Dynamical Systems and Control
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Lecture – 25
Phase Portrait of Linear Differential Equations - II

Hello friends. Welcome to this lecture. So in this lecture, we will continue study of phase portrait of the linear system. So in previous lecture, we have discussed 2 case. In both the cases, we have assumed that eigenvalues are real, unequal and may be of same sign or of say opposite sign. So in this case, we will just continue our study. So in this, let us start with the third case. The third case is that roots are real but equal. So roots are real and equal.

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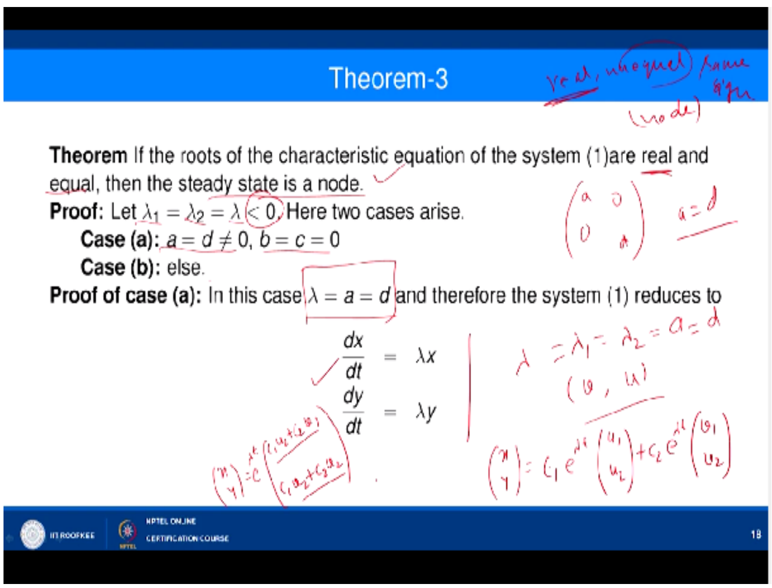

So now in that case, let us have the following theorem. It says that if the roots of the characteristic equation of the system 1 are real and equal, then the steady state is again a node. So here this thing we have already, so in the case when it is real, unequal and of the same sign, then also it is node. And now in this case when we have real but it is not unequal, it is equal, then also it is a node.

So let us see that how it is? So let us say that consider lambda 1 and lambda 2=lambda and let us assume that it is less than 0. So here we have 2 cases depending on a, b, c, d. So first case is that a=d is non-0 and b=c=0. Basically, it is corresponding to the case then we have a scalar matrix here. So it is a and d, diagonal matrix basically, right. Now in fact, in this case, a=d, so we can say that we have a scalar matrix.

And the other case when it is not a scalar matrix. So let us do proof one by one. So let us consider the case 1. In this case, we assume that lambda 1=lambda 2 and matrix is of this kind. So in this case, your lambda is nothing but your, the value of a and d. So lambda=a=d here. So we can write down that system is dx/dt=lambda x and dy/dt=lambda y. So in this case if you recall, your lambda 1=lambda 2 and equal to, let us say, this quantity, one of this quantity a or d.

So in this case when it is a scalar matrix, then this case we are assuming that since it is scalar matrix, your u and v are 2 linearly independent axis because it is a diagonalizable matrix. So it means that in this case, we must have 2 linearly independent eigenvector corresponding to lambda 1-lambda 2=lambda. Let us call this value as lambda. So in this case, your system is reduced to this dx/dt=lambda x, dy/dt=lambda y and corresponding to lambda, we have 2 linearly independent eigenvector u and v here.

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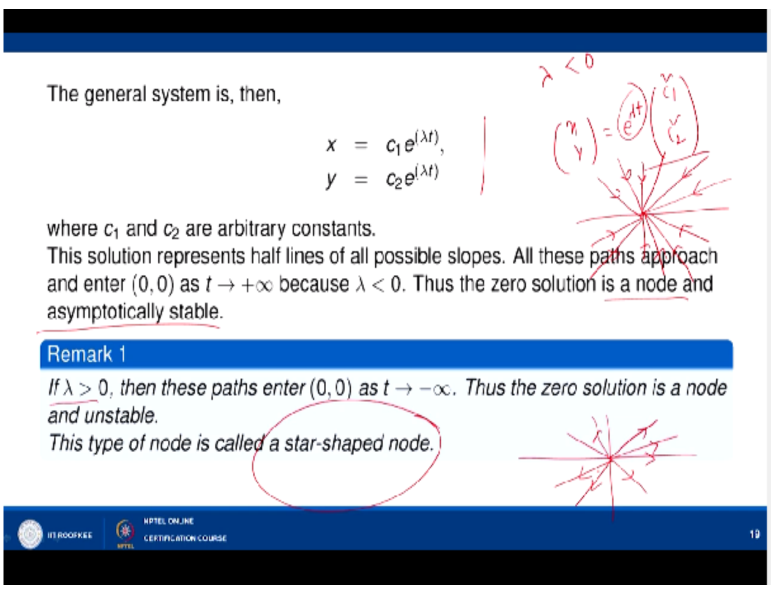

Because we are assuming that linear system is, since matrix a is diagonalizable matrix because it is a scalar matrix. Basically, it is a diagonalizable matrix. So the general system will be solution is then x,y=c1e to the power lambda rt and c2e to the power lambda t, where c1 and c2 are arbitrary constants. In fact, you can look at here, solution will be what? x, y=c1e to the power lambda t u1 u2+c2e to the power lambda t v1 and v2.

So if you look at here, what I can write it here? x, y=, now I can write it here, e to the power lambda t, you can take it out. Then it is what? c1u1+c2v1 and c1u2+c2v2, right. And these are some, since c1 and c2 are arbitrary, so it will take arbitrary values. So we can write down this solution as x, y=, so e to the power lambda t and here we have some constant c1 and c2 here. So c1~ or you can call it, these are some arbitrary constants c1.

Since c1 and c2 are arbitrary constants, so this c1~ and c2~ are also arbitrary. Let us call this as c1 and c2 here. So it means that here c1 and c2 are arbitrary constants. Now this solution represent half lines of all possible slopes. If you look at the slope of, this will be what? If you, just for the time being, let us forget this e to the power lambda t, then this will represent what? x, y=c1~ c2~ and c1~ and c2~ are arbitrary.

So it means that the slope of this will be c2~/c1~. Now these are arbitrary, so slopes are also arbitrary. So it means that this represent the half line with the slope c2~/c1~. Now these are arbitrary, so it means that I can draw several lines, several half lines passing through origin, right. And your solution is tending to 0 because lambda is, we have assumed that lambda<0. So as t tends to infinity, solution will tend to 0 along the line with the slope c2~/c1~.

So it means that your solution will tend to 0 along these lines here, right. So it means that in this case, your origin is a node because all the lines are approaching 0, 0 as well as entering 0, 0 along these lines, okay. So here 0, 0 is node and it is a stable node. Basically it is asymptotically stable node. And in this case, we call that 0 solution is node and asymptotically stable and we also have a special name for these kind of phase portrait and we call this that this type of node is called a star shaped node because every solution is entering to 0, 0 along the half lines, right.

So solution will enter to origin along these half lines. Now c1 and c2 are arbitrary, so these half lines are arbitrary, infinitely many half lines are towards origin 0, 0 here, right. And if lambda>0, then these path enter 0, 0 as t tends to -infinity or we can say that they move towards infinity as t tending to +infinity along these half lines.

So in this case, your half line will remain the same. The only thing is that solution is now moving away from origin. And in this case, 0, 0 is node but unstable node. And these kind of nodes are known as star shaped node. So there is a case when your matrix are having equal roots and basically diagonalizable, that is the case we have considered.

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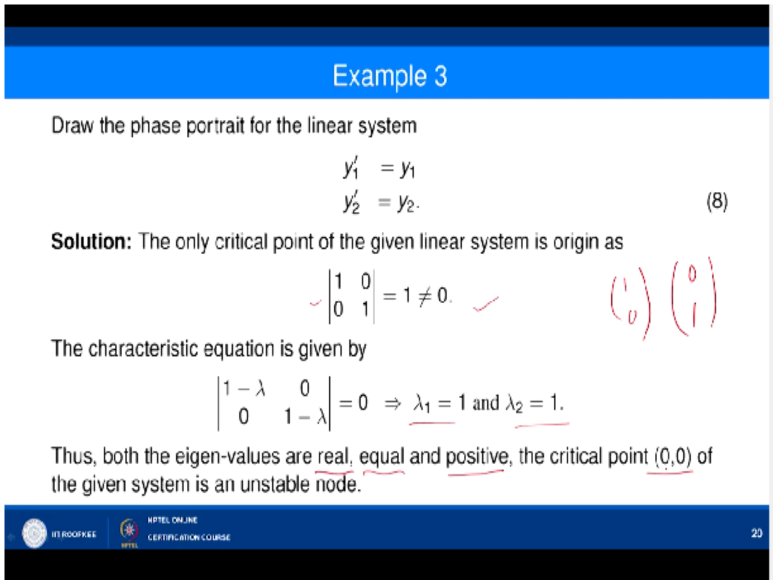

Now look at example based on this. So draw the phase portrait of the linear system y1 dash=y1, y2 dash=y2. The only critical point of the given linear system is 0, 0 because their determinant is coming out to be 1. So it is non-0. So 0, 0 is the only critical point and we can find out the eigenvalue of this. In fact, it is an identity matrix, so eigenvalues are nothing but 1, 1. And you can check that eigenvectors are 1 0 and 0 1 corresponding to lambda 1=lambda 2=1.

So here your eigenvalues are real, equal and positive. So critical point 0, 0 is node and it is unstable. It that okay?

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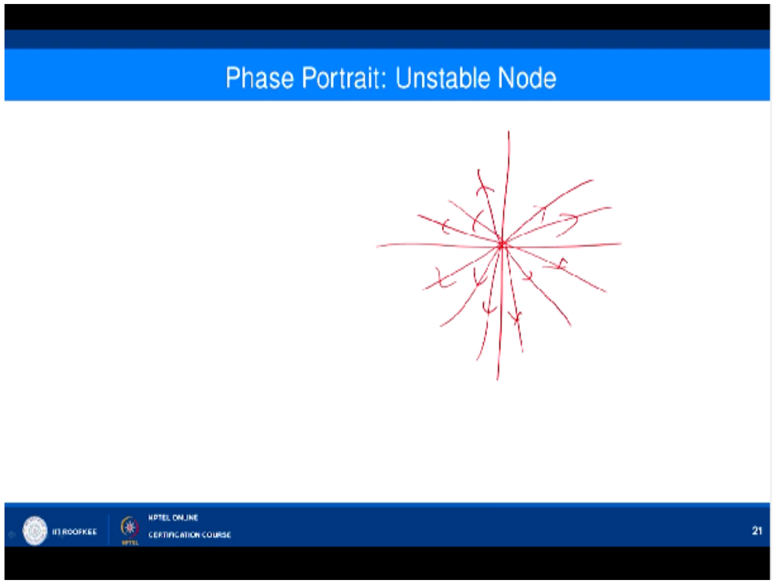

Now your phase portrait will be like this. So you draw several lines and along these lines, it is moving away from origin. So you can draw several lines because constants are arbitrary. So it is star shaped node.

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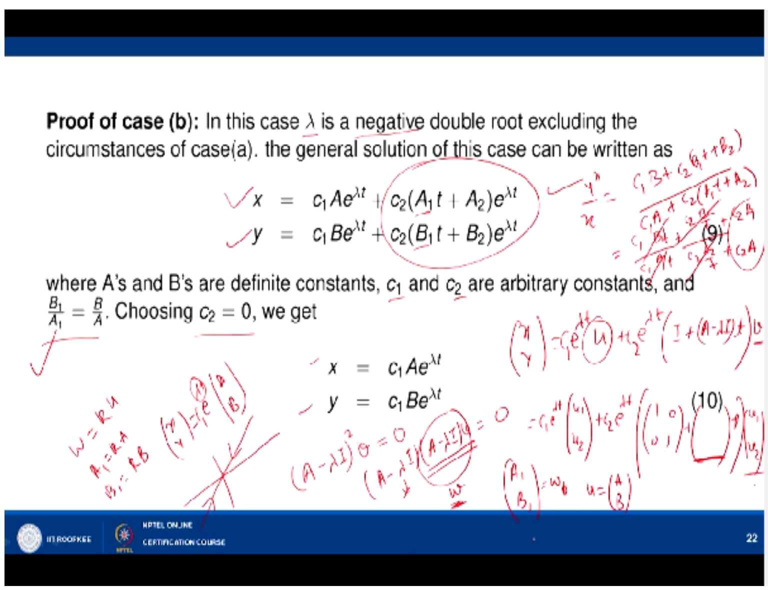

Now proof of case b. So in this case, lambda is negative. Let us assume that it is negative., double root excluding the circumstances of case 1. Now case a basically correspond to the case in which roots are equal but the eigenvectors are, say linearly independent, n linearly independent eigenvectors are there. So it means that the case 1 is basically correspond to the case where the matrix is diagonalizable.

Now this case when matrix is not-diagonalizable, it means that we are not having enough number of linearly independent eigenvectors, then what to do? So in this case, lambda 1=lambda 2. Let us assume that it is negative but we are not able to get your enough number of linearly independent eigenvectors. So in this case, your general solution is given by x=c1Ae to the power lambda t+c2A1t+A2e to the power lambda t; y=c1Be to the power lambda t+c2B1t+b2e to the power lambda t.

So let me write down your solution will be what? x, ye to the power lambda t, your u+e to the power lambda t, let us call this c1 and c2 here. Here we have I+A-lambda It, and here we have say, v here. So if you simplify this, it is what? c1e to the power lambda t, it is u1u2+c2e to the power lambda t, you can take it out, is 1 0 0 1+, this sum matrix, something and t, and this is v1 and v2.

So when you simplify, you will get this kind of form, right. So x, y is given by this. Now not only this, please observe that these AB's are constants and c1 and c2 are arbitrary constants. And we have this thing, B1/A1=B/A. So how we can get this? If you look at this, that basically if you recall how we are obtaining this v here? v is called as generalized eigenvector and we obtained this v as the solution of something that A-lambda I square v is 0, but, yes, it is A-lambda I square v=0. But A-lambda Iv is non-0, okay.

So we have obtained like this. So if you look at, I can write this as A-lambda I operating on A-lambda Iv=0. So if you look at this A-lambda Iv, if you call this as w, then this w is an eigenvector corresponding to the lambda. Now we already have that u is an eigenvector corresponding to lambda. So it means that w must be written as some constant multiple of u. So it means that w if I am writing as, w is basically this A-lambda I*this, so w is basically, if I look at here, w is basically given as A1 and B1.

So here it is w1, so it is w1. Now, sorry w here. Now u is basically your A and B. So if you look at, this w is a constant multiple of u. So it means that w is constant multiple of u. It means that this A1=k*A and B1=k*B here where k is some non-0. So it means that we can easily verified that B1/A1=B/A here. Now here again we have 2 possible cases. Let us say that choosing c2=0 and c1=0.

So if you choose c2=0, then this part is gone and we have x, y=c1Ae to the power lambda t, c1Be to the power lambda t. So corresponding to this, your solution will be x, y=c1e to the power lambda t and here we have simply u that is A and B here, right. So here A B is just a constant. So we can simply, this simply represents a line passing through origin with a slope B/A. So these are lines here. Now as lambda is negative, so solution is tending to 0, 0 here, right.

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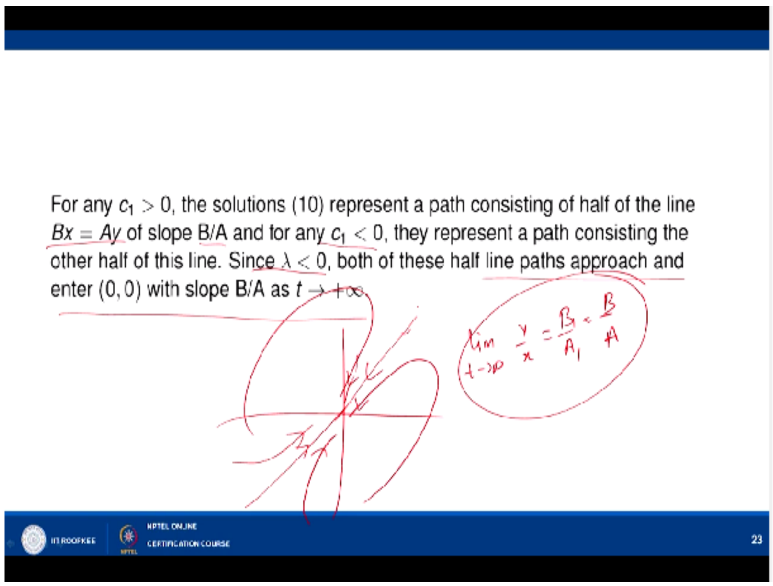

Now as for any c1>0, the solution 10 represent a path consisting of half of the line Bx=Ay. This is the line Bx=Ay, of slope B, A and for any c1<0, they represent the path consisting the other half of this line. Now since lambda<0, both of these half line paths approach and enter 0, 0 with the slope B/A as t tending to infinity. That we have pointed out here that this is the line having slope B/A and since lambda is negative, so as t tending to infinity, x, y both are tending to 0, 0 along the line having the slope B/A, okay.

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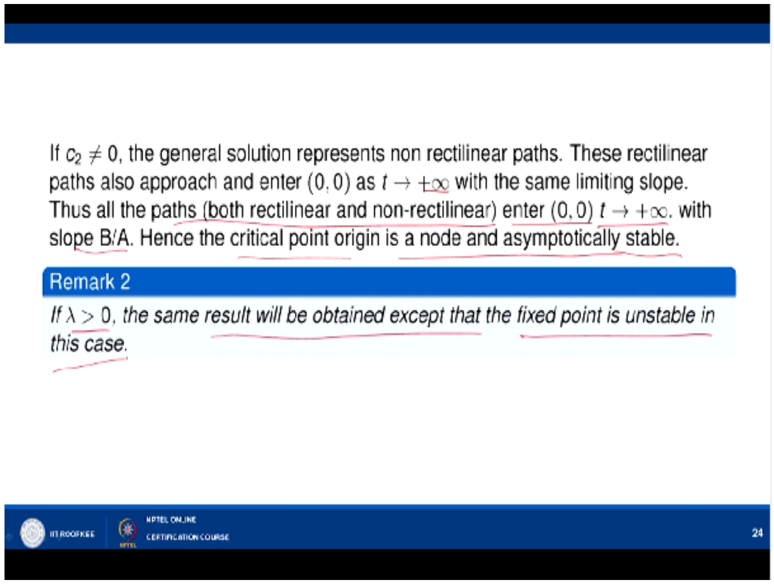

Now what happen if c2 is non-0? So in this case, the general solution represent non-rectilinear paths. And these rectilinear paths also approach and enter 0, 0 as t tending to +infinity and with the same limiting slope. Just look at here, the solution is this. And so let us look at the slope here. So slope will be what? y/x. So y/x will be what? e to the power lambda t, you can take it out. So we will get c1B+c2 and here we have B1t+B2, and here we have c1A+c2A1t+A2.

Now and if we divide it by t here, right, so what you will get? So I will get here c1B/t, here c1A/t, +c2B2/t, here also c2A2/t, +c2B1 and here it is c2A1. So these term will simply vanish. The only thing will left is, that B1/A1. So limit of, so you get limit t tending to infinity y/x=v1/A1. So it is B1/A1 which is same as writing B/A here, right. So it means that the earlier line along with your solution is tending to origin, so it means that this says that even with non-rectilinear path also entering 0, 0 along the same line which we have just drawn.

So it means that ultimately your solution are moving towards origin along the line this. Is that okay? So here, in this case, your all the solutions are entering 0, 0 along the line given by this which is the line containing the eigenvector corresponding to the lambda 1=lambda here. So it means that here we have only 1 eigenvector and solution will tend to 0, 0 along the half line given by this eigenvector here.

So in this case, all the paths, both rectilinear and non-rectilinear, enter 0, 0 as t tending to

+infinity with slope B, A and hence the critical point origin is node and asymptotically stable. Now the other case when lambda is positive, that the same result will be obtained except that the fixed point is unstable in this case. So in this case, your fixed point is unstable.

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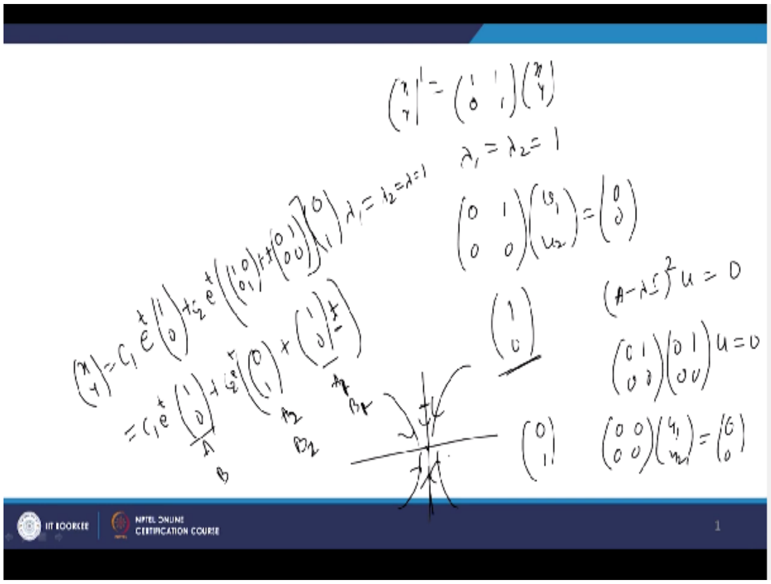

So just for say phase portrait of this, let us consider the following thing. you have x, y dash=1 1 0 1 x y. So here eigenvalues lambda 1-lambda 2=1, let us, okay. So here eigenvalues are same. Now corresponding to eigenvector, if you find out the eigenvector, eigenvector corresponding to lambda 1=lambda 2=lambda=1, it is 0 1 0 0. So and v1 and v2. So solution will be that v2=0. So your solution, eigenvector will get only this 1, 0.

So there is no other linearly independent eigenvector available. So in this case, you have to find out the other eigenvector, means generalized eigenvector which is a solution of A-lambda I square u=0. So if you look at A-lambda, lambda is 1 here, so it is 0 1 0 0 whole square, it means 0 1 0 0 and u=0. So it is nothing but 0 matrix, u1 and u2. So you can say that your, this will give you what?

This will give you that u1 and u2 may be any of choice. So we will choose the eigenvector which is linearly independent to the earlier ones. So it is let us say that 0, 1 is your generalized eigenvector. So solution will be written as x, y=say, c1e to the power t and eigenvector is 1 0+c2, your, e to the power t, and here we have I that is 1 0 0 1, +t*, A-lambda A that is 0 1 0 0, and here it is 0 1.

So you simplify c1e to the power t 1 0+c2, e to the power t you can still write and if you multiply, you will get 0 1 and here if you multiply, you will get 1 0 and t, right. So here it is if you recall this case, here. So here if you look at, A and B will be what? So here you can say we want to verify this B1/A1=B/A, so that we can verify here. So this is A and B and this is your A1B1, sorry, what is this?

It is A1B1, okay. So it is A1B1 and this is A2B2. No A1 corresponding to t, so here this is A2B2 and this is A1B1, because it is corresponding to. So we can easily see that here A and B and A1 B1 are same, exactly same. So of course ratio will be same, slope will be same. B1/A1 is same as B/A here. Is that okay?

So in this case, it is 0. So in fact, these are coming out to be same here, okay. So all the solution in this case will, so eigenvector is 1 0. So 1 0 means, 1 and 0. So 1 0 is a point here, x=1 and y=0, so it means that all these are, solution will tend to 0 and any other will tend towards 0. So your phase portrait will be like this, okay.

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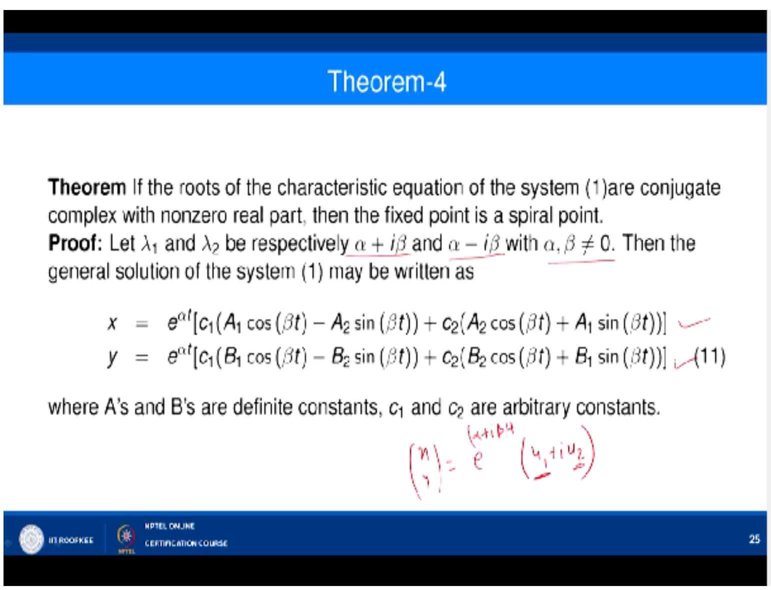

So now consider the next case, theorem 4. And it is corresponding the case when eigenvalues are complex and it is not purely imaginary. So it means that if the roots of the characteristic equation of the system 1 are conjugate complex with non-0 real part, then the fixed point is a spiral point. So let us have the proof of this. So let lambda 1 and lambda 2 be given by alpha+i beta and alpha-i beta and none of them is 0.

So since we are assuming that it is complex, so beta is non-0 and we are also assuming that it has non-0 real part. It means that alpha is also non-0. Then the general solution of the system 1 may be written as this. x=e to the power alpha t and so on, this one. So here how we can obtain this? If you recall, your solution will be what? x, y=e to the power alpha+i beta t and u1+i u2, right and when you simplify, you will get this, right.

So it means that, this I am leaving it to you that please verify that your solution is given by equation number 11 here. So here Ai's and Bi's are different, definite constants and c1 and c2 are arbitrary constants. So definite constants means your A1 B1 is coming from this and another A1 B2 is given by this.

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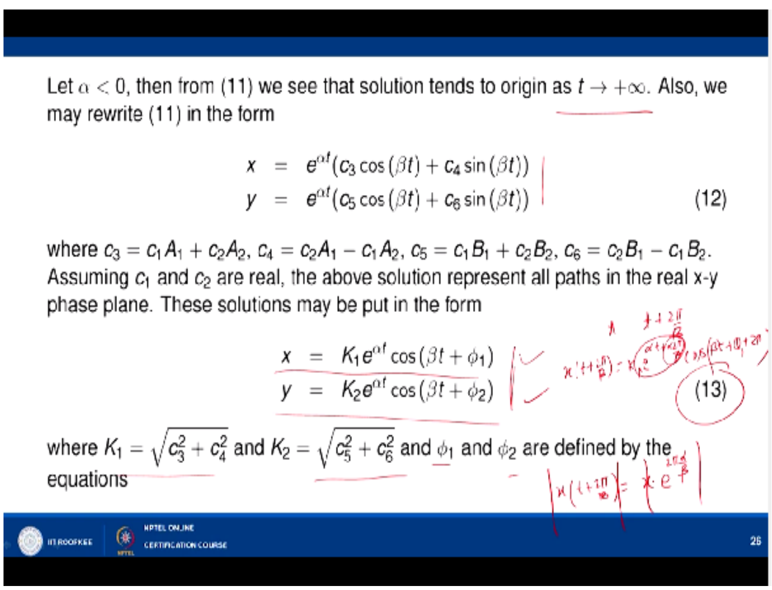

So it means that let alpha<0. Now let us assume that since alpha is non-0, so it may be positive, it may be negative. So let us consider that alpha<0. So let alpha<0, then from 11, we see that the solution tends to origin as t tending to infinity. So as alpha is negative, t tending to infinity means this term is tending to 0, similarly this term also. So x, y is tending to 0, 0 as t tending to infinity.

Now we can rewrite our solution x and y in this following format. x, y as e to the power alpha tc3cos of beta t+c4sin of beta t and y as e to the power alpha tc5cos of beta t+c6sin of beta t. And where c3, c4, c5, c6 can be given as the earlier c1, c2, and A1 A2 B1 B2, we can write it like that. This is not very difficult. You rearrange and you simply arrange the coefficient of cos of beta t and sin of beta t, you will get this.

Now again we can rewrite this in a very compact form and it is given by this, x=k1e to the power alpha t cos of beta t+phi 1 and y as k2e to the power alpha t cos of beta t+phi 2. Now here this is using geometric, here k1 is given by under root c3 square+c4 square and k2 is given by under root c5 square+c6 square.

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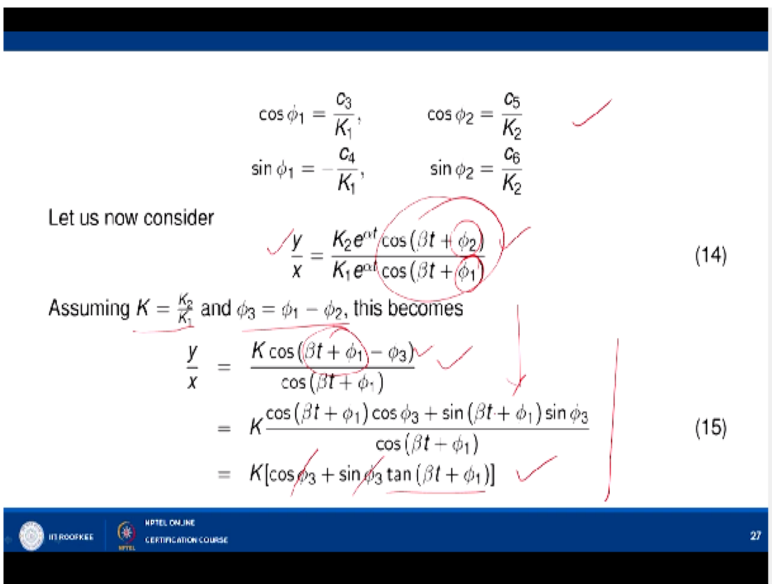

And phi 1 and phi 2 are defined by the equation, this that cos of phi 1 is c3/k1, cos of phi 2=c5/k2, sin of phi 1=-c4/k1 and so on. So it means that, I am assuming that you can rewrite equation number 11 as equation number 13 here. So this is the more compact form and we start working with this. Now idea is to say that here since t tending to infinity, your x and y, both component will tend to, will approach to 0, 0.

Now we want to know that what is the limiting slope. If limiting slope is finite, we call that origin as node. If they are approaching but not entering 0, 0, then the origin is known as the spiral point or focal point. So here let us find out the slope here. Slope is y/x=k2e to the power alpha t cos of beta t+phi 2 k1e to the power alpha t cos of beta t+phi 1. Now in this phi 2 and phi 1, since if c1 and c2 are arbitrary, then this phi 2 and phi 1 may not be the same thing.

So here by simple rearrangement, let us assume that k=k2/k1 and phi 3 as phi 1-phi 2. And assuming that phi 1 and phi 2, since c1 and c2 are arbitrary, so phi 3 is not ideally equal to 0 here. So y/x, we can rewrite like this. So kcos of beta t+, phi 2 I am replacing as phi 1-phi 3, so we are writing. And cos of beta t+phi 1. Now here we can simply simplify this using this as A-, so cos of A-B you can apply and it is written as kcos of beta t+phi 1 cos of phi 3+sin of this thing/this.

Or we can rewrite this thing. The idea is that if take the limit here, then this is independent of t, this is independent of t, this will depend on t. So as t tending to infinity, tan of beta t+phi 1 will tend to infinity. So it means that limit does not exist, right. So we simply say that limit t tending to infinity, y/x is not a finite quantity. So it means that in this case, origin is not a node here. So though we know that 0, 0 is stable but not a node.

And you can understand why we are doing all this thing. We can do it here also, but the thing is that here we simply say that here, these will cancel out. So k2/k1, this may be equal if phi 2 and phi 1 are equal. So here you simply say that limit t tending to infinity, limit does not exist. That is why we did these kind of say arrangement.

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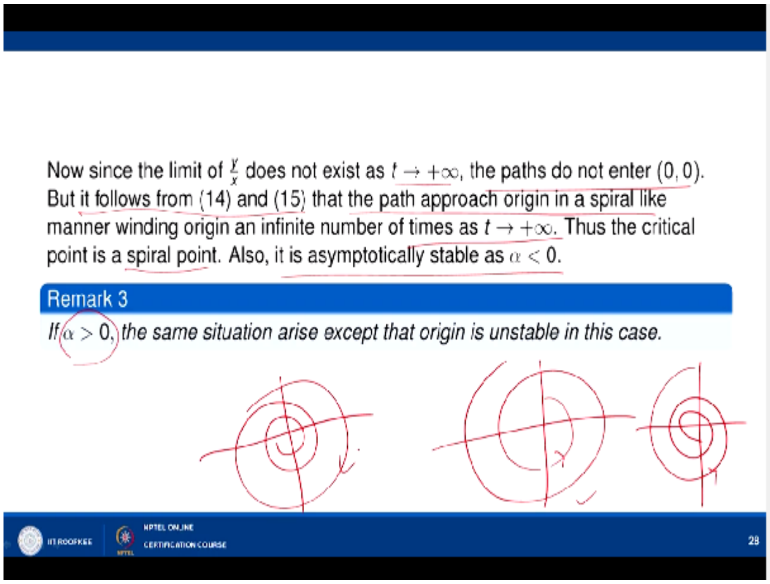

So here what we have concluded here? Now since the limit of y/x does not exist as t tending to infinity, the paths do not enter 0, 0 but it follows from the previous equation that the path approaches origin in a spiral like manner. So it means that it is tending to 0, 0 but it is not taking any say slope. So it means that there is no finite slope. So it means that there is a possibility that it will approach to origin in a spiral like manner, winding origin in an infinite number of times as t tending to infinity.

Thus the critical point is a spiral point. And also it is asymptotically stable as alpha<0. And if alpha>0, then situation will reverse. It means that it will be spiral but not a stable one. It will be unstable spiral. And here we simply say that origin is unstable spiral. Now here we can conclude the same thing from this. Here if you look at this, what you will get here? So we can write x=k1e to the power alpha t cos of beta t+phi 1, y=k2e to the power alpha t cos of beta t+phi 2.

So here we, in place of writing cos of beta t+phi 1, we can again simplify and from here also we can get that it is basically kind of a periodic if you look at here, it is periodic of 2pi/beta, right. So x is periodic with period 2pi/beta and y is also periodic with period 2pi/beta here. Is that okay? So here if I replace t by t+2pi/beta. So what you will get? You will get x of t+2pi/beta will be k1e to the power alpha t+alpha 2pi/beta and cos of beta t.

So beta t will be beta t and here beta will be cancelled. It is 2pi. So beta t+phi 1+2pi. So it is nothing but same. So it means that it will be coming out to the same thing. The only thing is that now radius is changed. It is not e to the power alpha t anymore. Now it is e to the power alpha, so it will be like, I can write it x of t+2pi/beta=x*e to the power 2pi alpha/beta, right. So here we have assumed that alpha is negative.

And let us say that beta is something. So here we can say that here your ray is changing, right. So here we simply say that I am assuming that now that beta is positive here. So we simply say that the distance is reduced from the earlier one, right. So here we simply say that it is tending to 0, 0 with say radius lesser than the earlier one. So it means that it is basically, it is something like this, oh, sorry, just opposite, or this way.

So again there are 2 possibilities. Whether it will follow this or it will follow this. It depends on, I think it is same, sorry, it is like this. So it is the, this thing side, okay. So here there are 2 possibilities.

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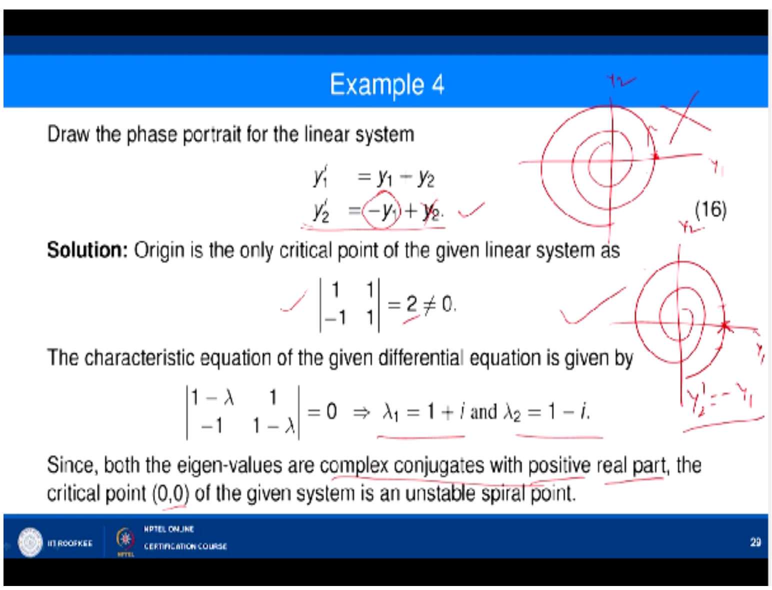

Now let us consider one example. So it will follow which path, that will depend on the equation itself. Now let us look at this example, example 4. Draw the phase portrait for the linear system y1 dash=y1+y2, y2 dash=-y1+y2 and we want to look at the behaviour of origin that is 0, 0. And we can check that origin is the only critical point of the given linear system that is because determinant is non-0.

And look at the eigenvalues here. So eigenvalues is coming out to be lambda 1=1+i and lambda 2=1-i. So eigenvalues are complex conjugate to each other with positive real part. So in this case, 0, 0 is the spiral point but it is unstable spiral point. And we want to find out the phase portrait in this case. So here there are 2 possibilities that the solution will move like this or solution will move.

So which one it will follow? Whether it is going the anticlockwise or it is moving the clockwise. It is anticlockwise here. So which path it will follow? So for that, choose a point on, let us say choose this point, right. Similarly, here. So let us say this is y1 here and this is y2 here. This is y1 here and y2 here. So at this particular point, your y2 is 0 and y1 is positive. And here if you look at above this, y2 is positive and below this, y2 is negative.

So basically we want to look at the behaviour of y2 at this particular point. So to look at the behaviour of y2, look at the second equation, differential equation in terms of y2. So here y2 dash=-y1+y2. So at this particular point, your y1 is positive. So this is negative. y2=0, so this is simply, so at this point your y2 dash=-y1. So y1 is positive. So y2 dash is negative. So it means that y2 is decreasing.

So it means that your y2 is coming from positive to negative side because y2 is 0 here and here it is decreasing. So it means that y2 is going to negative side rather than positive side. So it means that it will follow this path only, rather than this path. Because here, at this particular point, your direction is this way. So here at this particular point, your y2 is moving from negative side to positive side.

It means that here y2 dash has to be positive but from the equation, it is coming out that y2 dash is negative. So it means that this is not the correct phase portrait. This is the correct phase portrait, okay. So here we have seen that both the eigenvalues are complex conjugate with this positive real part, the critical point 0, 0 of the given system is unstable spiral point here. So with this, we will finish our lecture here.

So in this lecture, what we have discussed 2 cases, one is when eigenvalues are real but equal. In that case, we have shown that origin is node but whether it is star shaped or say it is only simple a node, that will depend on the eigenvectors corresponding to the eigenvalues lambda. And other case, we have considered is that when lambda 1 and lambda 2 are complex conjugate to each other and we have non-0 real part.

In this case, we have shown that it is a spiral point. Now whether it is stable or unstable, that will depend on the real part of the eigenvalue lambda. So if real part of the eigenvalues are positive, then it is unstable. If real part of the eigenvalue is negative, then it is stable spiral point. So in next class, we will consider one more case and we will discuss some more properties. So with this, we will conclude and thank you very much for listening us. Thank you.