Dynamical Systems and Control
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Lecture - 21
Properties of Phase Portrait

Hello friends, welcome to this lecture. In this lecture, we will focus our study towards the geometrical say behaviour of the system of linear equations, geometrical behaviour of the solution of dynamical systems. Now for simplicity, we restrict our discussion for n=2 means we are considering the system x dash = f(x), where x is 2 cross 1 vector, and we try to; because it is we need to represent on your xy plane, so that is why we restrict ourselves to n=2 itself. So, it means that we are considering the following system of differential equation.

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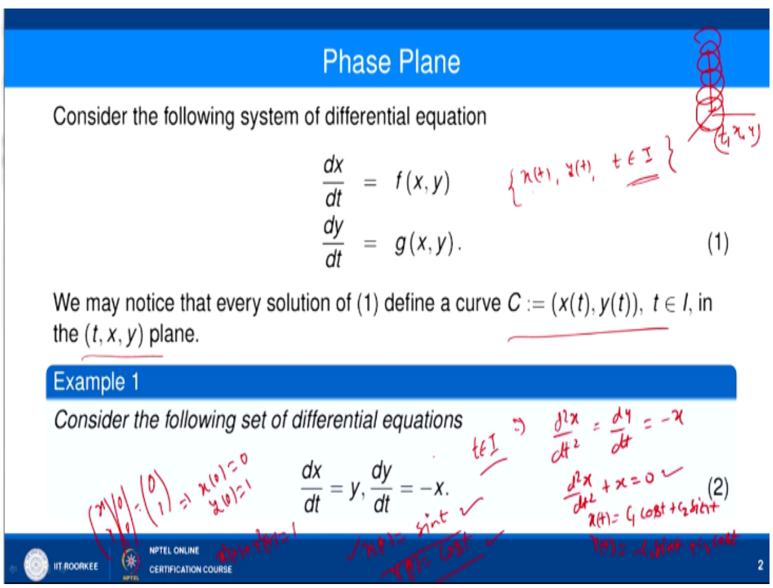

dx/dt = f(x, y) and dy/dt = g(x, y) here. And if we consider that f and g are nice enough in the sense that f and g have say partial derivatives with respect to say x and y and if they are continuous then we can talk about the solution of system 1. And we may notice that every solution of 1, solutions are given as x(t), y(t) where t is belonging to some interval. Then this x(t) and y(t) trace a curve in two dimension as well as in three dimension.

So if you look at, as t is running from; over I then x(t) and y(t) will trace a curve in three dimension plane that is (t, x, y). So let us take an example and then try to understand this. So consider the following set of differential equation dx/dt = y and dy/dt-x. And this we can solve here, and we can say that it is nothing but d2x/dt square = dy/dt and this is nothing but equal to –(x). So d2x/dt square + x=0. And we know the solution of this x(t)=c1 cos(t) + c2 sin(t).

And similarly, we can find out y(t), y(t) as –(x), so we can simply say that it is nothing but c1 sin(t) - c2 cos(t). So I can take any of the solution here. So in particular I can take that x(t) = cos(t) here and y(t) = sin(t) is one particular solution of this system of differential equation. So x(t) = cos(t) so is it a solution here. So we can take solution x(t) we can take sin(t) and y(t) = cos(t), if we impose certain condition, initial condition on this system.

So let us take a particular solution of system 2 that is x(t) = sin(t) and y(t) = cos(t). So it means that we are putting certain initial conditions along with this. Now these curves as t belongs to some interval say I, trace a curve in three dimensions.

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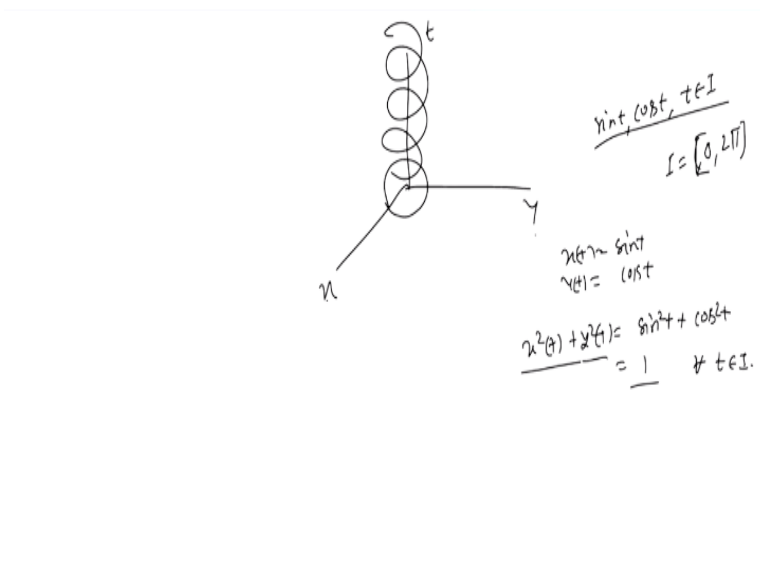

And if you look at here and represent these as x, y and there then as t is running from some A to B, this will trace like this kind of path here, right. This is kind of helix in three dimension plane here. So, it means that for t belonging to this I, this will trace this kind of a path here. Now the interesting path is this, that this sin(t) and cos(t) this solution will also trace a path in two dimension plane.

Like in x, y plane, if you consider here that t is running in some interval say I. For time being let us take this interval as say 0 to 2pi kind of thing, so closed interval 0 to 2pi. Then we can say that as, this is kind of traversing a path in your three dimension plane. But if you consider that if you look at the projection of this three dimensional figure into x, y plane then it is basically say, it is like this here.

So it means that if we can remove this and here since we have x(t) = sin(t) and y(t) = cos (t), then we can simply say that x square t + y square t which is nothing but sin square t + cos square t and this is coming out to be 1 here. So it means that not only this solution curve traces a curve in three dimension but it is also tracing a curve in two dimension plane x, y and the curve is given by x square (t) + y square (t) = 1 here for all t belongs to this interval I here.

So it means that the trace curve C in x, y plane has a spatial name and we call this curve as trajectory or orbit of the solution x and y here.

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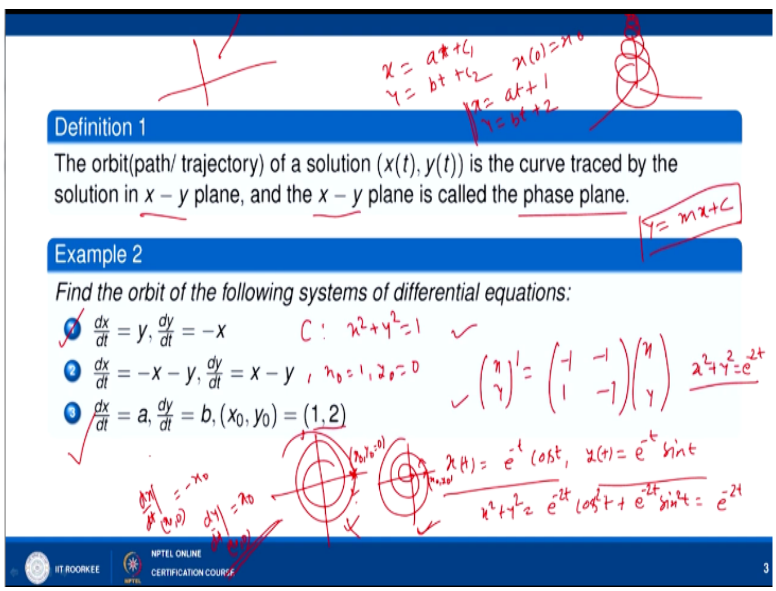

So we define the orbit path or trajectory as follows: The orbit, which is also known as path or trajectory of a solution x(t), y(t) is the curve traced by the solution in x-y plane. And x-y plane is also known as phase plane. And so it means that, here we are not considering the curve in three dimension, we are simply saying that what is the projection of the curve onto x-y plane and we are calling that projection as orbit of the solution x(t), y(t) here.

And now, let us consider few examples by which we can consider the orbit of the few system of differential equation. So first one is dx/dt = y and xy/dt = -x. And if you look at this, this is similar to the previous problem. In fact, it is the same as previous one. And here, we can say that the orbit is given as x square+y square=1 here. So orbit of this system of differential equation is given by dx/dt = y, dy/dt = -x as x square + y square = 1 here. Okay.

Now, look at the second equation dx/dt = -x-y, dy/dt = x-y. And so here to; we can simply say, that how we can solve this. It is nothing but, x y dash = -1 -1 1 -1 x and y here and we want to find out the solution here. So, we can find out the solution of dx/dt=y and dy/dt=-x as follows. We can simply say that d2x/dt = dy/dt square = dy/dt and this is nothing but –(x) from second equation, so we can write as d2x/dt square + x=0. And we can find out the solution of x(t)=c1 cos(t) + c2 sin(t). And y is nothing but dx/dt. So y can be written as –c1 sin(t)+c2 cos(t).

Now, so this represents the general solution of the system of differential equation. Now if you impose certain initial condition for example, let us say that x y at 0 0 it is equal to say 0 1 then we can find out a particular solution here. So we are imposing condition that x(0)=0 and y(0)=1 and we can say that, let us say that initial condition; solution is given as x(t)=sin t and y(t)=cos(t) here. And so here, this will trace a helix in three dimension.

So it means it will trace like this kind of path here. So it is a helix and t, x, y plane and not only it trace a path, a curve in three dimension. If you consider the projection onto x-y plane, then it will be a kind of a circle on x-y plane. So if you, basically you have a helix and then you simply put it down on x-y plane. So, you paste it together. So when you paste it together it will form a circle which is given as x square, y square; x square + y square = 1 and that you can find out as, you can say x(t) = sin(t) and y(t)=cos(t). You can find out, say x square t + y square t = 1 here.

So it means that it will form a circle on x-y plane. And that will represent say curve traced by solution on the x-y plane as t is running over some interval I here. So for time being let us take this interval as some 0 to infinity. When t is starting from 0 to infinity it keep on moving on the circle infinitely many times, okay. So it means that the curve traced by the solution as t is over I is known as the orbit of the solution in the x-y plane and the x-y plane is known as phase plane.

So, here, considering the second problem dx/dt = -x-y dy/dt = x-y, and some initial condition. For example, let us take initial condition x(0)=1 and y(0)=1, then this second system can be written as this system x y dash = -1 -1 1 -1 x y. And this we have already discussed, how we have to find out the solution and solutions are given as x(t)= e to the power -t cos(t), y(t)=e to the power of -t sin(t) here. Now, it will say, again give you helix kind of a figure in three dimension picture. The only thing is that as t tending to infinity your solution is merging towards 0.

Or you can see it is something like this figure, so here we have this then here it started from 1 and then this radius is keep on decreasing. So this is kind of a helix kind of a figure the only thing is that the radius is you can get it here. The radius is basically tending to 0 as t tending to infinity. So it means that, here this trace a helix kind of a figure. And t, x, y plane and if you project everything on x-y plane it will give you spiral kind of a figure, it is something like this either like this. Sorry, it is the other way round so it is like this or it is something like this, right.

So, now the direction will depend on the equation, that is, whether it will be this or this. So whether it is coming like this or it is coming like this. It is clockwise or anti-clockwise. But ultimately it is coming out to, tending to 0 as t tending to infinity. In fact, that we can look at here it is x square + y square = e to the power -2t, cos square t + e to the power -2t, sin square t here or this we can write e to the power -2t. So your x square + y square = e to the power -2t. So it means that as t tending to infinity it will trace one of the figure like this.

Now let us check, it will be exactly which one of the figure, so that we can verify from the equation. So first thing we get from the solution curve is that x square + y square = e to the power -2t means as t tending to infinity your solution will tend to 0 here. Now, let us look at the direction here. So direction will be what, just take one point here, so it is one point. Here it is your x0 is something, y0 is simply 0 here. So it is something x0, y0 here.

So if you look at dx/dt at this particular point x_0, y_0, so at this particular point $dx/dt = -x_0$. So at this point, your dx/dt at $x_0, 0 = -x_0$. And dy/dt at $x_0, 0$ is nothing but x_0 here. So here we can say that dx/dt is decreasing and dy/dt is increasing. So it means that dx/dt is decreasing means your x is decreasing. So it means x will be lowered down and y will be increasing, okay. So if you look at this will be what, suppose your point is x_0, y_0 here.

So x_0 is decreasing and y_0 is increasing. So here y_0 increasing means if y_0 is 0 so it means; increasing means it will go into a positive value. So as t is starting from this point onward it will go to something positive so it is something like this figure, right. So here your, the direction of the curve will be going anti-clockwise, right. So here, if you look at this, here your y_0 is moving say downwards, so it means that here y_0 is negative.

So it means dy/dt is decreasing. But here we have discussed that dy/dt is increasing. So this will not give you the orbit of the system $dx/dt=-x-y$ and $dy/dt=x-y$, so it will give you a figure like this, right. So, this figure is known as spiral figure, we will consider more about this in coming few lectures.

Now, in the third part $dx/dt= a$, and $dy/dt=b$. So solutions are constant solutions dx/dt=; or we can simply say that solution is say Ax; At + some constant x_0 some constant c_1 and $y=bt$ + some constant c_2. We can fix your this c_1 and c_2 provided that let us say that $x(0)=x_0$, right. So at $t=0$ your c_1 is given as 1 so $x=At+1$, $y=bt+2$, so this will give you; if you remove your t from this; you can removed it and you can get a relation like $y=mx+c$ kind of a thing.

So, in this case, in the third case, your orbits basically follows some kind of a line and it is originated from 1 and 2. So it will form a line starting from 1 to and somewhere going up. So it means that the orbit in third example is a line originated from (1, 2). Orbit in the second is nothing but a spiral converging to 0. And the orbit in the first case is basically circle of radius 1 here.

Now, here everywhere we have imposed initial condition x_0 and y_0 having certain different value. Now, if you remove your initial conditions then it will give you, say all possible path traced by the solution curves here. For example, if you look at the first case, dx/dt=y and dy/dt=-x then here, your initial conditions gives you, that the orbit of this solutions is basically unit circle x square + y square = 1. And if you take any arbitrary initial curve then it will give you the family of circles as a trajectory, as a path traced by the solutions of dx/dt=y and dy/dt=-x.

Now here the, we say that in all these problems we are able to find the orbit by solving the system first and then removing the parameter t. Now, we simply say that one very important, say benefit of considering the orbit is that.

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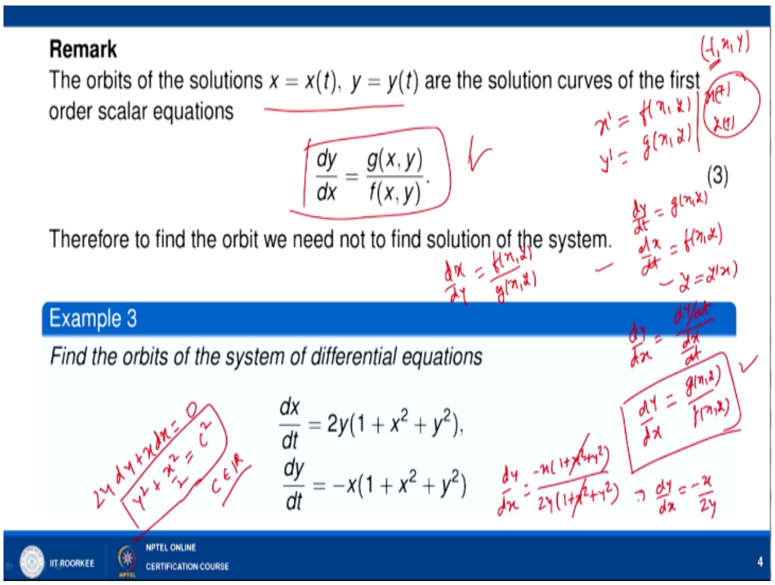

We can find out the orbit of the solution x=x(t) and y=y(t) without solving the system of differential equation. In fact, the orbit of the solutions x=x(t) and y=y(t) are solution curve of the first order scalar equations dy/dx = g(x, y)/f(x, y). Therefore, to find the orbit, we need not find the solution of the system here. Please look at here.

What we are doing here, x dash=f(x, y) and y dash=g(x, y) here. So here, we are getting the solution x(t) and y(t) here and this will trace a curve in (t, x, y) plane. And here, we simply say that orbit means, we are finding a relation between the solution component that is x and y without say considering the parameter t here. It means that we are finding a relation between the solution component x(t) and y(t).

Now here we simply say that, this is what, this is dy/dt=g(x, y) and dx/dt=g(x, y) and dx/dt=f(x, y). Now we can consider that y is a function of x when we are considering the orbit here. y=y(x). So orbit is the relation between solution component. Now when we say that we are finding the relation between x(t) and y(t) it means that we are assuming that either y is a function of x or x is a function of x.

So it means that I can consider this as let us say, without loss of generality, let us assume that y is a function of x here. Then we can find out dy/dx as dy/dt/dx/dt here. So it means that I can write this as dy/dt is given as g(x,y) and dx/dt is given as f(x,y) here. So it means that I can find out a relation between y and x, provided that dy/dx = g(x, y)/f(x, y) has a solution.

Now, the solution curve of this scalar differential equation; first order scalar equation is known as your orbit. So here, the one good thing happened here that here we can find out the orbit of the solution without actually finding the solution of the system x dash = f(x, y) and y dash = g(x, y) here. So it means that therefore, to find the orbit we need not to find the solution of the system here. And that is one good thing about the orbit of the solutions here.

So let us take one example and find out the orbit. So find the orbit of the system of differential equation dx/dt= 2y * 1 + x square + y square, dy/dt= -x * -1 + square + y square. So, if you look at in previous few examples we have considered certain systems of differential equation which are quite easy solve. But if you consider this kind of a system of differential equation it is not very easy to find the solution of this system of differential equation. But still, with the help of this we can find out the orbit of the solution of the system of differential equation as follows.

We simply calculate dy/dx=g(x ,y) that is -x 1 + x square + y square/ 2y 1+x square+y square. Now this 1+x square+y square is never 0, so it cancel out and we can have dy/dx=-x/2y. And it is basically separable equation. So we can write 2y dy+xd x=0 here, so we can say it is y square + x square/2 = c, let us say some c square. And we can say that this is nothing but ellipse equation.
So we can say that the orbits of the system of differential equation is given by ellipse where c is some arbitrary constant so c is some orbits constants here.

So orbits here is the; ellipses is for c in some number in r. Now here, one important thing we need to note down that a solution curve of the differential equation represent an orbit provided that both dx/dt and dy/dt are not simultaneously zero along the solution. So, here, if it happened, so here we are assuming that dy/dt or dx/dt because this expression dy/dx=g(x, y)/f(x, y) makes sense provided that this f(x, y) is non-zero.

And similarly, we can consider that in place of this I can consider dx/dy as f(x, y) / g(x, y). So it means that here we assuming that g(x, y) is non-zero. So if we are finding the orbit by this, that dy/dx=g(x, y)/f(x, y), we are assuming that f is non zero. And if we are using this differential equation dx/dy=f(x, y)/g(x, y), we are assuming that g is also non-zero. So it means that we can use this procedure to find out the orbit provided that none of the function f and g are 0 at the same time.

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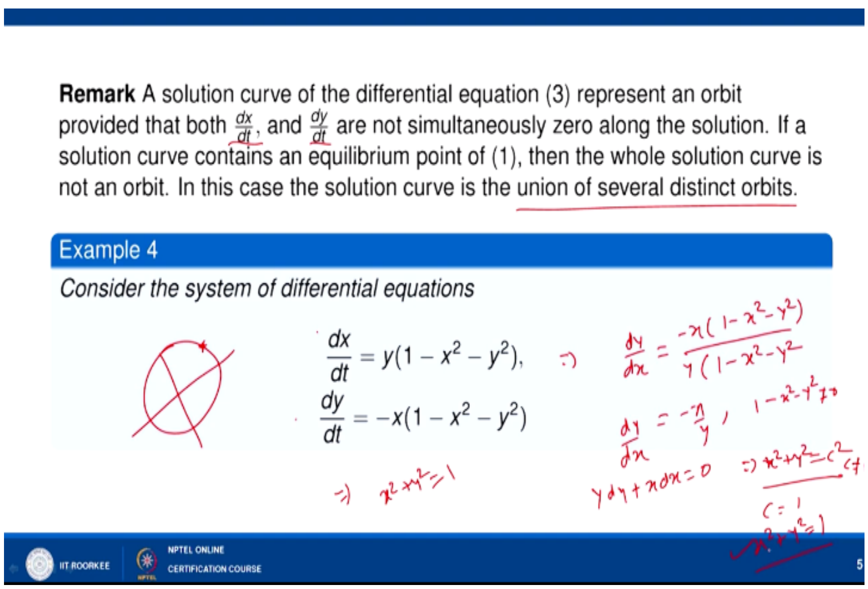

So it means that simultaneously dx/dt and dy/dt are not 0. It means that, if it happened, if a solution curve contains an equilibrium point of (1), equilibrium point means where f(x, y)=0 and g(x, y)=0 then the whole solution curve is not an orbit. Basically, in this case, the orbit, the solution curve is basically the union of several distinct orbits. So let us look at this concept here.

If you look at, if you consider the system of differential equation, dx/dt=y * 1-x square+y square and dy/dt=-x* 1-x square-y square. And if you find out the orbit of this, so we can simply say that the dy/dx = -x upon 1-x square-y square/1-y- x square- y square her. So here we are using this thing that dy/dx=g(x, y)/f(x, y) here. So here when you simply say that; if you assume that 1-x square-y square is non-zero, let us assume this then I can write this as dy/dx = -x/y provided that 1 - x square - y square is non-zero.

If it is non-zero, then we can find out the solution by ydy+xdx=0. So it means that it is x square + y square = c square. So orbits are given as x square + y square = c square and c is some constant here. But if we take that c=1, then it is what, the orbit is nothing but x square + y square = 1. So one of the orbit corresponding to c=1 is coming out to be x square + y square = 1. But this x square, at x square + y square = 1, this dx/dt =0 and dy/dt is also equal to 0. So it means that x square + y square = 1.

If I look at this, that at each point on this unit circle your dx/dt is 0 and dy/dt is also 0. So it means that here we are getting a constant solution. Constant solution will not raise any path other than the point itself. So it means that on the unit circle, every point is an orbit because it will not raise any path other than the point itself. So it means that here, the orbit of the system of the differential equations are x square + y square + c square where c is not equal to 1. And the point on the unit circle x square + y square + y square = 1.

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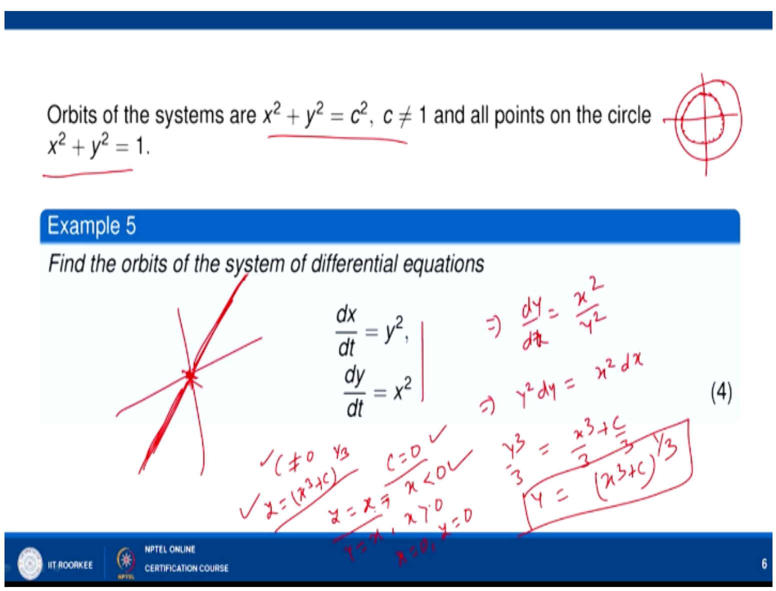

So it means that your system of orbits of the system are given as x square + y square = c square where c is not equal to 1 and all points on the circle x square + y square = 1. So basically you take any circle here then it will be an orbit. But if you consider the unit circle then every point on this basically act as an orbit here. So here we have infinitely many orbits available for the system of differential equation, right.

So here because on this orbit we have an equilibrium solution, so this x square + y square = 1 is not a single orbit but it is a union of infinitely many different orbit, so that will give you the orbit of the system given here. Now look at the orbit of the system of differential equations, dx/dt=y square and dy/dt=x square. So again if you find out, dy/dx=x square/y square and you can say that y square dy= x square dx here. So we can say that y cube/3 = x cube/3.

You can consider this as that, you can simply say that x cube/3+ some constant c. So we can simply say, I can simply write it as y= x cube + c /3 we consider; x cube + c 1/3 here. So orbits are given by this. So, now if you look at here, that if c is non-zero it will give you an orbit y = x cube + c upon 1/3. Now if I take c=0, then what you will get, you will get y=x. So it means that this y=x is also an orbit corresponding to c=0, but this orbit contains a say, so this is this line, but contains a point (0,0). And at (0,0) we have a equilibrium solution of this.

So it means that this y=x cannot be a single orbit rather than it is a union of three different orbits that is y=x when x<0, y=x when x>0 and the point x=0 and y=0 here. So it means all these curve y= x cube + c to the power 1/3 where c is not equal to 0 is your orbit. And corresponding to c=0, y=x is not a single orbit rather than it will be a union of different orbit because it contain an equilibrium solution.

So if it is an equilibrium solution, this will not trace any other point other than 0. So here 0 is contained on these orbits. So this orbit is not a single orbit rather than it is a union of different orbits. So union, basically it is y=x when x<0 means this line and y=x when x is positive means this line and the origin point itself. So the orbit in this case are the curve y = x cube + c, c non-zero and y=x when x<0, y=x when x>0 and origin itself that is x=0 and y=0. So by this we can find out all the orbits of this system of differential equation.

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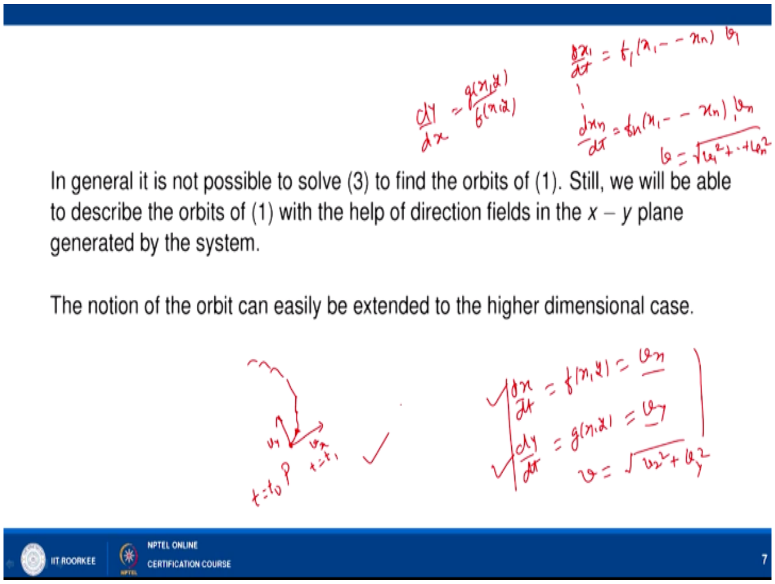

Now, it may happen in general in that, we may not be able to solve the system dy/dx=f(x, y)/g(x, y). In fact, we are able to find out the orbits by solving the equation dy/dx=g(x, y)/f(x, y). And we know that it is quite difficult to find out the solution of this system this scalar differential equation 3. This means that, what happen if this is not solvable, right solvable in the sense that we are not able to find out the solution in an explicit form and; still we will be able to describe the orbits of (1) with the help of direction field. What is this?

If you look at dx/dt is what, dx/dt=f(x, y) and dy/dt=g(x, y). And if you look at this is the velocity component in x direction and this is the velocity component in y direction. I can write this as Vx and this is Vy. So it means that if you start with any point let us say this is your point, and at this point it corresponds to some t= to some t0, so at t=t0 you know what is the velocity along the x component and what is the velocity along the y component.

So Vx, Vy and the total velocity is given as Vx square + Vy, sorry this is Vx square + Vy square; let me erase this, so it is Vy square. So it means that velocity at every point is given to you. So when you have this point you know how to proceed further and when you have this thing at this point you t is corresponding to say t1 and again you can calculate your Vx and Vy at t1 and again you have some point, some direction to go and then in this way we can find out different, different direction at a given point.

And using this, we can trace out your orbit in two dimension without solving your system dy/dx=g(x, y)/f(x, y). So here, when we are not able to solve in an explicit form then we can look at the direction field set up by this dx/dt=f(x, y) and dy/dt=g(x, y) and when we can still have the exact say description of the orbit generated by the solutions of this. And in this way, the notation of the orbit can be easily extended to the higher dimension case also.

It means that, if we have higher dimension case means we have dx1/dt=f1 say (x1 to xn) and dxn/dt = fn (x1 to xn). Then as we have pointed out here then here also we have; say velocity component v1 to vn, and the total velocity is given as under root V1 square to Vn square. And at each point you can find out the velocity and direction using this. So it means that just generalizing this concept we can have the notion of orbit in higher dimension also.

So, in this lecture, we have discussed the concept of orbit, which is the curve traced by the solution component in x-y plane if it is two dimension and in x1 to xn plane if it is n dimension plane. So basically, orbits are the relation between the solution component and we have discussed certain method to find out the orbit of the system of differential equation. And in the next class, we will discuss some more cases to find out this orbit and some properties of orbit here. So with this I end this lecture. We will discuss in next class. Thank you very much.