Dynamical Systems and Control

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Lecture – 26
Phase Portrait of Linear Differential Equations - III

Hello friends. Welcome to this lecture. In this lecture, we will continue our study of phase portrait for the linear system. If you recall, we were discussing the different cases. If you recall, we have this system of differential equation x dash=fxy and y dash=gxy. It is a nonlinear equation.

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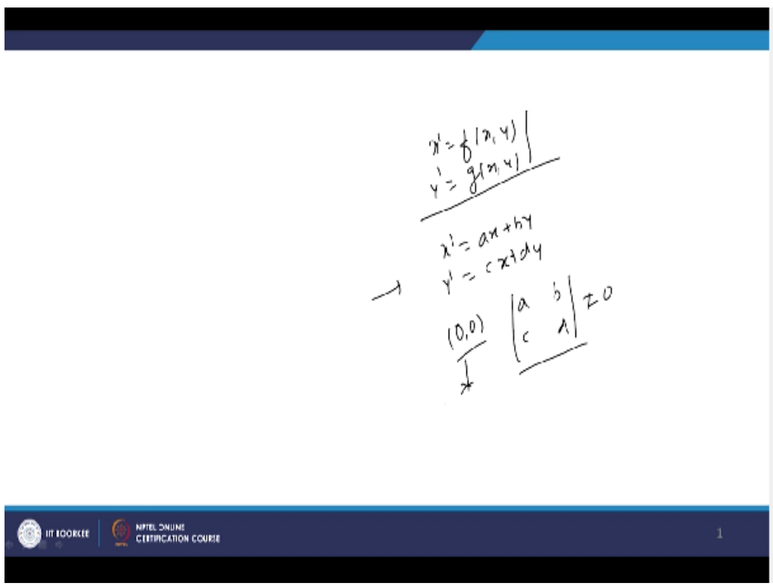

And then we consider the corresponding linear system like x dash=ax+by and y dash=cx+dy and here, it is clear that 0, 0 is a critical point of this. And if we assume that this condition that a b c d, determinant of this is non-0, then 0, 0 is the only critical point of this. And then we are checking the say behaviour of the critical point whether this critical point is node, say stable node or unstable node; saddle point; or say center or whatever. So depending on the eigenvalue of determinant of the matrix a b c d, we are trying to find out the behaviour of 0, 0.

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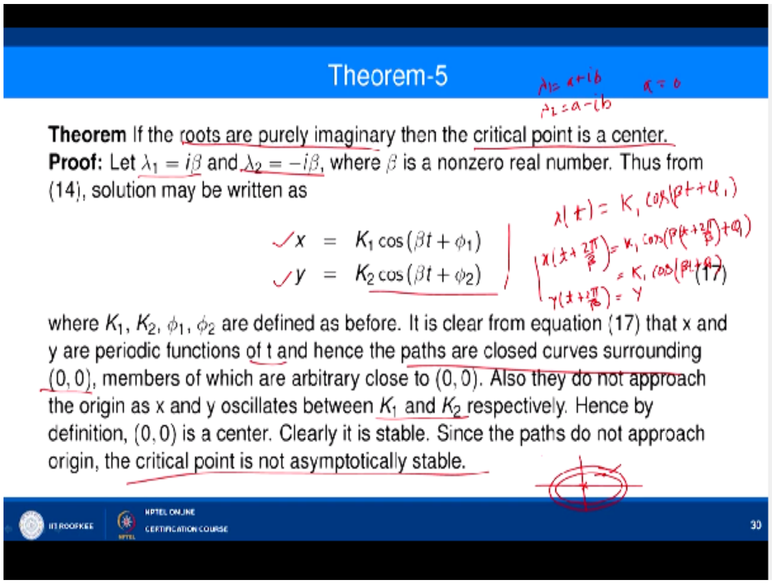

So we have discussed 4 cases based on this and now we are considering the fifth case corresponding to that result. So in this case 5, we are considering that the roots are purely imaginary, means first of all, they are complex which means that lambda you are writing as a+ib, lambda 1 and lambda 2=a-ib where a=0. So it means that real part is 0 here. So if the roots are purely imaginary, then we are claiming that the critical point is a center.

Now what is this center? We will look at here. So if real part of lambda 1 and lambda 2 is 0, then we can say that your eigenvalues are lambda 1=i beta and lambda 2=- of i beta where beta is a non-0 real number. And we may say that our solution may look at as this x=k1 cos of beta t+phi 1, y=k2 cos of beta t+phi of 2.

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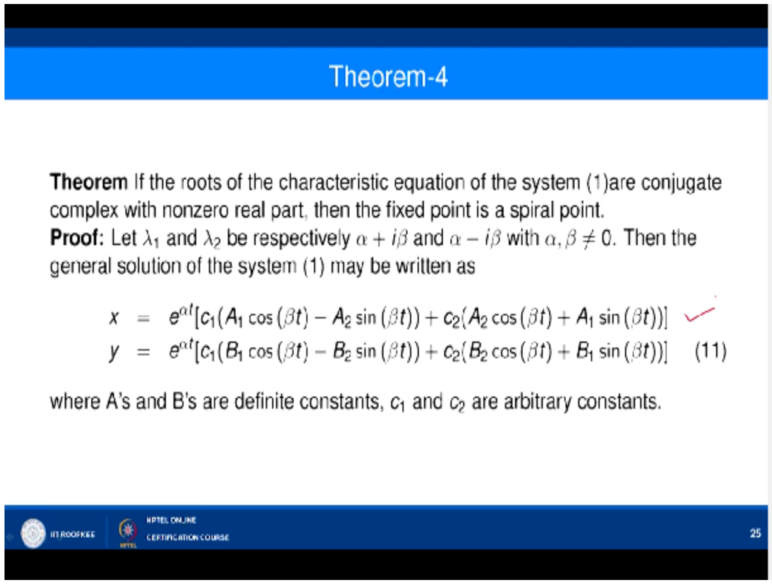

And for that, we have utilized our equation number 14 which we have already discussed, let me go to the equation number 14 here. First our solution will be looking like this as given in 11.

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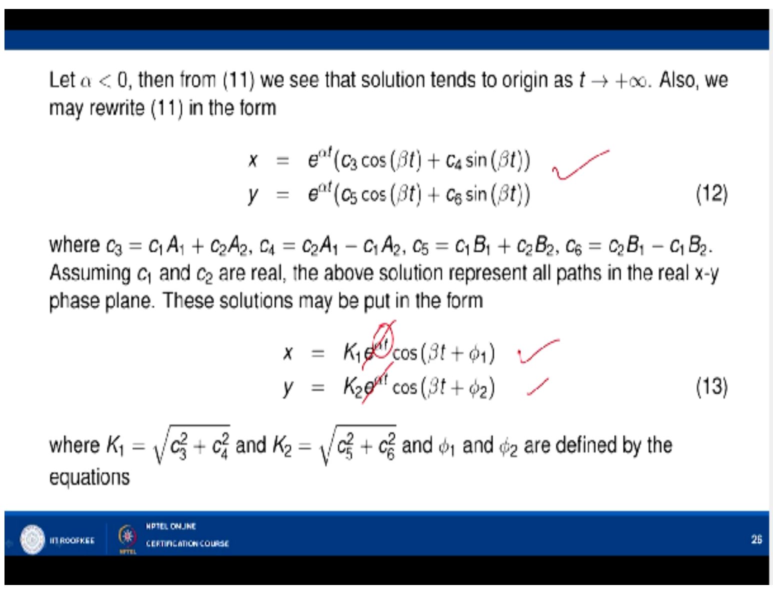

Then we simplify and try to convert into this format equation number 12 here. And then we further simplify and we can look at here as x=k1e to the power alpha t cos of beta t+phi 1 and y=k2e to the power alpha t cos of beta t+phi 2, where this alpha is a real part of the eigenvalue. So if this part is gone, then x is your k1cos of beta t+phi 1 and y=k2cos of beta t+phi 2 which is what we have written as equation number 17 here.

So x=k1cos of beta t+phi 1 and y=k2cos of beta t+phi 2 where this k1 and k2, phi 1 and phi 2 are already defined as before. And it is clear from this equation number 17 that x and y are periodic function of t. If you look at here and if you simply say that if you replace, this is your x of t, so x of t=k1cos of beta t+phi 1. So if I look at x of t+2pi/beta here, since beta is non-0, so we can write it 2pi/beta and this is what k1cos of beta 0t+2pi/beta here+phi 1.

If you simplify, what you will get? you will get k1cos of, and here it is beta t+phi 1. Similarly, we can see that y of t+2pi/beta is same as y here. So it means that 2pi/beta is a number, if we say that xt+2pi/beta is same as x and similarly y of t+2pi/beta=y, so it means that there exist 1, say t such that x of t+T=x of t and y of t+T=y of t. So it means that the minimum of such possible T will give you period of this solution x and y here.

So we simply say that from these equation number 17, we can conclude that x and y are periodic function of t. And hence the paths are closed curves surrounding 0, 0 here. So one is very much clear here that 0, 0 is a critical point and the solution will, say, form a closed orbit here. That we have already discussed that if they are periodic, they correspond to the orbit which is closed here. Now members of which are arbitrary close to 0, 0, that will depend on the values k1 and k2 here.

So also they do not approach the origin as x and y oscillates between k1 and k2. So it means that though they will very near to origin depending on k1 and k2, if k1 and k2 are very small, then they are very near to origin 0, 0 but they do not approach the origin as if you look at as t range from say initial point to infinity, you can say that the value x and y, both are oscillating value and x will take a value between -k1 to +k1 and y will take the value from -k2 to k2 depending on the value of t here.

And hence we can simply say that by definition 0, 0 is a center here. And now we simply say that this is a stable center. Why we say that if we take any solution near to your orbit here, if you take any initial point here and take a orbit say passing through this, then it will follow almost the this kind of path. So it means that there exist an orbit which is passing through this and it will remain always close to the orbit which we are considering as first solution, first orbit.

So it means that if you take 2 orbit which are close to each other initially, then it will always remain close to each other for future time also. So here in this case, your orbits are closed curves and critical point 0, 0 is the center which is a stable center. And since the path do not approach origin, the critical point is not asymptotically stable, though critical point is stable but it is not asymptotically stable because your paths are not tending to or approaching to 0, 0 here.

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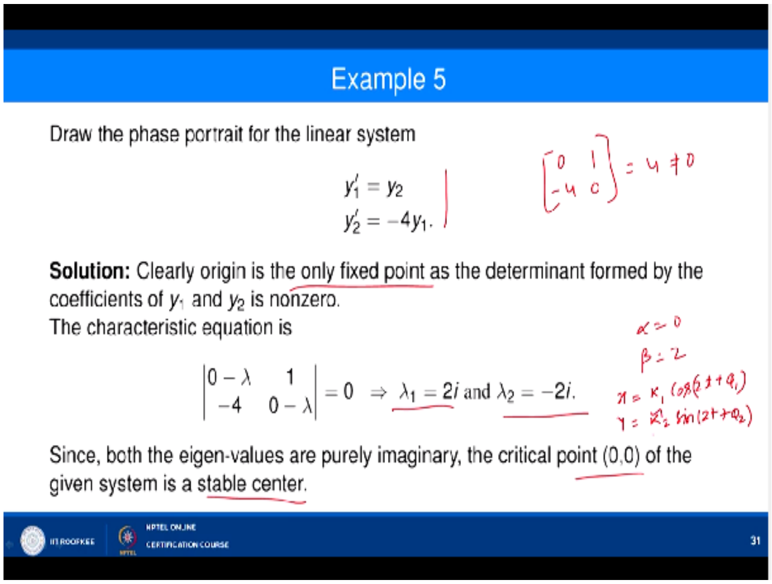

So let us have 1 example based on this. So draw the phase portrait for the linear system y1 dash=y2 and y2 dash=-4 of y1. So here we can simply say that 0, 0 is a critical point here and we can easily check that 0 1 -4 and 0 and if you simplify and find out the determinant here, determinant is coming out to be 4 and hence we can say that 0, 0 is the only fixed point, right.

Now, then the next thing we want to find out is the eigenvalue corresponding to this and if you find out, the eigenvalues are coming out to be lambda 1=2i and lambda 2=- of 2i. So here if you look at, your beta=2 here, right and alpha=0. So we can say that your solution x is written as k1cos of 2t+phi 1 and y is written as k2sin of 2t+phi 2, right. So here whatever we have given, then we can simply claim that since eigenvalues are purely imaginary.

And hence the critical point 0, 0 is a stable center here. Now let us look at the behaviour of the orbit here. So if you look at x=k1cos of 2t+phi 1 and y=k2sin 2t+phi 2, we can simply rearrange this cos and we can simply say that I can simply rewrite this y as k2cos beta t+phi 2 as k2sin 2t+phi 2 here. May be you can write it k2~ here.

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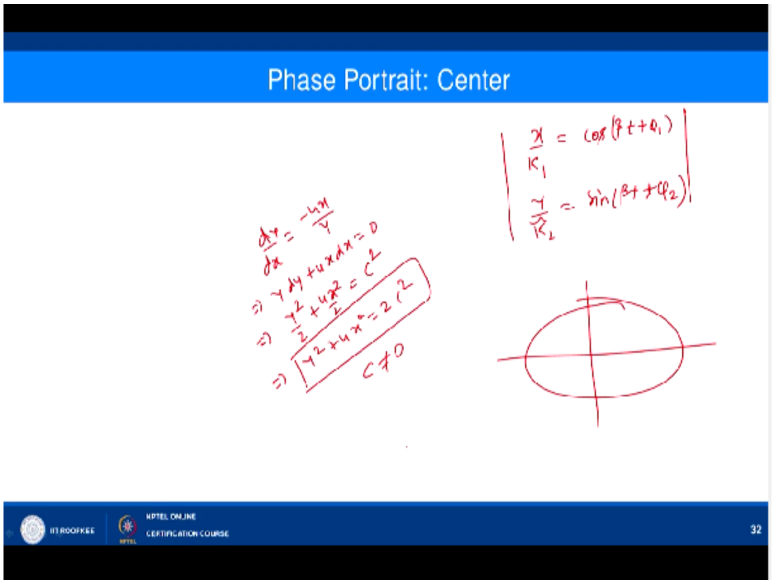

So here we can simply say that I can write x/k1=cos of beta t+phi 1 and we can write y/k2~=sin of beta t+phi 2 here. So here we can simply say that here the behaviour is, so the orbits are looking like the ellipse here, right. So I am not giving the exact say here, but you may try this. So in this kind of cases, your orbits are closed orbits and here we can see that the orbits are ellipse kind of curves here.

May be you can find out the orbit here. For example, if I want to find out the orbit here, let us look at here. Here we have your dy/dx; if I look at here dy/dx I can write it -4x/y here. So I can write it here ydy+4xdx=0. So it is what? y square/2+4x square/2=, say let us call this as c square here. So here we have y square+4x square=2c square. And if you look at carefully, if c is non-0, then this will represent the equation of ellipse here.

So it means that in these cases when origin is center, we can look at that the ellipse, that your orbits are nothing but say ellipse kind of a figure here. And we can consider the circle as kind of a, type of ellipse here, okay. So here we will conclude that if eigenvalues are purely imaginary, then critical point is a center which is stable here. And it is not asymptotically stable here and we have seen 1 example in this case also.

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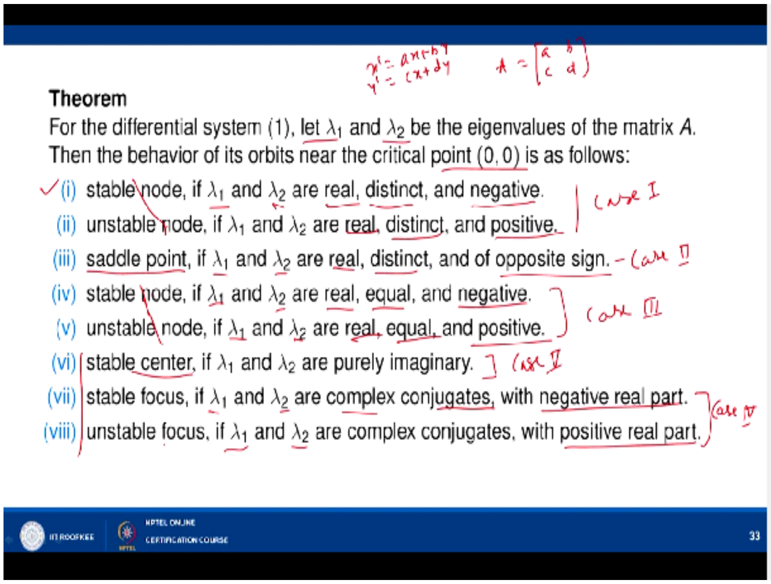

Now if we summarize whatever we have discussed so far, so we can summarize in the given theorem that for the differential system 1, that is x dash=ax+by and y dash=cx+dy, let lambda 1 and lambda 2 be the eigenvalues of the matrix A where A is the matrix given as a b c and d. Then the behaviour of its orbit near the critical point 0,0 is as follows. So here we simply write stable node, if lambda 1 and lambda 2 are real, distinct and negative. So here this will give you the case 1 here.

So here stable node if lambda 1 and lambda 2 are real, unequal and negative. And it is unstable node, if lambda 1 and lambda 2 are real, unequal and positive. So this will be your case, this was your case 1, okay. Now second case is what? That saddle point, if lambda 1 and lambda 2 are real, unequal and are of opposite sign and that corresponds to your case 2 which we have already discussed.

Now stable node, if lambda 1 and lambda 2 are real, equal and negative. And unstable node, if lambda 1 and lambda 2 are real, equal and positive. This was our case 3. Now and then stable center, if lambda 1 and lambda 2 are purely imaginary, that is what we have discussed as case 5 here. A stable focus, if lambda 1 and lambda 2 are complex conjugate with negative real part and unstable focus, if lambda 1 and lambda 2 are complex conjugate with positive real part that was our case 4 here.

Please if you look at these results, then stable, unstable depend on the real part whether real part is negative, then it is stable. If real part is positive, then it is unstable. And then we can simply say that it is node, if it is real, unequal. So if we look at 1, 2 and this thing, then node means if lambda 1 and lambda 2 are real, okay. And either with equal or having the same sign here. And it is saddle point, if they are real, unequal and of opposite sign.

And in center and focus, depending on whether they are complex. So if they are complex, then it will form a center if it is purely imaginary. Otherwise, it is focus here, okay. So that is what we have summarized which we have discussed as case 1, case 2, case 3, case 4 and case 5. So with this, we say conclude that for the linear system, your phase portrait and the critical point and behaviour of the critical point is done with this.

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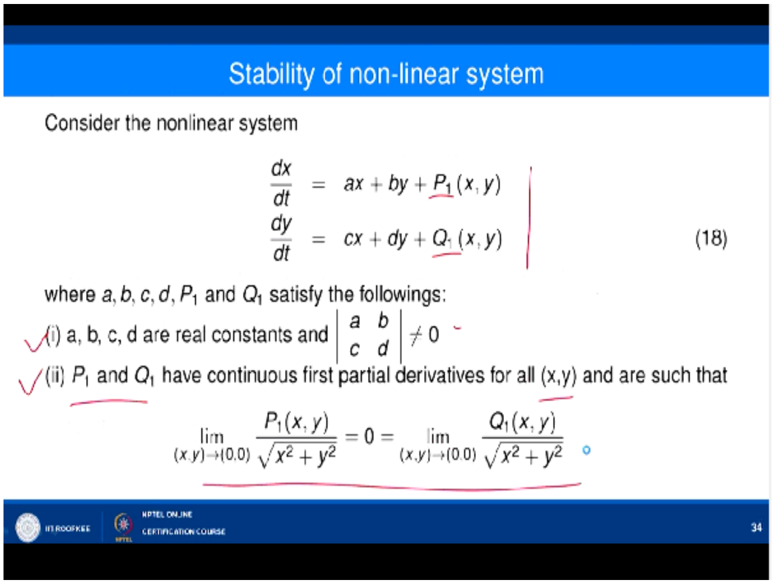

Now let us go to stability of nonlinear system here. So consider nonlinear system dx/dt=ax+by+P1x, y and dy/dt=cx+dy+Q1x, y here. Where a, b, c, d, P1 and Q1 satisfy the following condition. First thing is that a, b, c, d are real constants and determinant of a b c d is non-0. And second condition is that P1 and Q1 are having continuous first partial derivative of all xy and are such that limit x,y tending to 0,0 P1x,y/under root x square+y square=0.

And similarly, limit xy tending to 0,0Q1xy/under root x square+y square=0. And if you recall, this kind of a linear system is known as weakly nonlinear system here. And here, we are assuming that this P1 and Q1 are such that that it will vanish at origin here.

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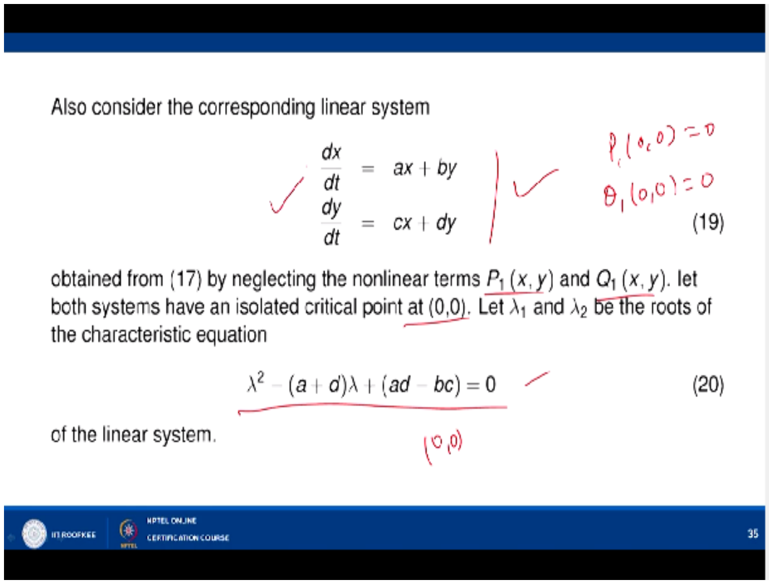

So also consider the corresponding linear system. So corresponding linear system is dx/dt=ax+by and dy/dt=cx+dy. What we want to conclude from this that though it is a nonlinear system but still the behaviour of the critical point 0, 0 is very much similar to the behaviour of the critical point of the corresponding linear system as we have already pointed out.

So here we simply say that consider the corresponding linear system by neglecting with the term P1 and Q1 here. And let both the system have an isolated critical point 0, 0 as we are assuming that P1 of 0, 0 is 0. Similarly, Q1 of 0, 0=0 here. And let lambda 1 and lambda 2 be the roots of the characteristic equation which is corresponding to the linear system that is lambda square-a+d lambda+ad-bc=0.

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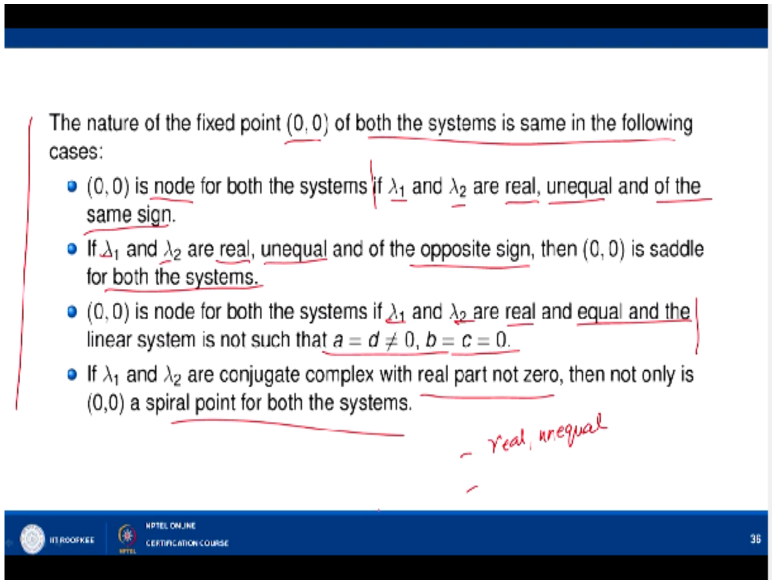

Then we have already discussed the behaviour of the critical point 0, 0 completely for the linear system. Now we want to conclude from the behaviour of the linear system the behaviour of the 0, 0 for the nonlinear system. So the, say, results are given as follows. The nature of the fixed point 0, 0 of both the system is same in the following cases. First that 0, 0 is node for both the systems.

It means that lambda 1 and lambda 2 are real, unequal and of the same sign. So if lambda 1 and lambda 2 are real, unequal and of the same sign, then 0, 0 is node for both the systems. And if lambda 1 and lambda 2 are real, unequal and of the opposite sign, then 0, 0 is saddle point for both the systems. So it means that if it is real, unequal, whether it is of same sign or opposite sign, behaviour of 0, 0 is same.

It means that if it is node for one system, then it will be node for the other system as well. And if it is a saddle point for one system, then it is saddle point of the other system. So for real, unequal, behaviour is same. Now 0, 0 is node for both the system if lambda 1 and lambda 2 are real and equal and the linear system is not such that a=d is not equal to 0 and b=c=0. And if you look at, this is the case corresponding to star shaped node.

So it means that if 0, 0 is node but not corresponding to star shaped shape, then also the behaviour of linear and nonlinear system is same. So it means that in the case when it is a node but not a star shaped node for the linear system, then also the behaviour of linear and nonlinear system is same. And last case is that if lambda 1 and lambda 2 are conjugate complex with a real part not 0, that is case 4, then not only 0, 0 is a spiral point for both the systems.

Then if lambda 1 and lambda 2 are complex conjugate with real part not equal to 0, then also 0, 0 is a spiral point for both the systems. And if you look at, these things we have already proved as a part of theorem. Here we are just visualizing it, okay. So let me recall this. So if node not corresponding to star shaped, then it is both for linear and nonlinear system. And corresponding to say focus, spiral point, we simply say that if lambda 1 and lambda 2 are complex with the non-0 real part, then also the behaviour of the critical point will correspond to in both the cases.

Like if it is say spiral point for linear system, then it will be spiral point for the nonlinear system.

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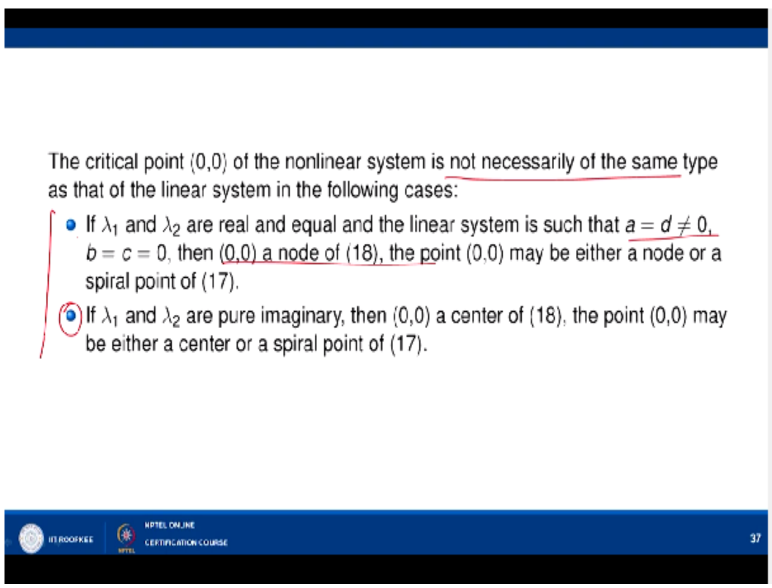

Now the critical point 0, 0 of the nonlinear system is not necessarily of the same type as that of the linear system in the following cases. So here in these 4 cases, they are exactly, similar, not exactly. It is similar to each other but in the given 2 cases, it may not be same. So if lambda 1 and lambda 2 are real and equal and of the linear system is such that a=d not equal to 0 and b,c=0.

So then 0, 0 is node of 18, the point 0, 0 may be either a node or a spiral point for 17. So it means that in this case when lambda 1 and lambda 2 are real, equal and it is corresponding to the case of diagonalizable matrix, then for linear system, it is a star shaped node but for nonlinear system, it may be node or may be spiral point. So this case we are not, here in this case when lambda 1 and lambda 2 are equal and corresponding to diagonalizable matrix, then behaviour may not be same.

And the second point, if lambda 1 and lambda 2 are purely imaginary, then 0, 0 is a center of linear system as we have already discussed as in case 5. The point 0, 0 may be either a center or a spiral point of nonlinear system. So in these 2 cases, your behaviour of critical point may not correspond, right.

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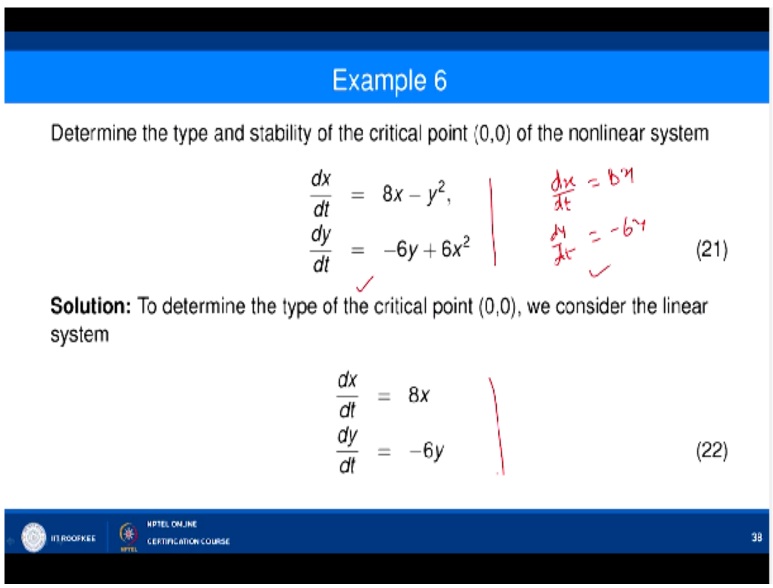

Now let us seek one simple example based on this. So determine the type and stability of the critical point 0, 0 of the nonlinear system here. So dx/dt=8x-y square and dy/dt=-6y+6x square. So here it is, if you look at here, 0, 0 is the critical point for both the linear as well as the corresponding linear part that is dx/dt=8x and dy/dt=-6y. So this is a nonlinear part and this is the associated linear part.

So determine the type of the critical point 0, 0. We consider the linear system. This is 8x and -6y.

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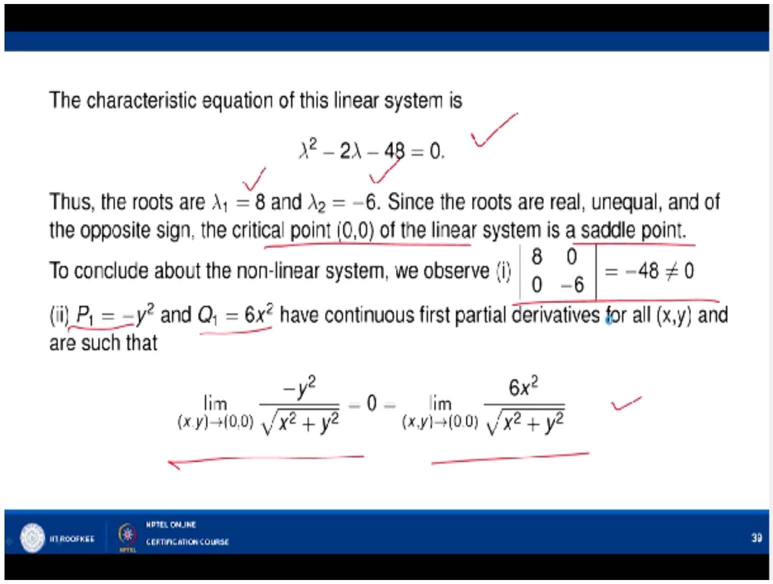

And the characteristic equation of the linear system we can find out like this, lambda square-2lambda-48=0 and it is coming out to be 8 and -6. So it is real, unequal and of opposite sign. So we simply say that for the linear system, your critical point is a saddle point. Now to conclude about the nonlinear system, we observe that the determinant is non-0 and P1=-y square and Q1=6x square have continuous first order partial derivatives for all x,y.

And such that limit x,y tending to 0, 0 -y square/x square+y square=0 and limit x,y tending to 0, 0 6x square/x square+y square=0. So it satisfies all the condition of the previous results. So it means that here we assumed that your weakly nonlinear system satisfy these equation, these condition that a, b, c, d are real constants such that determinant is non-0. Second P1 and Q1 have continuous first order partial derivatives with the condition. So here we are seeing that in example, it satisfies all the condition listed here.

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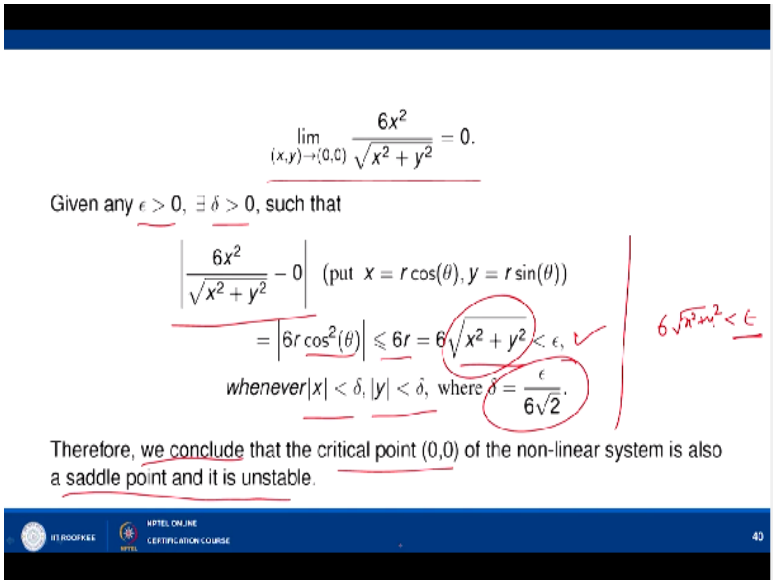

So it means that in this case, the behaviour of the critical point 0, 0 will correspond. It means that we have already seen that 0, 0 for the linear system is saddle point. So for the nonlinear system also, your 0, 0 is a saddle point. So to show that limit x,y tending to 0, 0 6x square/under root x square+y square=0, you may use the epsilon delta condition here.

So this, I am not putting here but you can simply say that to show that limit is equal to 0, you can say that given any epsilon>0, there exist a delta such that mod of 6x square/under root x square+y square-0. This we want to show that it is less than epsilon provided that the distance, means x square+y square is near to 0, 0, okay. So for that, you simply write x=r cos theta, y=r sin theta.

Whenever you have this term x square+y square, you can simply use x=r cos theta, y=r sin theta, that is polar coordinate system. So we can simply say that this is nothing but 6r cos square theta. Now cos square theta is something which is less than or equal to 1. So we can say that it is less than 6r, which is nothing but 6 under root x square+y square. Now if this quantity is less than epsilon and less than delta, then I can say that this is what?

6 under root x square+y square, is this we want that it should be less than epsilon. And so it means that if we simply say that whenever mod of x<delta, mod of y<delta where delta you can assume epsilon/6 root 2, then this quantity can be made less than epsilon. And hence we can say

that limit x,y tending to 0, 0 6x square/under root x square+y square=0 here.

So in a similar way, you can prove the other also that limit of x,y tending to 0, 0 -y square/under root x square+y square=0. So therefore, we conclude that the critical point 0, 0 of nonlinear system is also a saddle point and it is unstable, right. So with this, we conclude our lecture here and in next lecture, we will discuss something more about nonlinear system which is not weakly nonlinear system.

Here we have seen that the behaviour of the critical point in the linear system and in a similar way, we can conclude something about weakly nonlinear system. And what happens, if we have a general nonlinear system, what we can say about the critical point or can we say something or not. So that we will try to discuss in next lecture. Thank you very much for listening us. Thank you.