Dynamical Systems and Control
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Lecture – 23
Phase Portrait: Types of Critical Points

Hello friends. Welcome to this lecture. And in this lecture, we will discuss the different type of equilibrium points and how we define them and how these are useful in say geometrical study of solutions of differential system. So first let us consider critical point. We already know what is a critical point, stationary point, this thing.

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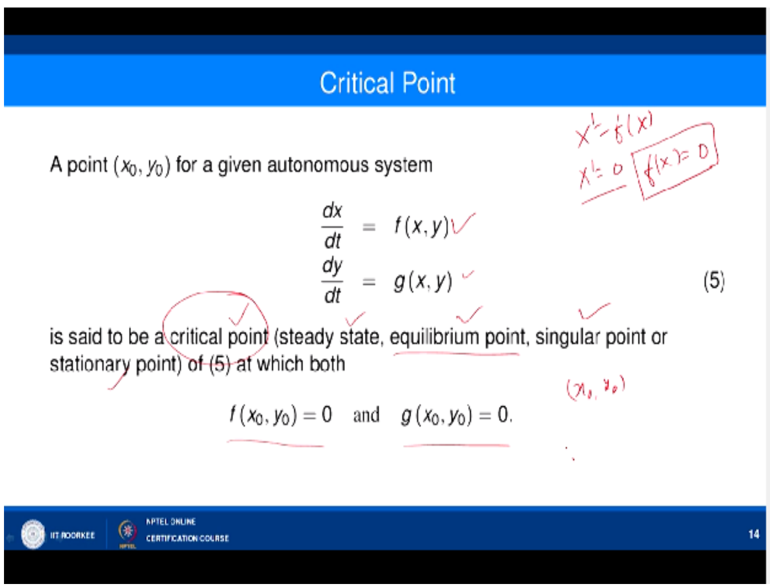

The only thing is that if we have a system x dash=f of x, your stationary points, equilibrium points, critical points, or singular points are known as those points for which x dash=0 here. And that we can obtain by solving f of x=0 and when we solve this, we simply say that the solution will be a stationary point or equilibrium points or singular point or stationary point. So there are several names for this, critical point, steady state, equilibrium points, singular point, or stationary point.

In fact, this critical point is something more general. In fact, we also call those points as a critical point for which even the derivative is not defined here. So we call these as critical points. So equilibrium points, let me give this definition that equilibrium points are those points for which dx/dt=0, dy/dt=0.

Or we can say that this point x0, y0 is called as critical point or steady state or equilibrium point provided that f of x0, y0=0 and g of x0, y0=0. So it means at these points, your right hand side is 0 and here your, the movement is stopped. AND you will simply say that the orbit containing the critical point is the point itself. So the orbit is the point itself, the critical point itself.

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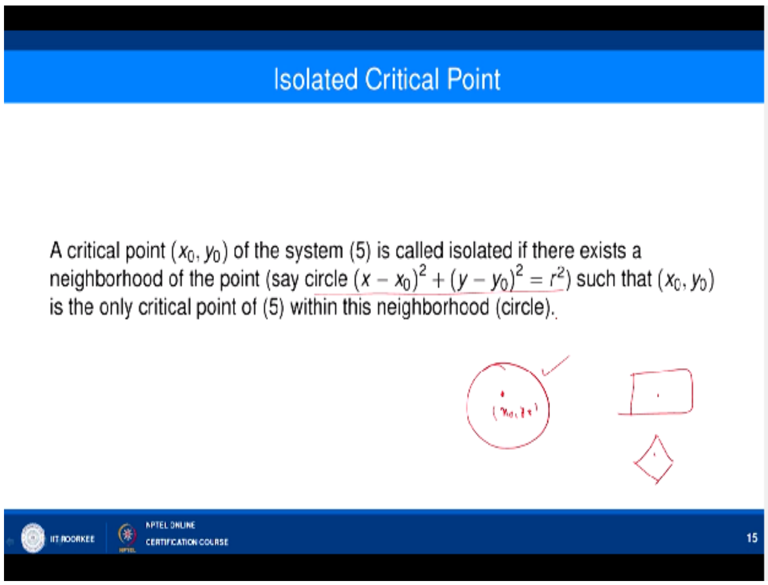

In next few slides, we will try to define various type of critical points and the type of critical points. So first a very important definition is the isolated critical point. Isolated means that critical point is say in the neighbourhood of critical point we do not have any other point of the system dx/dt=fxy and dy/dt=gxy. So the precise definition of isolated critical point is as follows.

A critical point x0, y0 of the system 5, 5 is the system, this is dx/dt=fxy and dy/dt=g of xy is called isolated if there exists a neighbourhood of the point, neighbourhood we can consider as a say circle around this point x0, y0 of the radius r, so we have a point x0, y0 here and we can define neighbourhood, let us say that this is a circle. You can define neighbourhood in the various possible way.

Some common neighbourhood is like this. So we will use, let us say this circle also. So we say that in a neighbourhood of x0, y0, we do not have any other critical point within this neighbourhood. So it means that in that case when we have a critical point and we have a neighbourhood in which we do not have any other critical point. Then we say that this critical point is an isolated critical point.

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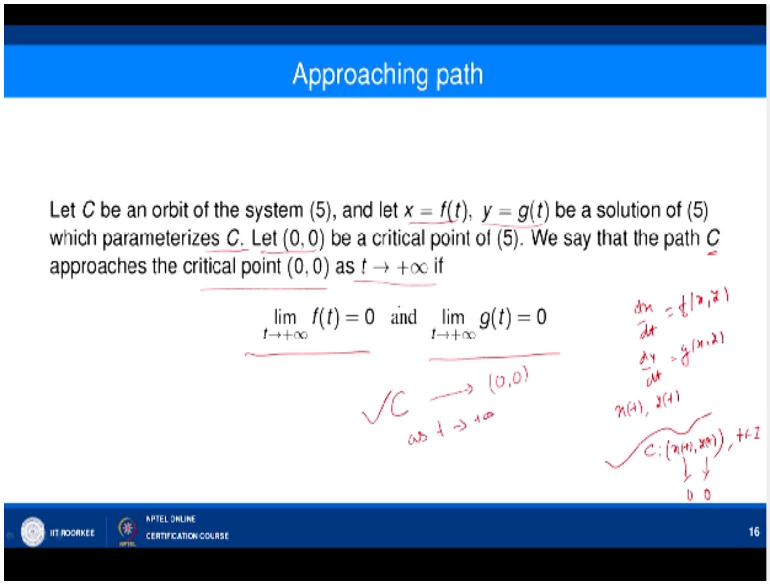

Next thing is the meaning of approaching path. When we say that a given solution is approaching towards some critical point. So that we are saying that let C be an orbit of the system 5 and let x=ft and y=gt be a solution of 5 which parameterizes C. So it means that here we have a system dx/dt=f of x, y; dy/dt=your gx, y here. Then we have a solution say xt and yt which trace a path, let us say this.

We call this as C. Then we can parameterize this curve C by the parameter xt, yt and we simply write xt, yt where t belongs to some interval, right. Then we simply say that let 0, 0 be a critical point of 5. We say that this path C approaches the critical point 0, 0 as t tending to infinity if the limit of xt at t tending to infinity is 0 and limit of yt as t tending to infinity is 0.

So it means that if limit t tending to +infinity ft=0 and limit t tending to infinity gt=0, then we say that this path C is approaching to the critical point 0, 0 as t tending to +infinity. Is that okay? So here we call this C as the approaching path towards 0, 0 here. And we simply say that the solution is approaching to 0, 0 along with this path C. So that we mean by this definition that this is the approaching path.

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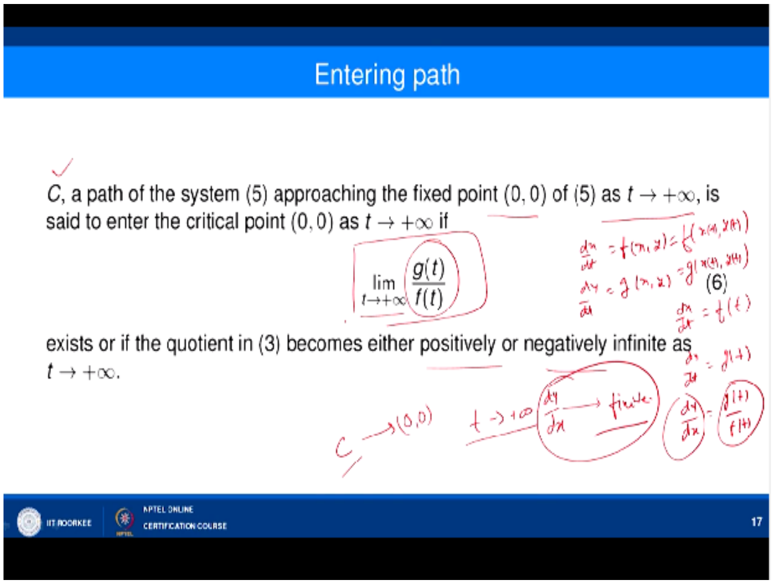

Next thing is the entering path. So given a critical point, say in this case, let us define everything in terms of the critical point 0, 0. If we have a non-0 critical point, then we can shift the origin and we can consider this case only. So let us say that C is a path of the system 5 approaching the fixed point 0, 0 as tending to +infinity. So it means that C is already an approaching path.

Now we call this approaching path C an entering path provided that limit t tending to +infinity, gt/ft exist or if the quotient in 3 becomes either positively or negatively infinite as t tending to infinity. So it means that it is not only the approaching path, but as t tending to infinity, +infinity, this gt/ft will take a, say constant value. What is gt and ft represent? If you look at dx/dt=f and dy/dt=f of x,y and g of x, y.

Now since x and y is a function of t, we can call this as f of xt, y of t and here it is g of x of t, y of t here, right. Or in short, I can simply say that I can consider this as dx/dt=, as a function of t and dy/dt as a function of, I will, gt. Then dy/dx is basically gt/ft. So basically this gt/ft represent the slope of the orbit here. So it means that as t tending to +infinity, if the slope dy/dx is say approaching towards a finite value, then we say that your path C is a entering path towards 0, 0 here.

If it is not happening, then we say that C is an approaching path only. So here we have to look at the slope. If slope is approaching a finite value, we say that the path is an entering path and if it is not happening, we simply, and it is, if the limit t tending to +infinity ft and gt, both are equal to 0, we simply say that C is the approaching path towards the equilibrium point 0, 0 here. So here what we have defined so far?

We have defined critical point. What do you mean by a critical point? And then we have defined isolated critical point, means that in the neighbourhood of critical point, we do not have any other critical point. So now we are talking about only the isolated critical point, okay. Then we have defined that how we reached to critical point in the neighbourhood of critical point. So there are 2 kind of path available.

One is approaching path that limit tending to infinity and the component will reach to the component of the equilibrium point. If it is true, then we say that it is approaching path. And if the slope of the path is tending to a finite value, then we say that our path is entering in the equilibrium point here. Then now let us define the types of critical point. So there are different types of critical points. Let us discuss one by one. So first critical point is the center.

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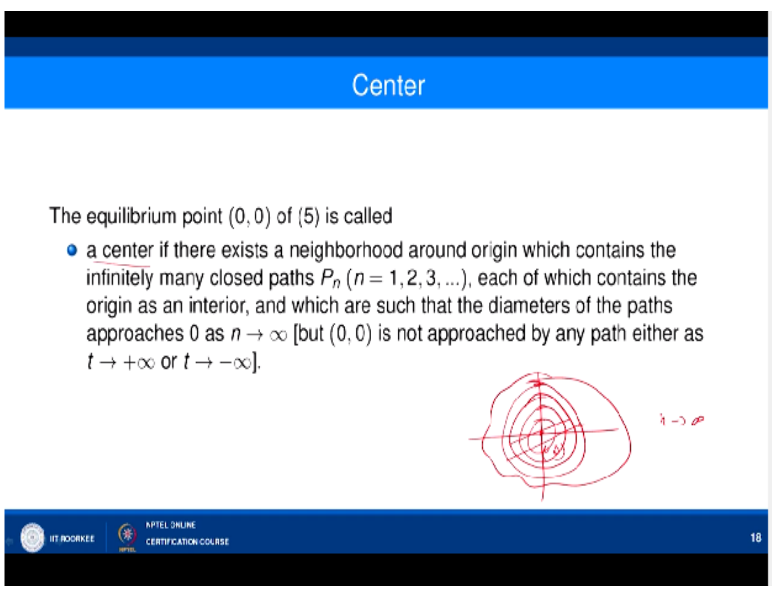

So the equilibrium point 0, 0 of 5 is called a center provided that, let us say this is your 0, 0. That in any neighbourhood of this, let us say you have a neighbourhood like this. Then in any neighbourhood of origin, we have infinitely many closed paths Pn, each of which contain the origin as an interior. So it means that here we have several path available for every neighbourhood of 0, 0 and none of the path, none of this path is say approaching towards origin 0, 0 but the path has a diameter.

Diameter means the maximum distance between any 2 points of this path, is tending to 0 as say t is tending to infinity. So it means that here, this path has diameter, let us say this and this path has a diameter this. So it means diameter is tending to 0 as n tending to infinity. And here we have infinitely many path contained in any of the neighbourhood of 0, 0. And none of the path is approaching towards 0, 0. In that case, we call this equilibrium point 0, 0 is a center, center equilibrium point.

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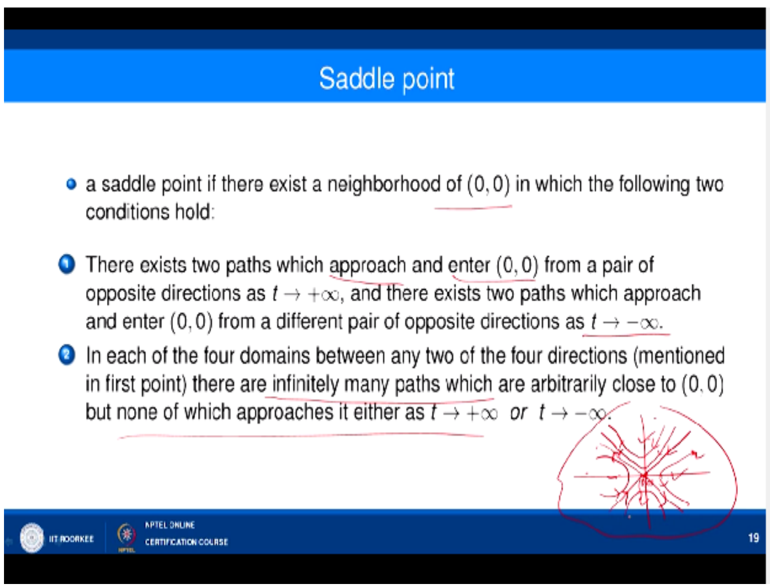

Next is saddle point here. So a saddle point, a critical point is said to be a saddle point if there exist a neighbourhood of 0, 0. By the way, we are talking about equilibrium point 0, 0 and in each definition, we are defining the type of the critical point 0, 0 only. So here we call this 0, 0 a saddle point if there exist a neighbourhood of say 0, 0. We have a small neighbourhood like this in which the following 2 condition holds.

First that there exist 2 paths which approaches and enters 0, 0 from a pair of opposite direction as t tending to +-infinity. So it means that there are, say this is 0, 0. So here, one path is entering to 0, 0 and other path is leaving this 0, 0 or we say that this path is entering to 0, 0 as t tending to

-infinity. So we say that it is leaving as t tending to infinity or we simply say that it is entering as t tending to -infinity, both have the same meaning.

So if we say entering as t tending to infinity means it is like this. And if we say entering towards 0, 0 as t tending to -infinity means it is leaving in the positive time, right. So it means that it will represent the saddle point if there exist 2 paths which is entering your 0, 0 as t tending to infinity and leaving to opposite path as t tending to -infinity. Now in each of the 4 domains, now it will create 4 domains, 1, 2, 3, 4. In each of the 4 domains between any 2 of the 4 directions, this direction is coming here going out, coming here going out.

In each of the 4 domains, they infinitely many paths which are arbitrary close to 0, 0 but none of which approaches it either as t tending to infinity or -t tending to -infinity. So it means that if you look at this domain for example, your orbits are reaching towards 0, 0 but leaving along this direction. So it means that here we have this kind of behaviour. Here we have this, yes, this thing.

So here it is like this. So it means that your path, if you look at any orbit, whether it will start from this domain or this domain or any of the domain, so if it is starting at this domain, it will try to reach to origin along this direction but after some point, it will be leaving or moving away from origin along the direction of the, along the other direction available. So it means that here none of the path area actually, they are approaching for some time, but after some time, it will move away as t tending to infinity. So in this kind of case, you call 0, 0 as a saddle point.

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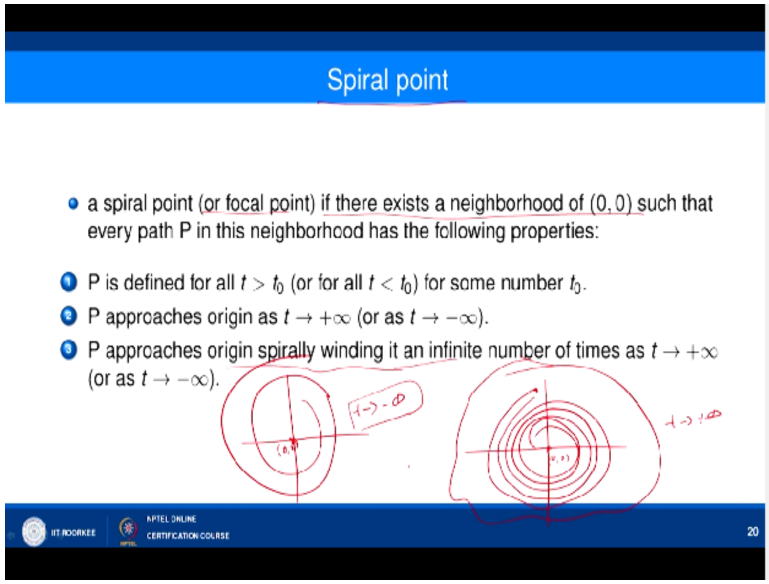

Now the spiral point. Now spiral point is a critical point, we have another name for the spiral point is that is focal point. So in this case, if there exist a neighbourhood of 0, 0, let us say that we have a 0, 0 here, such that every path P in this neighbourhood has the following properties. So you take any neighbourhood. Let us, not drawing any neighbourhood. Neighbourhood is something like the better neighbourhood.

Now in this neighbourhood, you take any point here and now we say that we have a path which is defined for all time t>t0 and P approaches this point 0, 0 as t tending to infinity or -infinity. So P approaches origin spirally winding it an infinite number of times as t tending to +infinity or as t tending to infinity. So it means that if it starts from this, then as spirally it is moving around origin here.

It is something like this and some more path available. So if you take any path, it is reaching to origin but it is never entering the critical point 0, 0. So every path here will approach 0, 0 and approaching path but never entering the point 0, 0 here. And in any of the neighbourhood, you can find several infinitely many path of this kind. Then in this case, there are another possible cases like this.

Here we have, sorry, if we have this possibility, then we say that here, this path is entering your 0, 0 as t tending to -infinity. So if it is moving closure to 0, 0, we say that your path is approaching 0, 0 as t tending to +infinity. And if it is, this path is moving away from 0, 0, then we say that this path approaching 0, 0 as t tending to -infinity. So if it is happening, we call this critical point 0, 0 as a spiral point or focal point.

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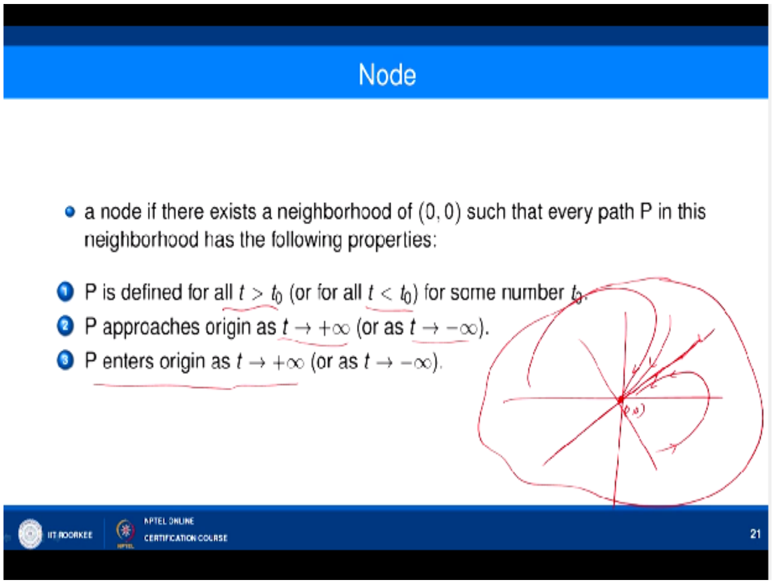

Next is node. So a critical point is said to be node if there exist a neighbourhood, let us call this as a neighbourhood 0, 0. So neighbourhood is something here, something like this, bigger one, such that every path P in this neighbourhood has the following properties. First, it is defined for all t>t0 or all t<t0, this is also true here also in the spiral case. And P approaches origin as t tending to +infinity.

So it means that you take any path, it will be an approaching path towards 0, 0 as t tending to +infinity or t tending to -infinity. And not only this, it is entering to origin as t tending to +infinity. So what is the difference between the spiral point and the node point here. In a spiral point, every path is an approaching path towards the critical point 0, 0 but none of the path is entering the critical point 0, 0 here.

But in node, every path is not only the approaching path but also an entering path. So it means that we have a say something, direction here and along some direction, it is trying to reach to the origin 0, 0. So here we have something like this. We will discuss more about it when we consider certain example here. So it means that you start from anywhere and it is coming back to this and it will take the direction of some finite line.

So here suppose this is a line say passing through 0, 0. And this orbit is entering 0, 0 along the direction of this line which we have denoted here, right. So in this case, when every path is not only approaching path but it is also an entering path, we call this as a, this critical point as the node, right. So right now it is coming towards this. So we call that it is a node here.

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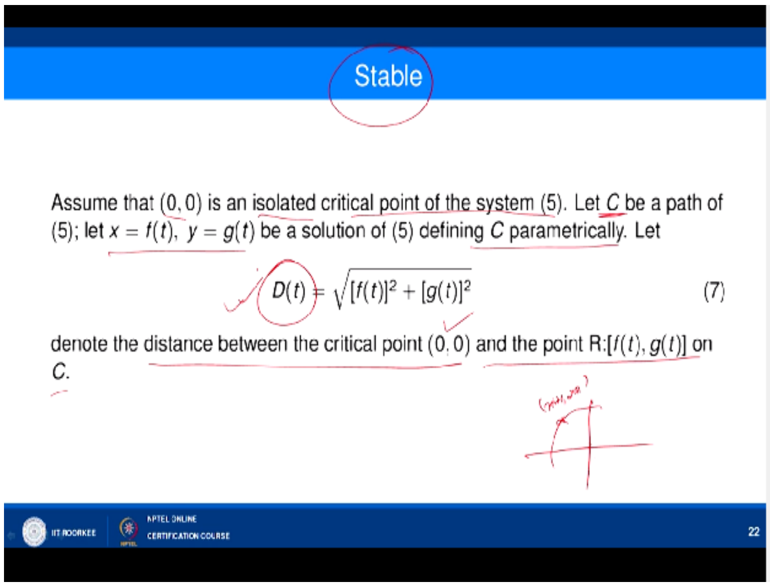

And we, in the sense of stability, we say that we have a critical point and we want to know whether this critical point is a stable critical point or unstable critical point. So how we can define stability of a critical point. So look at here. So assume that 0, 0 is an isolated critical point of the system 5. And let C be a path of 5 and let x=ft and y=gt be a solution of 5 which define your C in a parametric manner.

So it means that you have a 0, 0 here and there is a path say somewhere here. So path has a parameterization, let us call this xt and yt where these parameter say define or solve the solution of the system 5 here. Then we can define the quantity Dt which is under root ft square+gt square. Basically this represents the distance from origin to the point, say some point on the curve C here. So Dt denote the distance between the critical point 0, 0 and the point R which is denoted by ft,gt on this curve C.

So now we say that distance is given by Dt here and now we can define the stability of 0, 0 in the sense of distance function.

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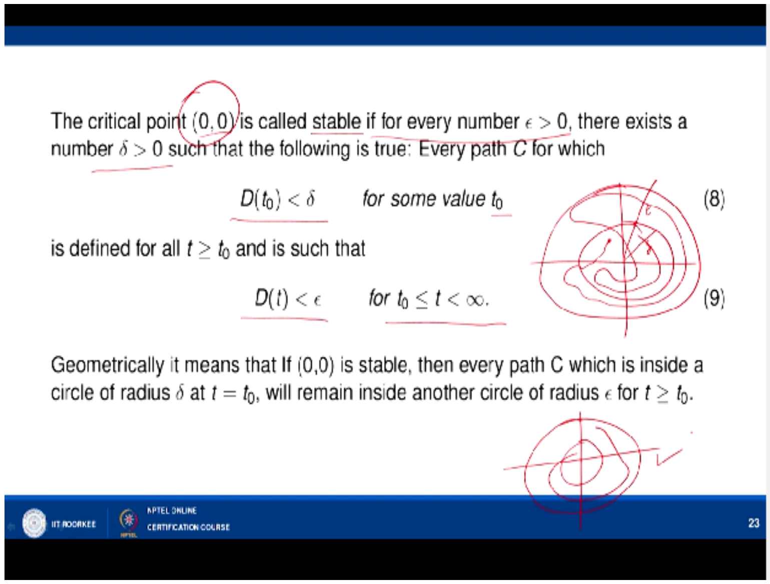

And we say that the critical point 0, 0 is called stable if for every epsilon>0, there exist a number delta>0 such that Dt0<delta for some values of t0, implies that Dt<epsilon for all t greater than or equal to t0 here. It means that you have a Dt, say, 0<delta means suppose you have say 2 circle like this. So we simply say that this is the circle of the distance say delta and this is circle of distance epsilon. So it means that if initial distance is somewhere here, it is inside this. So it means that your curve is starting from this point.

Now if it is starting at this point, then it will go only in, say, within this circle only. So it means that it is started inside this and it can go outside your circle but it will always remain in a bigger circle of the radius epsilon. So it means that the distance between origin and any point on the curve C, path C is always less than epsilon. Then we say that our solution is basically remains there in a bigger circle. Or we say that our critical point 0, 0 is a stable critical point, right.

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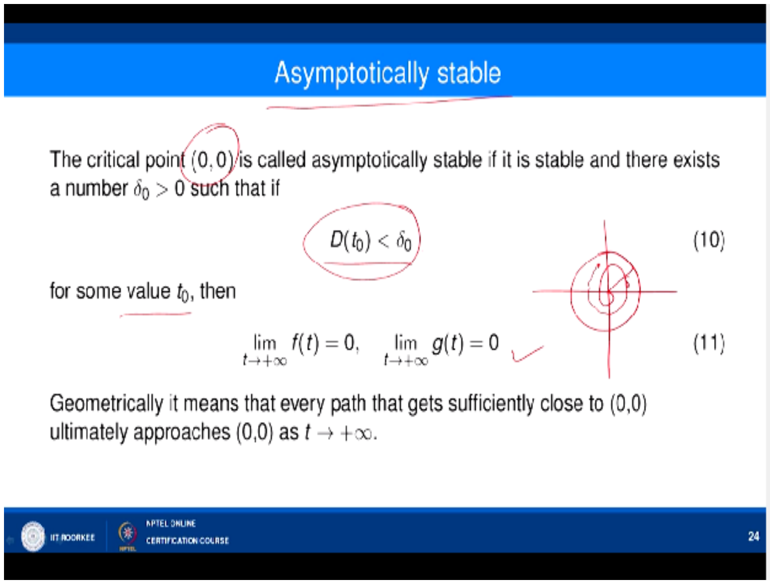

Now we have a asymptotically stable critical point. So here along with this, we have some more thing that if Dt0<delta 0 for some values t0, then limit t tending to infinity ft=0 and limit t tending to infinity gt=0. So it means that when it starts inside your inner circle of delta, then as t tending to infinity, your distance keeps on decreasing and ultimately it is tending towards origin here, right.

In this case, we simply say that your critical point 0, 0 is asymptotically stable here. So look at the difference between asymptotically stable and stable here. Here your solution which started inside your circle of delta 0, it keeps on decreasing and ultimately it is tending to 0. And here, your solution may not go to 0, 0 but will remain inside your circle of radius epsilon. It may go to 0, 0 and may not go to 0, 0.

So in this case we call it stable solution. And in this case when it is entering, when it is approaching 0, 0, we call this as a say asymptotically stable critical point. So here this path is approaching towards 0, 0. So we call, 0, 0 is a critical point which is asymptotically stable critical point.

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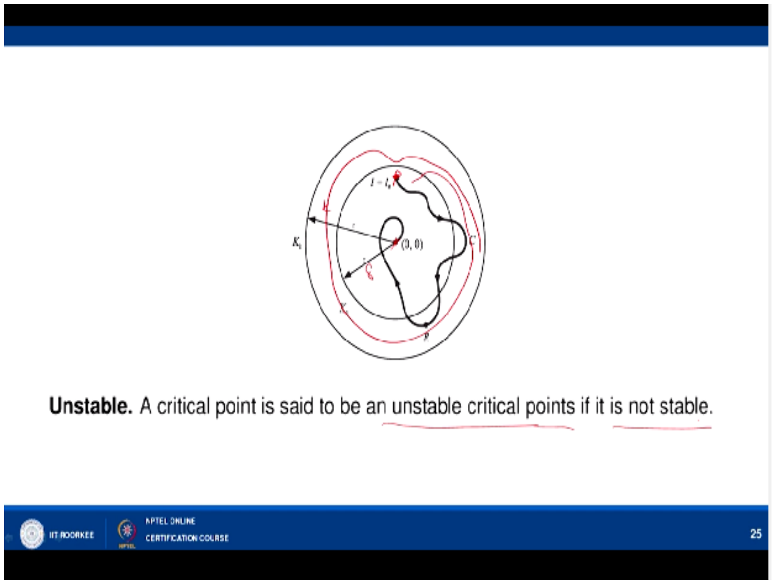

So this is the picture here. So here we simply say that your point starting at this point, it is starting here and then it will again come into 0, 0 here where the inner circle is of radius delta and the outer circle is of radius epsilon. So if it is, it will keep inside your bigger circle, then we call it stable. It means that it is starting from this and will move somewhere and it will remain there only, right.

Then we say it is stable. If we simply say it is starting from this point and ultimately t tending to infinity, it is tending towards 0, 0, then we call this as asymptotically stable critical point. And a critical point is called an unstable critical point if it is not a stable critical point. So we know what is stable critical point. If it is not a stable critical point, we call this as unstable critical point.

So we have discussed in this lecture the, what is critical point, what are isolated critical point and there is one more, if critical point is not isolate, we call this as non-isolated critical point. So we have defined critical point, isolated critical point and the path which is approaching towards the critical point and entering the critical point. And then we have discussed several types of critical point, that is center, spiral, node, saddle.

This we have discussed. And in next lecture, we will discuss that under what condition, a given critical point is one of these critical points. So that we will continue in next lecture. So today, we will say stop here only. And we will continue in next lecture. Thank you very much for listening. Thank you.