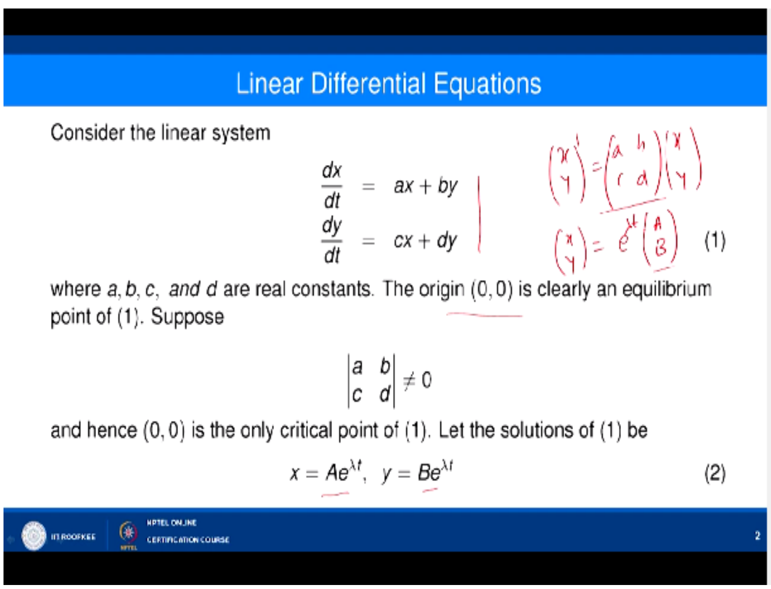
Dynamical Systems and Control
Prof. D. N. Pandey
Department of Mathematics
Indian Institute of Technology - Roorkee

Lecture – 24
Phase Portrait of Linear Differential Equations - I

Hello friends. Welcome to this lecture. In this lecture, we will continue with study of phase portrait. So in previous lectures, we have discussed what do you mean by phase portrait and we have discussed the different types of critical points. So in this lecture we will discuss more about phase portrait. In fact, we will focus on linear system of dimension 2 so that we can portrait the phase portrait on your xy plane.

And we also try to see that the critical point of the linear system is of what type. So for that, let us consider the following linear differential equation. So consider the linear system dx/dt=ax+by.

(Refer Slide Time: 01:11)

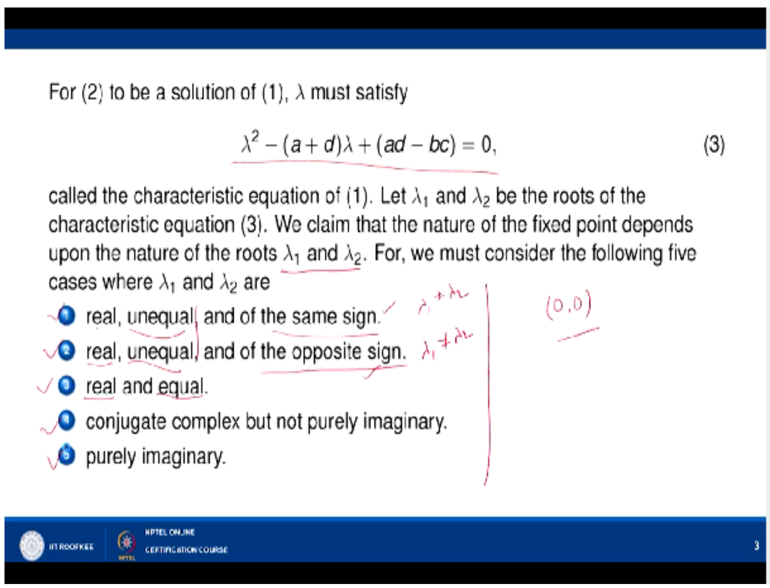


And dy/dt=cx-dy where a, b, c, d are real constants. And here we can say that this system has 0, 0 as an equilibrium point. And if we impose one more condition that determinant of a b c d is non-0, then 0, 0 is the only critical point of this system 1 here. Now let us find out the solution of 1 here. So for that you simply assume that x=Ae to the power lambda t and y=Be to the power lambda t be 2 possible solution of this.

And then try to find out the solution of this. In fact, if you look at, here if I simply denote this xy as vector, then it is nothing but your a b c d and x y it is written here. So it is a system like this and your solution which we are assuming is the following that x y=e to the power lambda t and this A and B here. So here we need to find out this vector A B and vector AB and the constant lambda such that it will work as a solution here.

So that is why we have writing that x=Ae to the power lambda t and y=Be to the power lambda t be the possible form of the solution. And we need to find out the condition, the values of A B and the lambda. So here we have already seen that this lambda is the characteristic note of the matrix A and AB is the corresponding eigenvector for eigenvalue lambda here.

(Refer Slide Time: 03:02)



So for that, look at here. Then for 2 to be a solution of 1, lambda must satisfy this equation that is lambda square-a+d lambda+ad-bc=0 which is nothing but the characteristic equation of 1 in terms of lambda here. So here we call this equation number 3 as the characteristic equation of 1. And since it is a quadratic equation, then there is a possibility of having 2 roots lambda 1 and lambda 2.

So let lambda 1 and lambda 2 be the roots of the characteristic equation 3. And we claim that the nature of the fixed points that is 0, 0 depends upon the nature of the roots of lambda 1 and lambda 2. So here we already know that 0, 0 is a critical point or say equilibrium point or stationary point or fixed point. And the type of the critical point whether it is say node or say stable node, unstable node, focus, spiral, it will all depend on the behaviour of lambda 1 and lambda 2.

So based on the root, lambda 1 and lambda 2, we have following 5 cases and we will consider each and every case and based on each case, based on the conditions given in each case, we try to find out the behaviour of 0, 0 whether 0, 0 is what kind of critical point. So first case based on lambda 1 and lambda 2 is that both lambda 1 and lambda 2 are real, unequal and of the same sign.

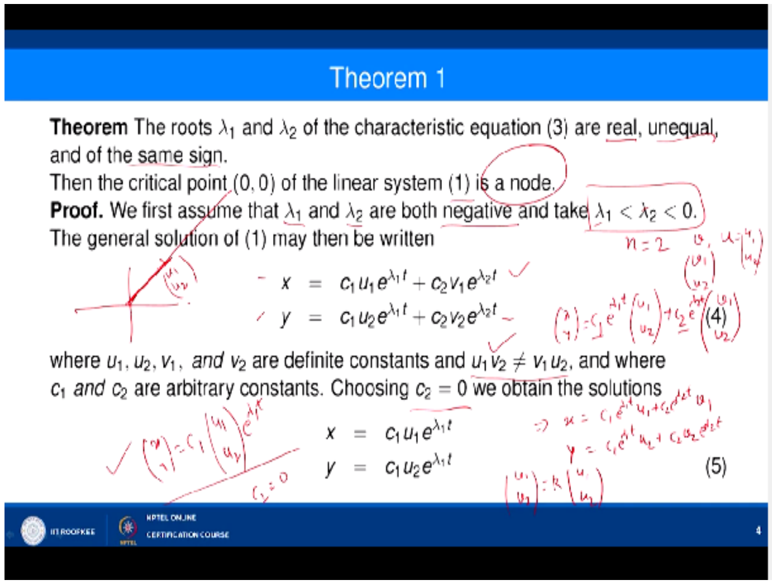
So it means that here lambda 1 is not equal to lambda 2 but the sign is same whether, it means that both may be positive or both may be negative. And the second case is real, unequal but in this case, they are having opposite sign. So it means lambda 1 is not equal to lambda 2 and it may happen that lambda 1 is positive but lambda 2 is negative and it may be the otherwise also. So this is the case 1 and 2.

Here lambda 1 is not equal to lambda 2. One is that they are having the same sign and the other case is they are having the opposite sign. Now third case that they are real and equal. So it means that the first 2 case are corresponding to unequal roots and the third case is corresponding to equal. So 1, 2, 3, they are all real. Lambda 1 and lambda 2 both are all real. And it may be unequal or equal.

And the fourth case and fifth case are corresponding to the case when lambda 1 and lambda 2 are complex. So they are not real. So again, in complex, we have 2 subcases, that is conjugate complex but not purely imaginary. So it means that they are having both real part as well as the imaginary part. And fifth case is corresponding to purely imaginary part. So it means that real part is missing here.

So these are the 5 cases and based on each case, we have 1 theorem which tells us that under these condition, the behaviour of 0, 0 is what.

(Refer Slide Time: 06:25)



So let us consider the first theorem which corresponding to the case 1. So here the theorem says that the roots lambda 1 and lambda 2 are of characteristic equation 3 are real, unequal and of the same sign, right. Then the critical point 0, 0 of the linear system 1 is a node. So here the case 1 which is for lambda 1 and lambda 2, both are real, unequal and of the same sign. Then the critical point 0, 0 is a node.

Now whether it is a stable node or unstable node, that will depend on the sign of lambda 1 and lambda 2. So first let us assume that lambda 1 and lambda 2 are both negative and let us assume this that lambda 1<lambda 2<0. Since they are real and unequal, so we have, we can take this condition lambda 1<lambda 2<0. So here we, just for simplicity, we are assuming that lambda 1<lambda 2 and greater than 0.

So it may be otherwise also. Now in this case, your solution of 1 may be written as x=c1u1e to the power lambda 1t+c2v1e to the power lambda 2t and y as c1u2e to the power lambda 1t+c2v2e to the power lambda 2t. In fact, in this case when your dimension is 2 and we have 2 distinct eigenvalues, then we have 2 distinct eigenvectors, this we have already run. So let us call this as v and u are 2 eigenvectors.

So u is given as u1 and u2 and v, I am writing as v1 and v2 here. So let me write down the solution here. Solution xy is given as c1e to the power lambda 1t. Now corresponding to lambda

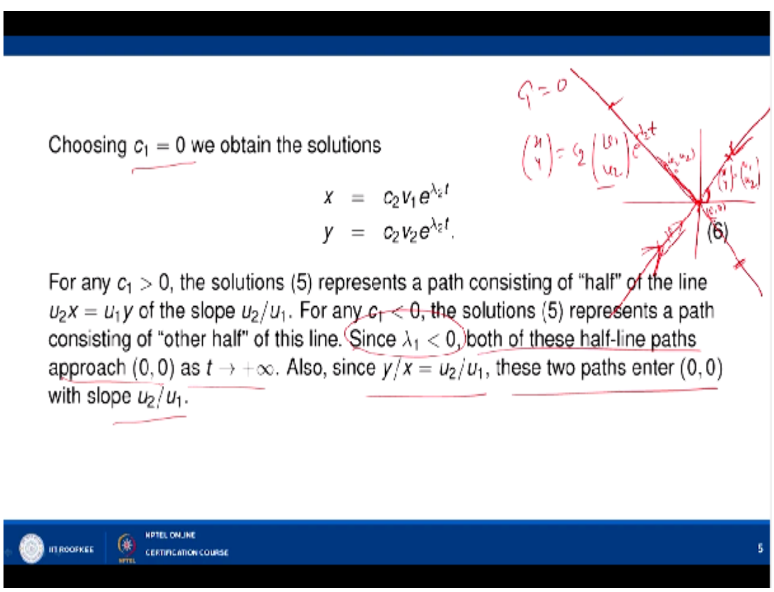
1, we have eigenvectors say u1u2+c2e to the power lambda 2t and corresponding to lambda 2, we have v1 and v2 here. So when you simplify, you will get what? x=c1e to the power lambda 1tu1+c2e to the power lambda 2v1.

Similarly, y, I can write it, c1e to the power lambda 1tu2+c2v2e to the power lambda 2t here. So the solution, form of the solution is given from this, okay. So here, this u1 u2 v1 v2 are some constants and here we are assuming that u1v2 is not equal to v1u2. So that is very obvious from here because here v and u are linearly independent to each other. So it means that I cannot write v as some constant multiple of u here.

So it means that, so v cannot be written as constant multiple of u means, let us take v1v2 as say k*, say, u1 and u2. And we can say that this is, this condition is easily verified here. And now here, since c1 and c2 are some arbitrary constants and we can write down the general solution as xy here. Here now we consider different cases corresponding to this c1 and c2. So now let us assume that c2=0.

So when c2=0, then we have the solution x=c1u1e to the power lambda 1t, y as c2u2e to the power lambda 1t or I can say that it is nothing but xy=, say c1 and here we have u1 and u2 and e to the power lambda 1t here, right.

(Refer Slide Time: 10:19)



So now similarly, choosing c1=0. Here we have assumed c2=0, so we have solution like this. And if we assume that c1=0, then your solution will be xy=c2, here we have v1 and v2e to the power lambda 2t, right. So now once we have, this is corresponding to c1=0. And the earlier one is corresponding to c2=0. Now here consider this case when c2=0, but c1 is non-0, then c1 may be positive and c2 may be negative here.

So here if we consider that c1 is positive, then this u1 and u2 simply represent a vector here and we can write here that, let us consider this. So u1 u2 represent a point here somewhere, right. So now this c1u1u2e to the power lambda 1t is basically, this is your u1 u2. Now c1 is just a scaling vector and e to the power lambda 1t is some scaling vector. So it will give you some point on this line.

So it means that this will represent a line passing through the point u1 and u2 here, right. And it is passing through this. Similarly, if you look at this, this may represent, suppose v1v2 is somewhere here, v1, v2 here. And this line represent the vector v1 and v2. And if c2 is positive, then it will represent the line like this. And the earlier one line is like this. Now if c1 is negative, then this will be here and if c2 is negative, then this will be here, right.

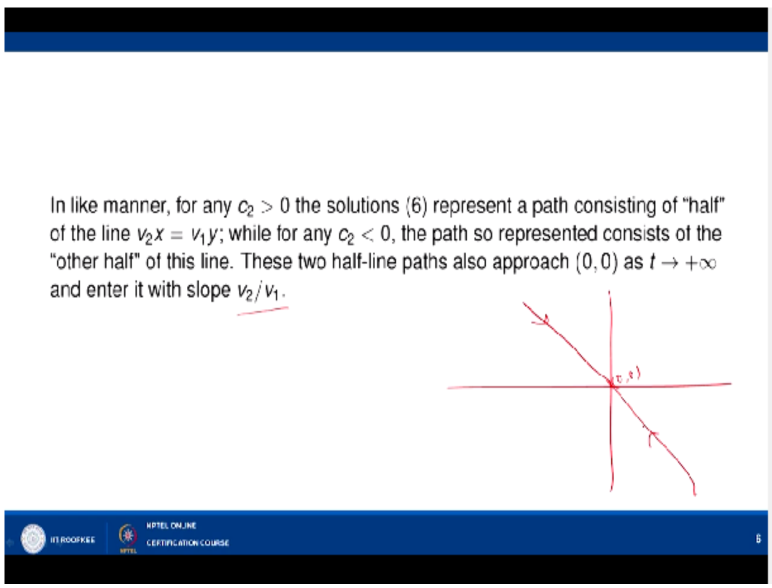
So it means that when c2=0, then depending on c1, whether c1 is positive, then it will represent this line. And if c1 is negative, this will represent this line. And similarly, when c1=0, then it will be along your vector v1 and v2 and c2 positive will give you this line and c2 negative will give you this line. So it means that for any c1>0, the solution 5 represent a path consisting of half of the line u2x=u1y.

So this line is equation is given by xy=, say, u1 and u2. So we can write down this line as u2x-u1y. And the slope is given by u2/u1. So this line has a slope given by u2/u1 here, right. Now for any c1<0, the solution 5 represent a path consisting of the other half, like other half, this. Now here, one thing we may note down that lambda 1 is negative. So it means that as lambda 1 is negative, so as t tending to infinity, your xy is tending to 0 along the line given by this u2x and u1y.

So it means that as t tending to infinity and c2 is 0, then your solution will tend to 0, 0 here along the line this with slope u2/u1, right. So it means that your solution will come to here. Now similarly, we can say that solution will tend to 0, 0 if c1 is negative and as t tending to infinity, solution will tend to origin through on this line itself.

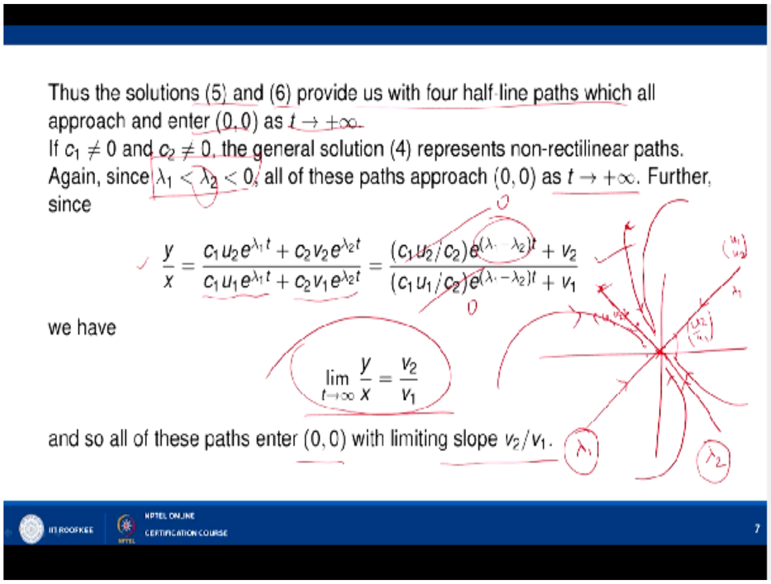
So we simply say that both of these half line paths approach 0, 0 as t tending to infinity. Now we already know that y/x=u2/u1, so these 2 paths enter 0, 0 with the slope u2/u1 here. So it means that as t tending to infinity, your solution will enter origin 0, 0 as t tending to infinity with the slope u2/u1 here.

(Refer Slide Time: 14:46)



Now the other, in the same manner, we can say that this is your line, right. So here also, your solution will enter to origin with the slope v2/v1 in the same way. So if c2 is positive, then it will come through this line. If c2 is negative, then it will come through this line, right. So it means that solution will enter 0, 0 along a particular line with the given slope, right.

(Refer Slide Time: 15:22)



Now let us assume that, so it means that the solution 5 and 6 provide us the 4 half line path which all approach and enter as t tending to +infinity. Now let us assume that none of the constants c1 and c2 are 0. So it means that if c1 and c2 are non-0, then the general solution 4 represents non-rectilinear path. Means your x is written as c1u1e to the power lambda 1t+c2v1e to the power lambda 2t and y is written as c1u2e to the power lambda 1t+c2v2e to the power lambda 2t.

So again, since lambda 1<lambda2<0, all of these path approach 0, 0 as t tending to infinity but not along the given half line path. So let us find out that whether it will enter to origin 0, 0 along some line or not. So for that, you just find out the slope y/x here. So y expression is given, x expression is given, so you simply find out the ratio of y/x and ratio of y/x is given by c1u2e to the power lambda 1t+c2v2e to the power lambda 2t/cau1e to the power lambda 1t+c2v1e to the power lambda 2t.

Now since lambda 1<lambda 2, so let us divide by e to the power lambda 2t. So when you divide by e to the power lambda 2t, you will get this c1u2+c2e to the power lambda 1-lambda 2t+v2 and c1u1/c2e to the power lambda 1-lambda 2t+v1. Now as t tending to infinity, this quantity is now negative. So this term will tend to 0, this term will tend to 0. So limit of t tending to infinity y of x is given by v2/v1 here.

Now what is v2 and v1? Basically it is an eigenvector corresponding to the eigenvalue lambda 2 here. So here, let me look at here. Here, yes, this v1 and v2 is the eigenvector corresponding to lambda 2. So here we simply say that you look at this line. So this is the line corresponding to lambda 1 and the eigenvector u1 and u2. So slope is u2/u1, the inverse of u2/u1. And similarly, this is the line which is having slope v2/v1.

So it is having the slope v2/v1 and it is passing through v1, v2, right. So we already know this thing. Now it is seen that if we take any path and consider the path t tending to infinity, then this represents that your solution will tend to 0, 0, approach to 0, 0 along the line this, right. And since it has a definite slope, so we say that your every solution will enter 0, 0 with limiting slope v2, v1.

So it means that if we consider a solution which is starting from this, then ultimately as t tending to infinity, it will try to come to 0, 0 and towards the line having slope v2, v1. So if we have some point here, it will try to come to something like this. And similarly here, if we have any path and it will try to tend to 0, 0 along this line, along these half lines. So if we have a point starting from this, it will try to have, say tend to origin along the line this, right.

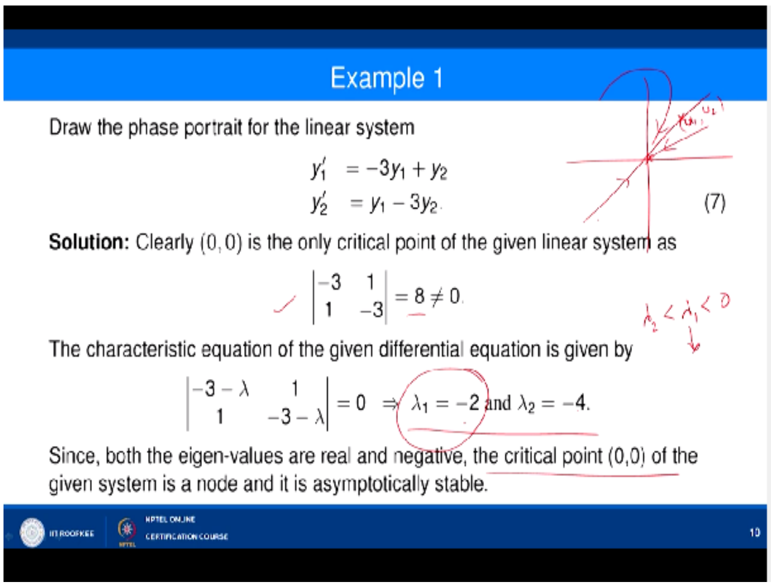
So if we consider the situation, this is the eigenvector corresponding to lambda 1 and this is the eigenvector corresponding to lambda 2. And since lambda 2 is bigger than lambda 1, then solution will tend to origin along the eigenvector corresponding to lambda 2. So it means that this is the eigenvector, this is the line corresponding to the eigenvector v1, v2 here. Is that okay? So it means that in this case, your origin 0, 0 is a stable node because all the solution is tending to 0, 0.

In fact, entering to 0, 0 along a line, it means having a limiting slope v2, v1 here. So we can say that origin 0, 0 is node. In fact, it is stable node or we can say since it is entering 0, 0, so it is asymptotically stable node. So in this case, when lambda 1 and lambda 2, both are real, unequal and negative, then solution will tend to 0, 0 and enter 0, 0. So in this case, origin is asymptotically stable node.

So thus in summary, we can say that thus all the path both rectilinear and non-rectilinear, rectilinear means along the straight lines. And non-rectilinear means along the curves. Enters 0, 0 as t tending to +infinity and all except the 2 rectilinear ones defined by 4 and enter with slope v2/v1, according to the definition, the critical point 0, 0 is a node and clearly it is asymptotically stable node here.

So this is the statement theorem 1. And let us consider one example based on this. So rather phase portrait for the linear system y1 dash=-dy1+y2, y2 dash=y1-3y2. And here we can identify that 0, 0 is the critical point and then now look at the determinant of the linear part, that is -3 1 1 -3 and you can see that determinant is coming out to be non-0.

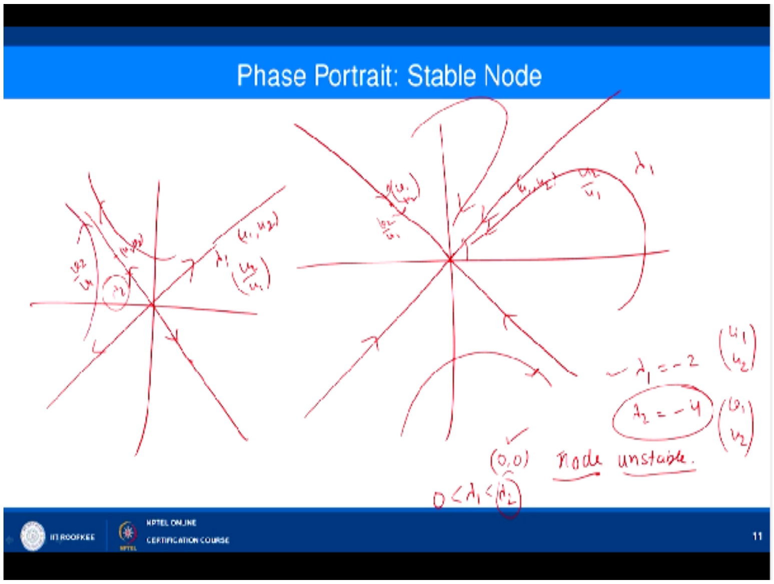
(Refer Slide Time: 22:14)



So since determinant is non-0, so we can say that 0, 0 is the only critical point. Now let us look at the eigenvalue corresponding to this linear matrix. So matrix is -3-lambda 1 1 -3-lambda and we can find out the eigenvalues as lambda 1=-2 and lambda 2=-4. So both are unequal and real and negative. So it means that we can apply the previous theorem and we can say that the critical point 0, 0 is node and it is asymptotically stable.

So it means that solution will tend to 0, 0 along a slope corresponding to this lambda 1=-2. So here I can write it lambda 2<lambda 1<0 here. So here we need to find out that it will, so corresponding to lambda 1, if it is the eigenvector, so eigenvector corresponding to this means u1 and u2. So it is passing through u1 and u2. So it means that your solution will ultimately enter to 0 along this line. Is that okay? So the phase portrait will be like this and the critical point 0, 0 is a node and which is asymptotically stable node.

(Refer Slide Time: 23:50)

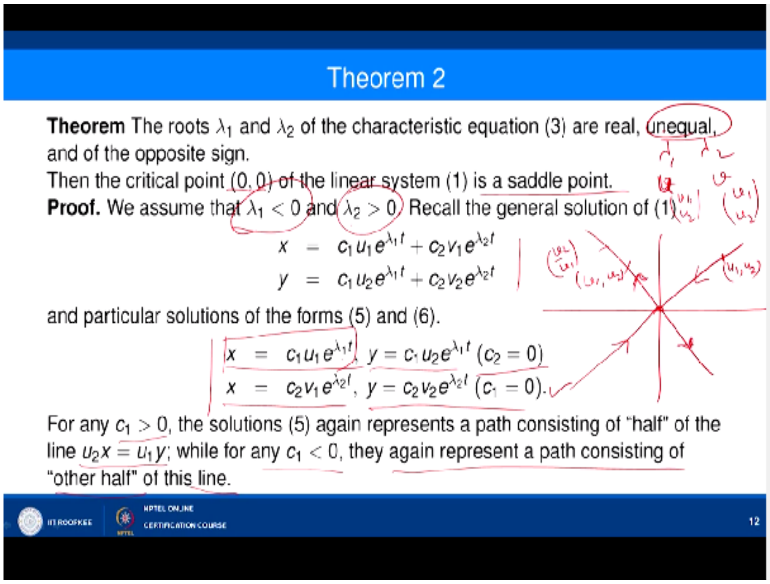


Now this is the phase portrait we can draw here. So let us say we have lambda 1=-2, lambda 2=-4 and let us say that lambda 1, corresponding to lambda 1 we have u1 and u2 as the eigenvector and corresponding to lambda 2, we have v1, v2 as the eigenvector. So let us first find out u1 and u2. Let us say that u1 and u2 here, you can find out u1 and u2. So this is the line which represents, line segment which represents the eigenvector corresponding to lambda 2=-2.

So it means that now line segment you simply extend which continue u1 u2. And this will represent the half lines having the slope u2, u1. And similarly, we can have v1 and v2. Let us say v1 and v2 is here. I am just giving a pictorial representation. You can actually find out the values of u1 and u2 and you can draw it here. So here suppose your v1 and v2 is here. So this will represent eigenvector.

And if you extend these line segment here, then it will represent the line segment having the slope v2, v1 here, right. And along this also, your solution will tend to 0, 0 here. So here the slope of this is v2/v1 and slope here is u2/u1, right. And these are the line segment and since this is corresponding to your lambda 1, so your solution will ultimately tend to 0 along this line, okay. So somewhere we have a solution and it will tend to 0, 0 along the line this. Is that okay? So here we have a, the origin is asymptotically stable node.

(Refer Slide Time: 25:46)



Now consider next theorem, theorem 2. Here the roots lambda 1 and lambda 2 of the characteristic equation 3 are real, unequal and of the opposite sign. Now one more thing which I have missed here in the previous lecture that here what happens if lambda 1 and lambda 2, both are real, unequal but positive. So in that case, the only difference is your sign will be reversed.

Because in that case, 0, 0 is the repulsive point. It means that all the solution will tend to infinity, say going away from origin 0, 0. So in that case, your 0, 0 will be node but in this case, it is unstable node. So this is the case corresponding to lambda 1<lambda 2 and it is greater than 0. So in this case, your node, origin will remain the node but in this case, it is unstable and the phase portrait will look like exactly the same.

The only thing is that now solution will move towards the infinity along the, say other sign. So it means that it is moving away along the line, say having the slope u2/u1 and it will go towards infinity along the other line corresponding to lambda 2 equal to the bigger one. So corresponding to, say lambda 2 here. So it means that it will tend to infinity along this line. Is that okay? So let me write it here.

So corresponding to this, we have this line. So this is corresponding to lambda 1 and we have your u1, u2 on this line and slope of this line is u2/u1 and this is corresponding to lambda 2 and slope is v2/v1 and v1 and v2 are 2 points running on this. And now all solutions are moving away from origin and now it will, this lambda 2 is bigger than lambda 1. So as t tending to infinity, the component corresponding to lambda 2 is bigger.

So it means that all the solution will go to infinity along this line. Is that okay? So here your origin 0, 0 is node but unstable node, right. Now move on to theorem 2. So the roots lambda 1 and lambda 2 of the characteristic equation 3 are real, unequal and of the opposite sign. So in the previous case, it was of the same sign. And then this case, critical point 0, 0 of the linear system is a saddle point.

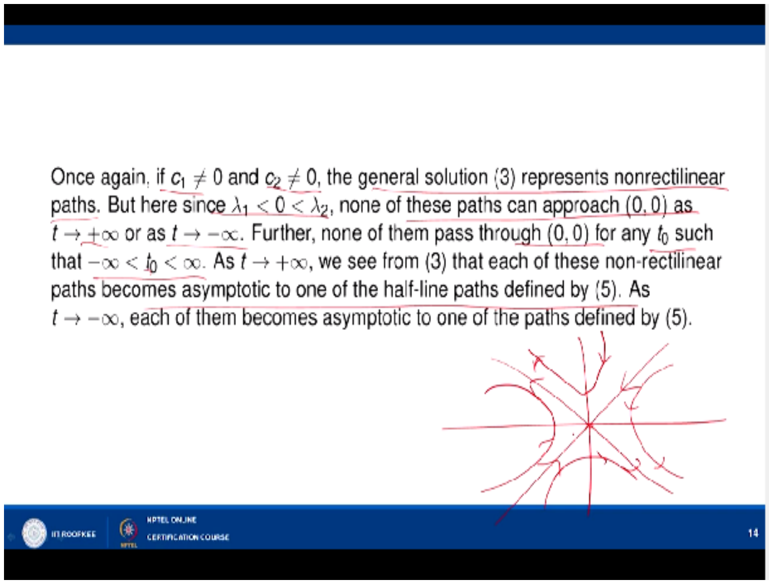
So it means that if eigenvalues are opposite in sign, then we call origin a critical point as a saddle point. So in the same way, we recall the general solution as x=c1u1e to the power lambda 1t+c2v1e to the power lambda 2t; y as c1u2e to the power lambda 1t+c2v2e to the power lambda 2t. And the particular solution in the case when c2=0, we call x=c1u1e to the power lambda 1t, y=c1u2e to the power lambda 1t and similarly, in the case corresponding to c1=0.

So in this case, we can simply write it that you have lambda 1 and lambda 2. Since these are unequal, so it means that we have say u eigenvector and v eigenvector. So u and v are linearly independent eigenvectors, right. And let us say this u is given by u1 u2 and v is given by v1 and v2. And the line, this is the line corresponding to u1, u2 which contain this vector u1, u2 and slope is u2,u1 and this line is passing through the point v1,v2 with the slope v2,v1 here.

And depending on the c1 and c2, it will be this thing. Now here, we are assuming that lambda 1<0 and lambda 2>0. So it means that if you look at the component x here, which is corresponding to c2=0, so since lambda 1<0, your x will tend to 0, right. So it means that x is tending to 0 but y is tending to say infinity, I am sorry. x and y tending to 0 along the line c2=0. So it means that here solution will tend to origin as t tending to infinity.

Now as lambda 2>0, so in this case when c1=0, your x and y, both are tending to infinity. So it means that it will move away along the line passing through v1, v2 here. Is that okay? So it means that for any c1 positive, the solution 5 again represent a path consisting of half of the line u2x=u1y while for any c1<0, they again represent the path consisting of the other half of this line.

(Refer Slide Time: 31:17)



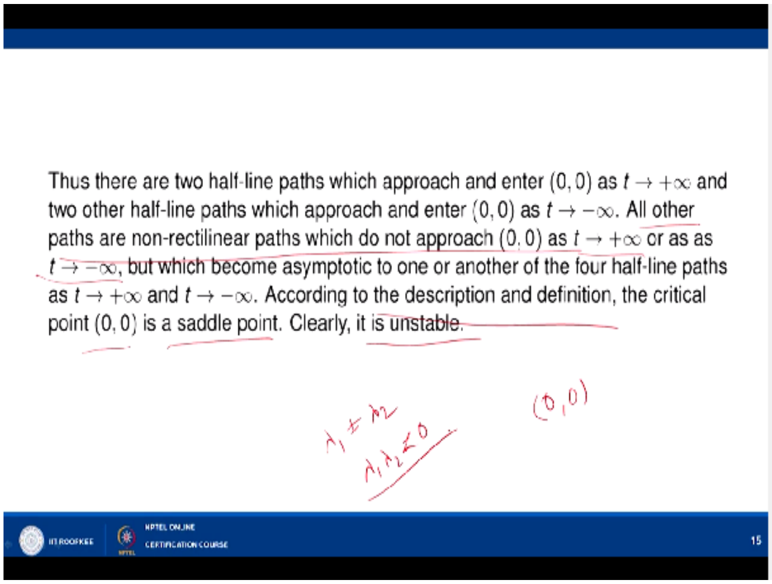
So since lambda 1<0, both of these half line path approach and enter 0, 0 as t tending to infinity. So as we have pointed out here, the solution will enter to 0, 0 as t tending to infinity depending on the behaviour of c1 whether the c1 is positive or c1 is negative here. Now for any c2>0, the solution 6 represents a path consisting of half of the line v2x-v1y, that is this line here. This line is v2x=v1y.

And for any c2<0, the path which they represent consist of the other half of this line. But in this case, since lambda 2>0, both of these half lines now approach and enter 0, 0 as t tending to -infinity, right. So in this case, it will enter 0, 0 as t tending to -infinity. So it means that as t tending to infinity, they are moving away from origin. So we are putting like this. So once again if c1 and c2, both are non-negative, then general solution 3 represents non-rectilinear paths.

But since lambda 1<0<lambda 2, none of these path can approach 0, 0 as t tending to +infinity or t tending to -infinity. And none of them pass through 0, 0 for any t0 such that t0 is lying between -infinity to infinity, right. As t tending to infinity, we see that each of these non-rectilinear path

become asymptotically to one of the line paths defined by 5. As t tending to -infinity, each of them becomes asymptotic to one of the paths defined by 5 here.

(Refer Slide Time: 33:02)



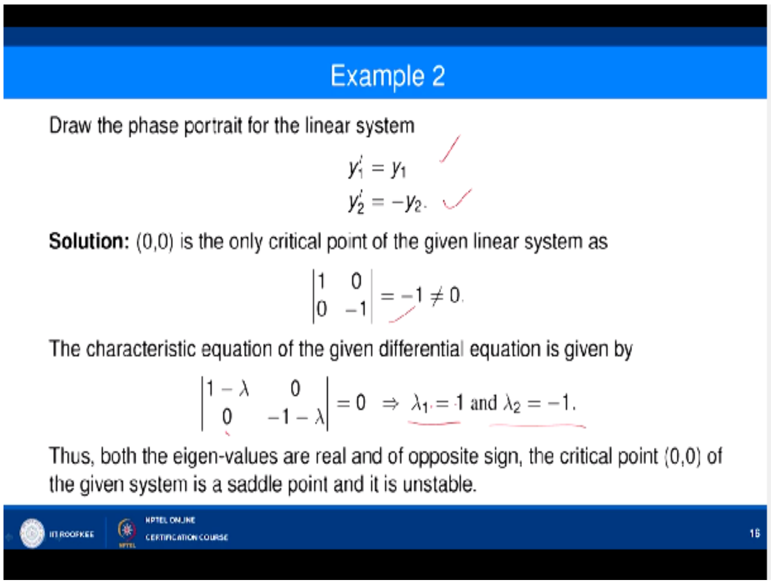
So it means that if you look at here, we have a path line like this. So it is solution are tending to origin along this line and moving away from this. So if you take any point here, solution starting from this, then it will try to come to origin along this line but as it is near to origin, then it will tend away from 0, 0 along this line, right. So it is coming along the line this which is tending towards origin 0, 0 and as it is near little bit origin, near to 0, 0, it will move away from 0, 0 along the other line, right.

So in a similar way, if we have this thing, then it will behave like this and here it is like this, here it is like this. So none of the path, none of the solution is tending towards 0, 0 here. So thus there are 2 half line paths which approach and enter 0,0 as t tending to infinity. These are half line paths. So one half line path is tending to origin and other half line path are moving away from origin, right.

All other paths are non-rectilinear path which do not approach 0, 0 as t tending to +infinity or as t tending to -infinity. But which become asymptotic to one or another of the 4 half line paths as t tending to +infinity and t tending to -infinity. So according to the definition, the critical point 0, 0 is a saddle point. And saddle point is always a unstable point. So in this case, 0, 0, the case is what? When lambda 1 and lambda 2 are not equal and having opposite signs, that is lambda 1 lambda 2 is less than 0.

So when product is less than 0 and lambda 1 and lambda 2 are real and unequal, then in this case 0, 0 is a saddle point and unstable point.

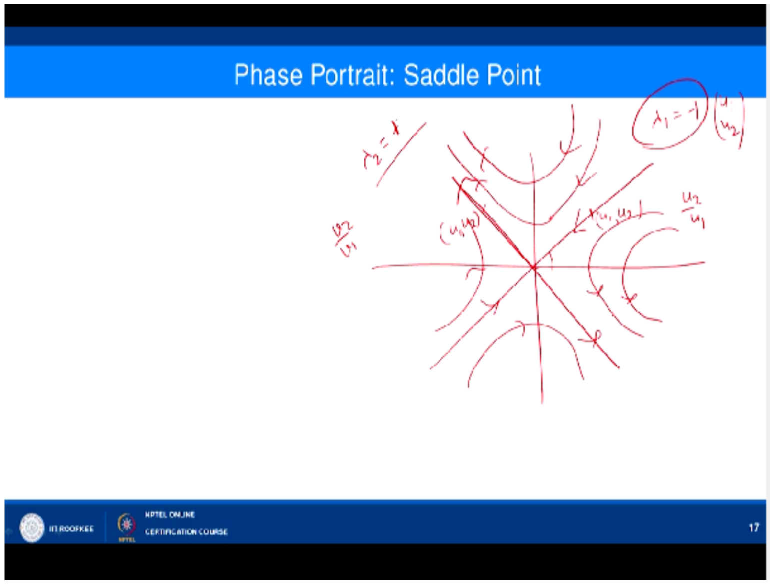
(Refer Slide Time: 35:19)



So now let us consider one example based on this. Draw the phase portrait for the linear system y1 dash=y1 and y2 dash=-y2. So we can see that 0, 0 is the only critical point and we can check that the determinant is -1, so we can simply check. Now the characteristic equation of the given differential equation is given by lambda 1, this equation and which gives the roots lambda 1=1 and lambda 2=-1.

So both are real, unequal and having opposite sign. So in this case, the critical point 0, 0 is saddle point and it is unstable here.

(Refer Slide Time: 35:59)



And the graph or the phase portrait, let us say that this is the eigenvector corresponding to, so this is the eigenvector corresponding to say lambda 1=say -1 and this is the lambda 2=+1. So corresponding to this, we have an eigenvector say u1 and u2. So let us say this is the point u1, u2 or we can say the line passing through u1, u2 with the slope u2, u1 here. Similarly, this is the line passing through origin 0, 0 and passing through v1, v2 and the slope is v2,v1 here, right.

So here since lambda 1 is negative, so solution will tend to origin as t tending to infinity. And since this is corresponding to positive eigenvalues, so it means the solution will move away from origin as t tending to +infinity or we can say that solution will tend to 0 as t tending to -infinity, okay. So now these are 4 rectilinear paths and for non-rectilinear paths, you should take any solution, then it will tend towards origin along this line and as it is near to origin, then it will move away from origin along the other line here.

So your phase portrait will be like this, something like this. So this is the phase portrait of saddle point and so here we stop. So in this we have considered case corresponding to that eigenvalues are real, unequal. One case where it is of equal sign and in other case, it is of opposite sign. And here we have seen the case in which they are of the same sign, then origin is node and stable, unstable depending on the sign of the eigenvalue.

But in the other case, when lambda 1 and lambda 2 are unequal, real and are opposite sign, 0, 0 is a saddle point. So here we will stop and we will continue in the next lecture. Thank you very much for listening. Thank you.