell you that they exist a closed orbit are on which all the solution are spiralling into it as t tending to infinity or t tending to – infinity.

And that existence of closed orbit gives you the existence of periodic solution of the nonlinear system, okay, so we are doing this, we are finding the region R, so dr/dt of x square + y square is coming out to be 1- x square – 2y square upon * y square. Now, with the help of this, we will try to find out the region here, now look at that y square is always positive right, so now everything will depend on this 1- x square – 3y square.

So, I can write 1- x square – 2y square into 2 following ways; one is this, 1- x square – y square – y square, another way I can write it 1- 2x square -2y square + x square, so if this quantity is negative, then the entire thing is negative and if this quantity is positive, the entire thing is positive, so it means that this quantity is negative means, x square + y square is bigger than 1, so then this quantity is negative, right.

So, it means that 1- x square – y square is positive, now this quantity is positive means, this 2 times x square + y square is < 1, right, so it means that if x square + y square is > 1/2 and < 1, then this x square + y square, sorry, < 1/2 then, it will remain, sorry, this is so we can simply say that dx square + y square is bigger than 1, then this will be negative thing and if 2 times x square + y square is < 1, then it is always positive.

But if we take this region, x square + y square which is < 1 and > 1/2 then, what should be the behaviour here? So, we take our region bounded by these things, it is a circle of radius 1/root 2 and 1.

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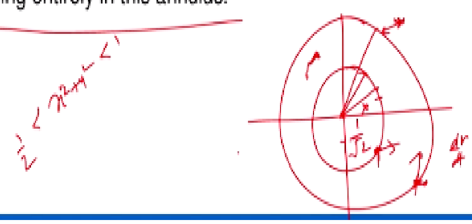
Clearly, $1 - x^2 - 2y^2$ is positive for $x^2 + y^2 < \frac{1}{2}$ and negative for $x^2 + y^2 > 1$. It follows that $x^2 + y^2$ is increasing along any solution of (27) when $x^2 + y^2 < \frac{1}{2}$ and decreasing when $x^2 + y^2 > 1$.

Thus any solution starting in this annulus remains there for all time. Also, it has no equilibrium point and hance by Poincare Bendixsen theorem, there exists at least one periodic solution lying entirely in this annulus.



So, here this clearly 1- x square – 2y square is positive for x square + y square < 1/2 and negative for x square + y square > one that is what we have pointed out here that x square + y square is < 1, it is all positive and negative for x square + y square is > 1, so it follows that that x square + y square is increasing along any solution of 27, when x square + y square is < 1/2 and decrease in when x square + y square is > 1.

So, it means look at here, this is an inner cycle, and this is outer circle, now this circle is correspond to radius 1/root 2 and this circle is corresponding to the radius 1, so if I take any solution here right, if we take any solution here, then we know that your 1- x square – 2y square is negative here because x square + y square is bigger than 1, so it means that your solution will try to; radius of the solution will try decrease.

It means that it will always come to say, closer to the unit circle x square + y square = 1 and if we take any solution here in the inner circle, then your 1- x square - 2y square is positive and your radius vector is trying to increase and somehow it will try to close to your circle, x square + y square = 1/2 so, idea is that if you take any solution in this region which is bounded by x square + y square < 1 and 1/2.

So, if we take any solution here, my claim is that that solution will remain there for this time because if somehow this solution at some point it is trying to cross this region, then as soon as it crosses, then your x square + y square is bigger than 1, right and as soon as it is say on the say, x square + y square = 1, as soon as it try to move out of this region, then your dx/; dr/dt will be negative and it will come to; it will force to be back to say, circle, inner to the circle here.

So, it means that now, there is another possibility that it will try to cross this inner circle and try to move to region here but as soon as it is trying to cross this inner circle, then on this boundary of the inner circle, your x square + y square is < 1/2 and hence your radius will be increasing, so it means that when x square + y square is < 1/2 then, x square + y square is increasing, it means that it radius will try to increase and it will force thrown again inside your annulus region.

So, it means that if a solution is starting inside this annulus region, it will always remain there in this annulus region for all the future time t, so it means that any solution is starting in the annulus remains there for all the time t and also it has no equilibrium point and hence because the only equilibrium point, we can check that for this, it is 0 0 here, right, you can check that, it is 0 0 here. So, it means that your annulus region is not containing equilibrium point 0, 0.

And hence this must exist a closed orbit onto which all the other solutions are spiralling inside this, so it means that by Poincare-Bendixson theorem, we can say that they exists at least one periodic solution lying entirely in this annulus, it is not telling which solution is say, periodic solution or which orbit is corresponding to periodic solution, it just tells you the existence of at least one periodic solution.

We have seen in this particular example that with the help of Poincare-Bendixson, we can say give and guarantee that they exist a one periodic solution lying entirely in this annulus region, so here we can see that the Poincare-Bendixson theorem gives a guarantee of existence of nontrivial periodic solution but it will not give any information how to find out that periodic solution, it just gives you existence.

So, how to find out the that exact which one is a periodic solution and how to find out that periodic solution is not covered by Poincare-Bendixson theorem, so here we will stop our lecture and in next lecture, we try to look at the possibility of finding the say periodic solutions orbit which is missing here. So, in next lecture we will continue the study here and we try to focus on finding the orbit of the say, periodic solution, so that we will discuss in next lecture, thank you very much for listening this, thank you.