Dynamical Systems and Control
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Lecture - 20
Stability of Non Linear Systems Using Linearization

Hello friends, welcome to this lecture. In this lecture we will continue our study of stability of an equilibrium solution of non-linear system here. So if you recall let me look at here. In previous lecture, we have discussed this problem.

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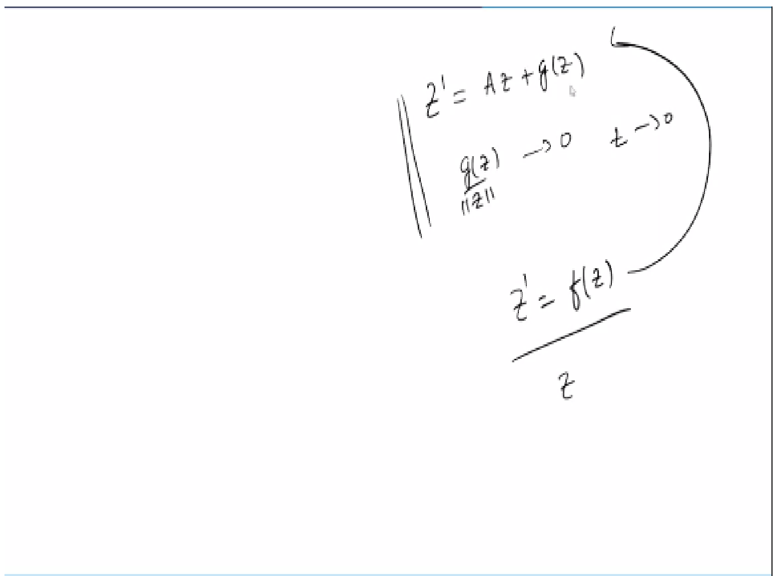

That we have z dash = Az + g(z) system of linear equation, we call this a system of weekly non-linear system. Here g(z)/norm of z is a continuous function of z and tending to 0 as z tending to 0 here. And we have seen that, that this can be utilized the result discuss for this weekly non-linear system may be utilized for a non; for a general say non-linear system z dash = f(z) provided that f has some certain nice features that the second order partial derivatives are continuous.

Then we can see that this can be written as this and here this can be written as z dash = Az + g(z) where g(z) satisfy condition of the theorem which we have discussed here. So let us utilize this to find out the stability of equilibrium solution of a arbitrary for a general non-linear system here. So based on the observation we have done earlier.

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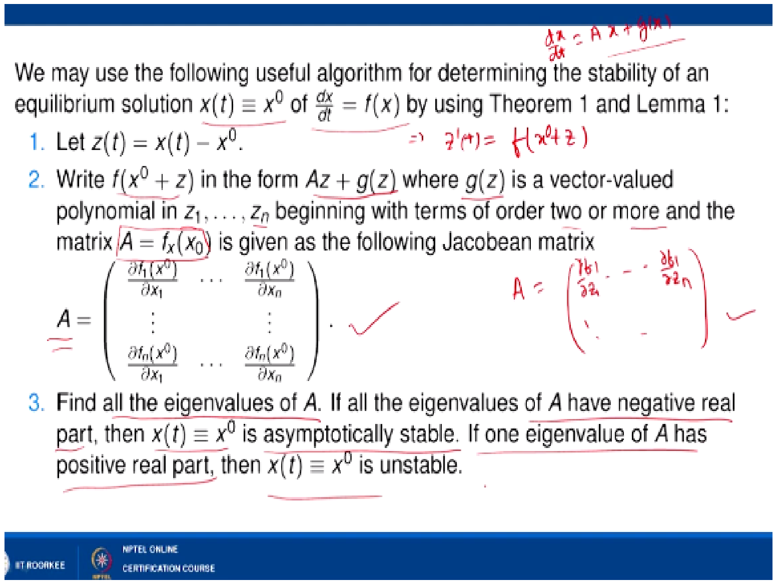

We have the following algorithm to determine the stability of an equilibrium solution x(t) identical equal to x0 of dx/dt = f(x) by using Theorem 1 and Lemma 1. So Theorem 1 is a result for dx/dt = Ax + gx and Lemma 1 says that if f has a second order partial derivative continuous then this f(x) can be written as Ax+g(x) kind of thing.

Now, let us look at the algorithm and how we can utilize algorithm to find out the equilibrium; stability of equilibrium solution Xt = identical equal to x0. So here first thing we shift our origin or we can write down z(t) = x(t) – x0, where x0 is the equilibrium solution of dx/dt = f(x). Then when we do this then our dx/dt = f(x) is reduce to z dash t = f(x0+z) here. So now once we have a f(x0+z) then we try to write f(x0+z) in the terms of Az+g(z) here.

Where g(z) is a vector-valued polynomial in z1 to zn beginning with terms of order two or more and the matrix A is basically Jacobean here and that we already seen that f(x0+z) is reduce to Az+g(z), where A is basically your dou f1 dou z1 to dou f1 dou zn and so on. So we denote this matrix as a Jacobean matrix and we A as dou f1 dou x1 at x0. So this is evaluated at x0. So we can, in summary we can write A as fx at x0 evaluated at x0.

Now, once we have written it like this then look at all the Eigenvalues of A. Now if all the eigenvalues of A have negative real part then x(t) identical equal to x0 is asymptotically stable as we have pointed out in theorem we have discussed. And if any one of the eigenvalues of A has positive real part, then x(t) = x0 is unstable solution.

So this is our working method to deal with an equilibrium solution of A non-linear system here. So please note this thing, down that we can discuss the stability of every solution of n linear problem, that is x dash = Ax. But regarding the non-linear system we can discuss the stability of only equilibrium solution because it is may not be possible all the time to find out say every solution of a non-linear system.

So in case of non-linear system methods are available to find out stability or unstability of equilibrium solutions. So here is one of the method to provide the stability of equilibrium solution. So let us consider few example to illustrate the working of previous few lectures.

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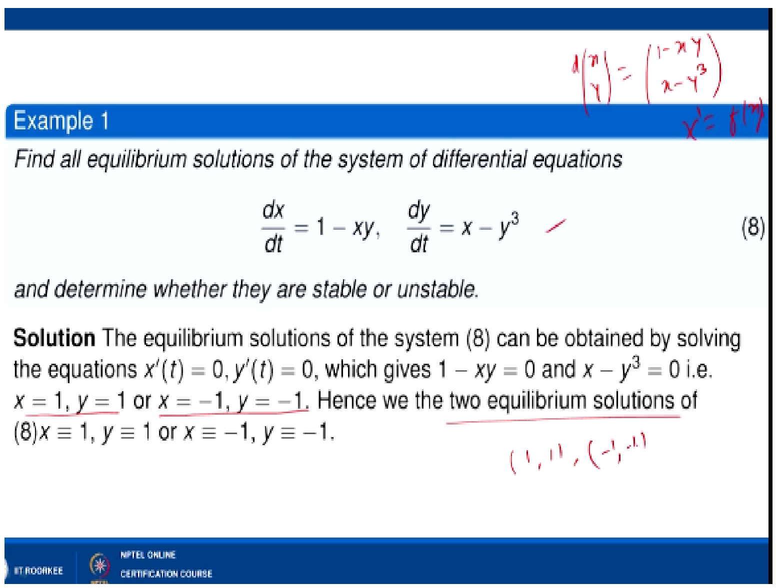

So in this example find all equilibrium solution of the system of differential equations dx/dt = 1 – xy, dy/dt = x – y cube and determine whether they are stable or unstable. So first thing is that it is a non-linear system that we can write down this as dx/dt so it is xy=; and it is what 1-xy and x-y cube. So this is a system x dash = f(x) is given to you. Now we want to find out the stability of equilibrium solution. So it is a non-linear system.

So first thing we need to find out the equilibrium solution. To find out the equilibrium solution we simply put f(x)=0 and try to find out the solution here. So here to find out the equilibrium solution we have to assume here that.

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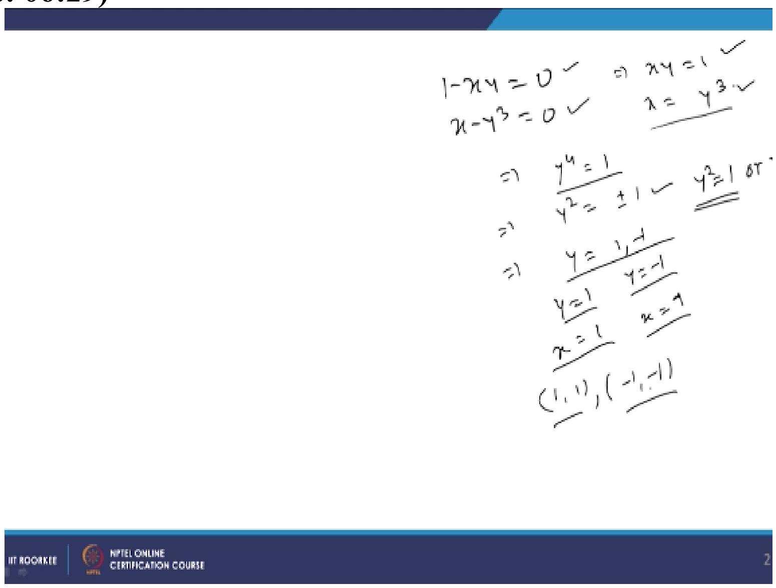

So 1-xy=0 x-y cube = 0 implies that xy=1 and x=y cube. Now putting x=y cube in first equation we have y2 power 4=1 which gives you that y square = +-1 here. So if I take the y square = 1 or y square = -1, now this will give you an imaginary root so will consider this. Now considering y square = 1 we have to roots y=1 and -1. So if I take y=1 then this xy=1 give you that x=1 and if you take y=-1 then xy=1 gives you x=-1.

So if we consider the pairs the pairs are 1 1 and -1 -1. So here equilibrium solution we obtain as two equilibrium solution as 1 1 and -1 * -1. So here we have two equilibrium solution 1 1 and -1 -1. Now let us find out the stability of each equilibrium solution.

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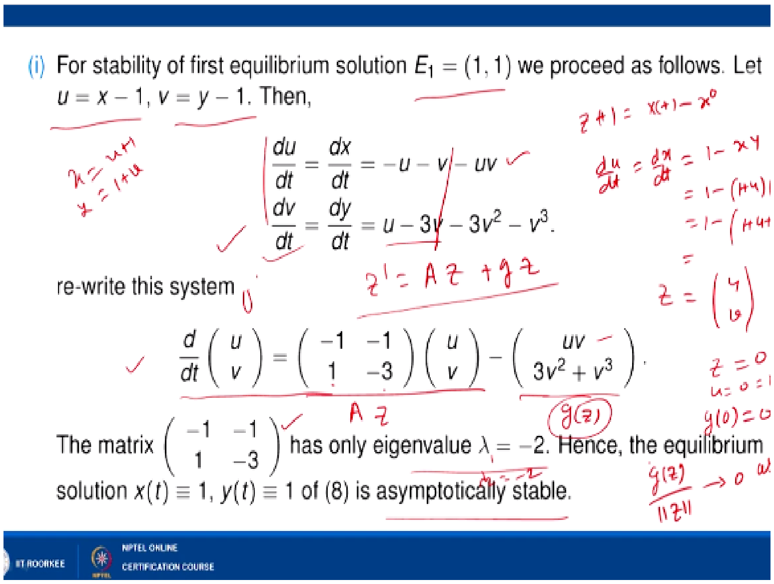

So first consider the first equilibrium solution that is 1 1. And we proceed as we have done in previous algorithm. So first thing we consider as z(t) = x(t)- your x0, x0 is 1 1 here. So you consider u as x-1 and v as y-1. And we can write down du/dt in terms of dx/dt as we simply write du/dt is what du/dt = dx/dt =. Now in case if it is 1-xy, now what is x here, x is 1+u and v is 1+y. So we can write down 1-x is 1+u and y as 1+v here.

So we can simplify and we can write down 1- this is what 1+u+v+u of v here. So we can simply write –u-v-uv. Similarly, we can write down dv/dt as dy/dt x-y cube, so x we already know x is basically u+1 here and y=1+v here. So using this we write down du/dt as –u-v-uv and dv/dt is equal to u-3v-3v square – v cube. Now using this we rewrite our system as du/dt =.

Now we write down, we take out the linear part out and non-linear out, so we can write this, this is a linear part here and this is a non-linear part, so we can write du/dt=-u-v* - u*v. And this second equation dv/dt I can write it u-3v-3v cube-v cube here. So we can write down this as u dash or you can say z dash = Az+g of z here and z here is basically u and v here.

So we can write down this is a linear part Az and this is the part g of z here. So we can say that, g(z) if you take that for z=0 means u=0=v g(0)=0. So first thing we have seen that g(0)=0 here. And second thing we simply say that as g(z)/say norm of z is tending to 0 as z tending to 0 here. So uv/norm of u square, so here we will take the norm as two norm and let me prove that.

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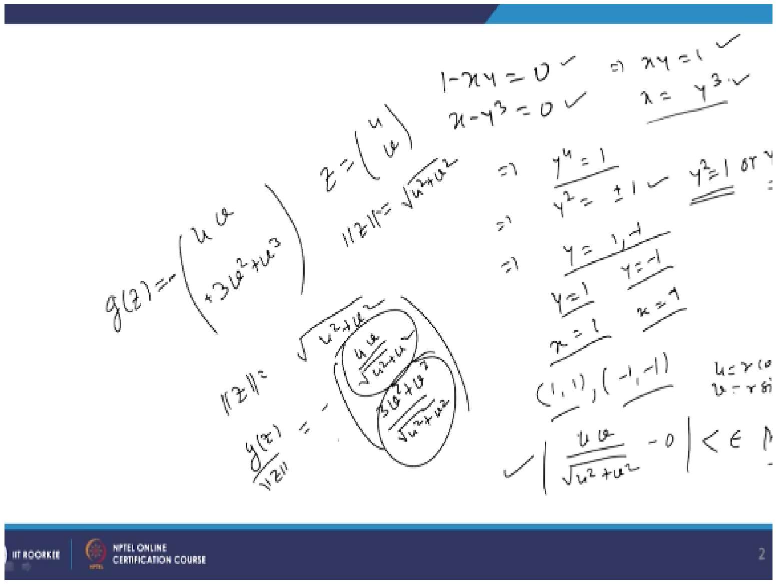

Here you have g(z)=uv and here in the second term what we have -3v square + vq – 3v square + v cube here. So now we can write down here that if we take norm of z= under root u square + v square, so g(z) is written as – of uv and 3v square + v cube. And here your z is basically u and v. So norm of z we are writing as equality norm and it is u square + v square here. And we want to show that g(z) upon norm of z is a continuous function of z here.

So if you look at g(z) of – g(z) upon norm of z = -uv upon norm of u square + v square and here it is 3v square + v cube / u square + v square here. Now we simply say that this is your g1 this is your g2 and both are tending to both are continuous function of uv as at 0 and it will tend to 0 as z tending to 0 here.

So for continuity of g1 you can simply say uv/norm of u square + v square we can check using polar coordinates you can use u=r cos theta and v = r sin theta we can say that this quantity can be made arbitrary small as r tending to say mod of r is < delta. So when r is less than delta we can make this quantity arbitrary small. Similarly, we can look at the second terms second term is 3v square + v cube.

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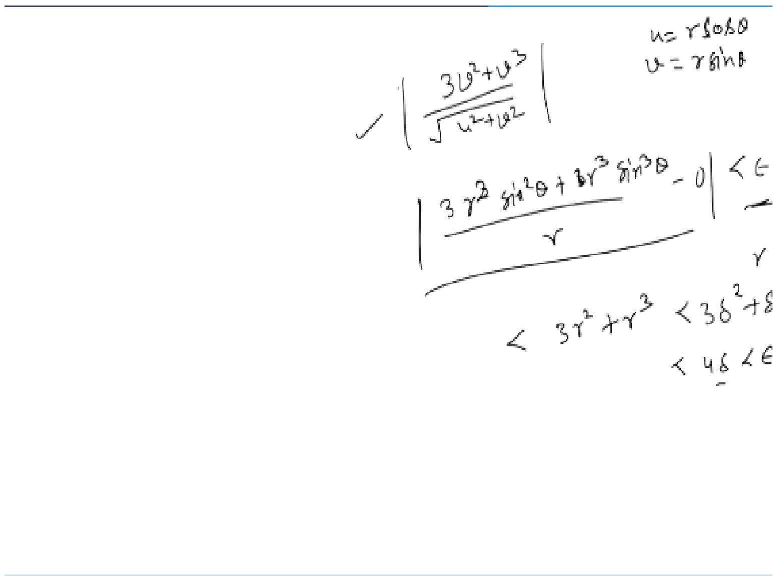

3v square + v cube divided by u square + v square and again you can take u=r cos theta and v as r sin theta. So we can say that this quantity modulus of this is written as 3r cube r square sin theta say sin square theta + 3r cube sorry r cube sin cube theta divided by r – 0; I want to make this quantity less than epsilon as r is less than delta. So if you look at this thing is less than say 3r square + r cube here, right. And we can make this quantity less than 3 delta square + delta cube or we can say this is less than 4 delta if we assume that delta is less than 1.

And we can assume that if we choose delta l as a epsilon/4 then this quantity is continuous at 0 0. So what we have proved here that the system this can be written as z dash = Az+g(z) where g(z) satisfy the conditions of theorem that is g(z)/norm of z is a continuous function of z and vanishes as z tending to 0 here. So it means that our theorem is applicable. Now look at the say eigenvalues of A. So look at the linear part, linear part is this. It is z dash = Az here.

Now the matrix A is given as -1 -1 1 and -3. Now you can find out the eigenvalues of this matrix and here we can check that it has only the eigenvalues lambda = -2 and -2. And here we can say that, that the real part of eigenvalues are negative. So it means that lambda 1 = -2 and lambda 2=-2. So both eigenvalues are having negative real part. And hence we can say that equilibrium solution x(t)=1 and y(t)=1 is asymptotically stable solution.

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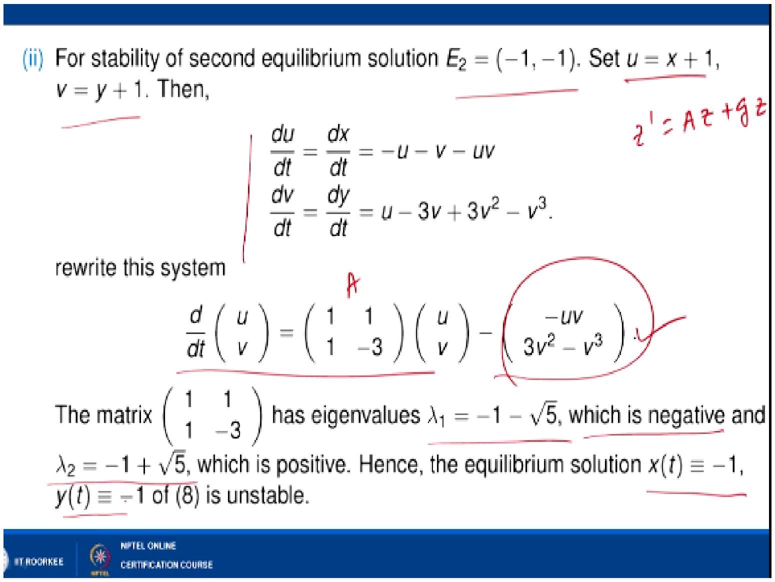

So now in same way we can look at the; consider the stability of the second equilibrium solution that is -1*-1. So here you set u=x+1 v=y+1 and write down system as z dash = Az+g(z) here. Now du/dt is written as –u-v-uv and dv/dt is u-3v+3v square – v cube. And it is written like this. Now it is a linear part here and as we have pointed out in a similar way we can prove that g(z) satisfy the condition mentioned in theorem that we have similarly prove this.

So we can say that the stability of z(t)=0 is equivalent to stability of the associated linear part. Now look at the eigenvalues of matrix A. And if you look at the eigenvalues are lambda 1 = -root 5 which is negative. And second one is lambda 2 = -1 + root 5 which is a positive one. So here, we have seen that they exist an eigenvalues with positive real part and hence by part b of the theorem we can say that the equilibrium solution x(t)=-1 and y(t)=-1 is an unstable equilibrium solution here.

Now here, by this we have completed the analysis of stability of equilibrium solution of E1 and E2 here. Now here one thing we noted down, that here we have done this kind of analysis that u=x+1 and y=v+1 and similarly we have done in the first case also that we assume u=x-1 and v=y-1. But we have already pointed out that this A is nothing but the Jacobean of the function f(x) here.

So here you can directly calculate your matrix A without this translation. In fact, you can write your f(xy) as follows.

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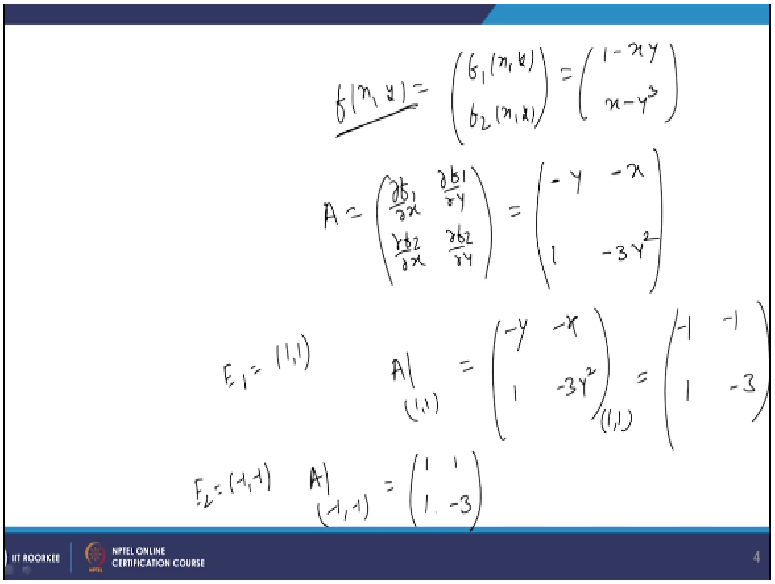

Your f(xy) is basically what f(xy)=f1(xy) f2(xy) here and this is what 1-xy it is x-y cube here. So your A is basically Jacobean of this so it means that dou f1 dou x dou f1 dou y dou f2 dou x dou f2 dou y here and this is nothing but, this is –y here and is –x here and this is what it is 1 and it is -3 y square here. So this is the Jacobean of function f here. Now to find out the stability of find out the stability behaviour of say E1 that is 1 1 here.

We have to look at the matrix A at 1 1 here. So if you look at this is what –y –x 1 -3y square at 1 1. So if you evaluate this -1 -1 1 -3. And if you look at it is same as the matrix here. -1 -1 1 -3 and if you look at this is -1 -1 1 -3 here. So here you can look at the eigenvalues of this and we can get it. So without doing this say exercise we can directly find out the matrix A and we can look at the eigenvalues and we can get, we can infer the information about the stability or unstability here.

Similarly, we can look at here A at -1 and -1. So A at -1 and -1 it is equal to what is 1 1 and 1 -3 here. And it is similar to your; here 1 1 1 -3 it is 1 1 1 -3 here. So here you can avoid this long calculation and you can directly find out the matrix A here and we can say that analysis is based on the eigenvalues of the matrix A. So here you need not to do all these exercise you can directly find out your matrix A and we can get our information about the stability of equilibrium solution.

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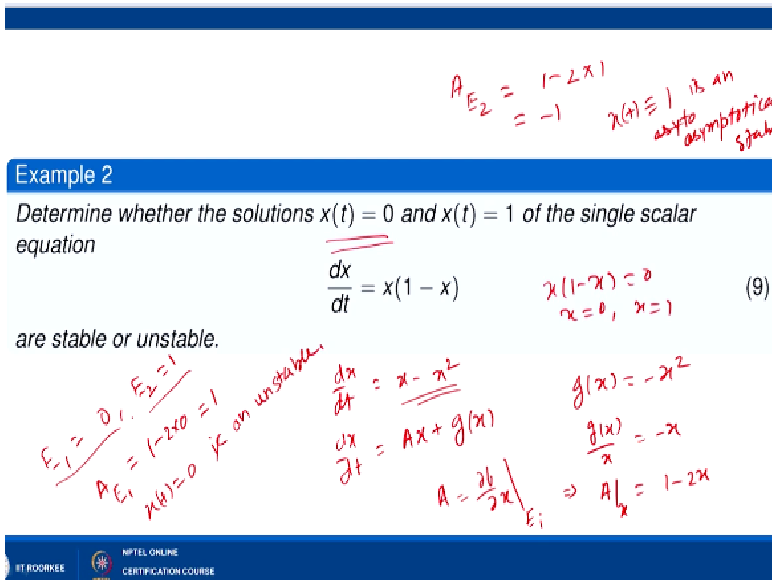

Now look at next example, and this example we have already discussed and here we want to find out determine whether this solution x(t)=0 and x(t)=1 of the single scalar equation dx/dt=x(1-x) are stable or not. So first thing that equilibrium solution of what x*1-x=0 so it is x=0 and x=1. Now I can write this as dx/dt= this is x-x square. So here this is your dx/dt, if we compare with your; this is then it is Ax+g(x) and it is written in this particular form where g(x)=-x square so g(x)/norm of x here that is this is a continuous function with respect to x and vanishes at x=0 here.

So it satisfy the condition of the theorem. And then we can say that stability of equilibrium solution is depending on the eigenvalues of the matrix A here. So A is basically say, say Jacobean of this matrix x is x-x square so in the scalar case it is nothing but dou f/dou x at equilibrium points say Ei. So let us calculate, look at the stability of 0 solution first. So it means that what is dou f/dou y, dou f/dou x is given as A at x is given as 1-2 of x. now if you want to calculate E1 that is the stability of 0 solution.

Then A at E1 will be what, it is 1-2 times 0 that is 1 here. So it is a matrix of 1 cross 1. And if you look at the eigenvalues here is nothing but 1. So it means that eigenvalues have positive real part. So we can say that x(t)=0 is unstable solution and that we have already verified here. Now we want to check the stability of 1 for that you look at A at E2 this is nothing but 1-2 times 1 here so it is -1. So here eigenvalues is -1.

So it means that eigenvalues are having negative real part and hence this is a stable solution asymptotic stable solution. So we can say that x(t) ideally equal to 1 is an asymptotic, an asymptotically stable solution. So here we can directly get without doing all these exercise. Here we simply calculate the Jacobean that is dou f/dou x at Ei. And Ei's are E1=0 E2=1 and we can directly get the stability of 0 solutions one solution without actually calculating this any every solution of dx/dt=x*1-x.

So you can compare now, the solution given here and the solution given there and we can say that here we can directly calculate the stability of 0 solution and 1 solution.

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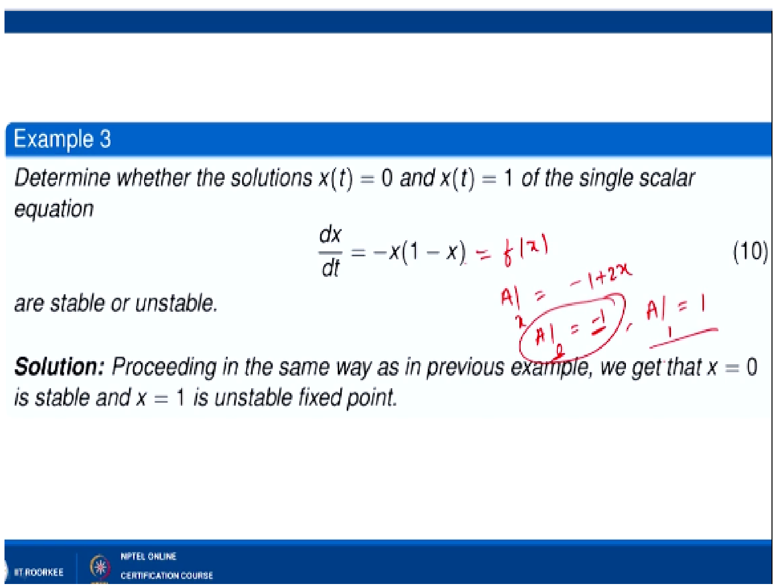

And similarly look at here, depending whether the solution x(t)=0 and x(t)=1 of the single scalar equation at dx/dt=-x(1-x) are stable or unstable. So for that this is your f(x), so A is at any point x is given as -1+2(x). And we can say that A at say 0 it is basically -1 and A at 1 it is given as 1 here. So here we can say that 0 solution is asymptotically stable solution and x(t)=1 solution is unstable solution. And this is how we have done in this particular part.

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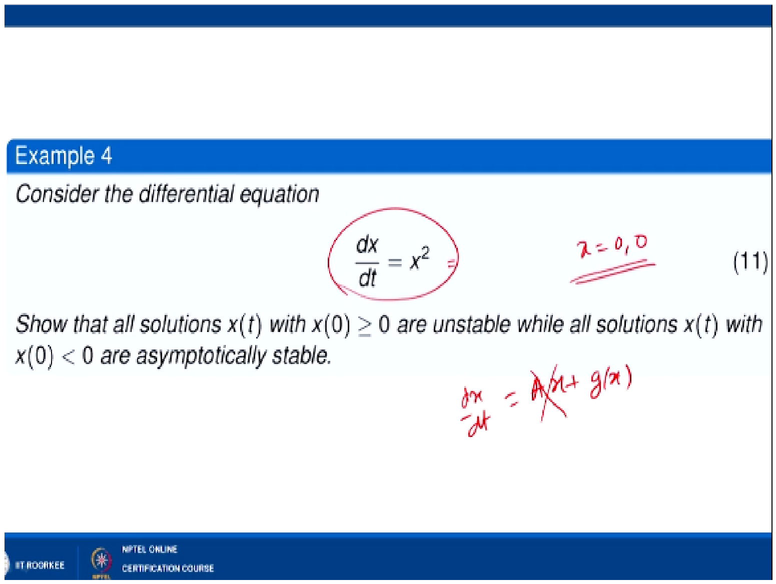

Now let us look at one more problem that is consider the differential equation dx/dt=x square and here the we want to show that all solution x(t) with x0 equal to or not 0 are unstable while all solutions x(t) with x(0) < 0 are asymptotically stable solution. So here if you look at the dx/dt=x square and here if you look at your equilibrium solutions are x=0 and 0 here. So here it is non-linear system where 0 solution equilibrium solution is already given as 0 solution.

So here we cannot transfer into a system dx/dt=Ax+g(x) kind of thing. So here we cannot apply our; so here we can simply say that here if we, we cannot write it because equilibrium solutions are 0 itself. So we can say that here the linear part is missing. So it is written as dx/dt=g(x) though g(x) satisfies all the conditions that g(x)/norm of x is a continuous function of x and it is vanishes at 0.

But since eigenvalues of A are say 0 real part having 0 real part so we cannot apply the previous theorem. So this falls under the category C here. And we can say that the stability here cannot be determined using the previous result here. So here we have already discussed the stability of 0 solution and as we have discussed in previous lectures.

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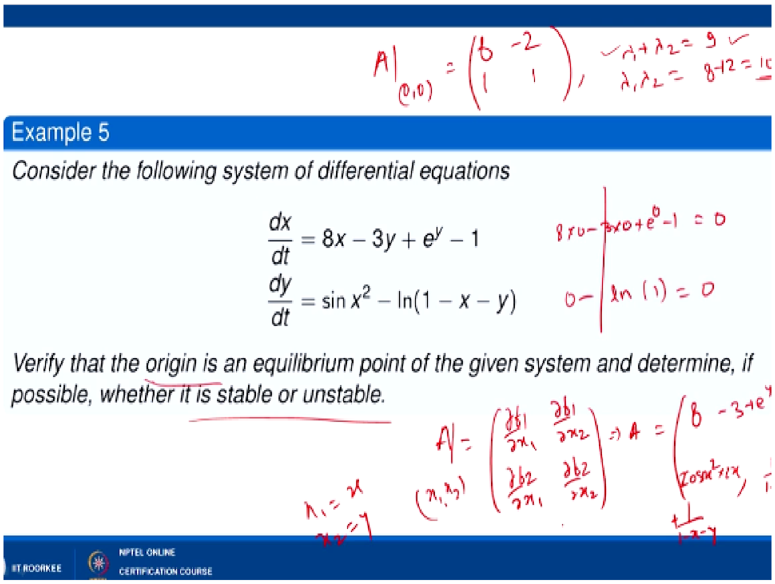

Now consider the following system of differential equation dx/dt= 8x – 3y + e to the power y-1 and dy/dt= sin x square – ln(1-x-y) here. And we just verify that origin is an equilibrium point of the given system and determine, if possible, whether it is a stable or unstable solution. So here, we can say that 0 is a equilibrium solution because if you put 0 0 you can get that it is nothing but 0. So it is 8*0-3*0+e to the power 0 – 1 it is nothing but 0 here.

Similarly, we can check 0-ln of 1 that is again 0. So 0 is an equilibrium solution. Now we want to find out the stability of equilibrium solution 0. So for that we need to find out the linear part of this. So rather than doing this we simply assume, we call this result that A is nothing but Jacobean of f with respect to xy. So A I can write it dou f1/dou x1 dou f1/dou x2 dou fx/dou x1 dou f2/dou x2. Now here x1 is x and x2 is your y.

So you can find this is your x1 and x2. So we can find out here A as, if you look at the differentiation of f1 with respect to x you will get 8 that is all and corresponding to y you will get -3+e to the power y and we will get that is with respect to x so it is 2 say cos of x square first of all, right. So sin of x square so cos of x square * 2(x)*2x-1/1-x-y here and then +1 here. So this is the first part. And second thing it is simply 1/1-x-y here.

So; and you can look at the matrix A at 0 0 so A at 0 0 will be what, it is the 8, it is -2. You will get 0 and here it is 1 here, so 1 and here it is. So now A at 0 0 is 8 -2 1 and 1 here. So we can find out the eigenvalues of A, but here you can note down that trace it means that is a lambda 1+2 is 9 here and lambda 1*lambda 2 = 8+2 it is 10 here.

So we can say that here lambda 1, lambda 2 is equal to 10 and lambda 1 + lambda 2 equal to 9, so we can observe this say that either each one is having positive sign or negative sign, both will have positive sign and negative sign. So lambda 1, lambda 2 is either positive or both negative. And it is given that lambda 1 + lambda 2 = 9 so we can say that both the eigenvalues are positive in fact we can verify that it is you can find out the eigenvalues using this.

But we can say that eigenvalues are positive here, okay. So it means that the 0 solution for this system is an unstable solution here. So here we have proved that by this observation one of the eigenvalues has to be positive. So we can say that 0 solution is an unstable solution.

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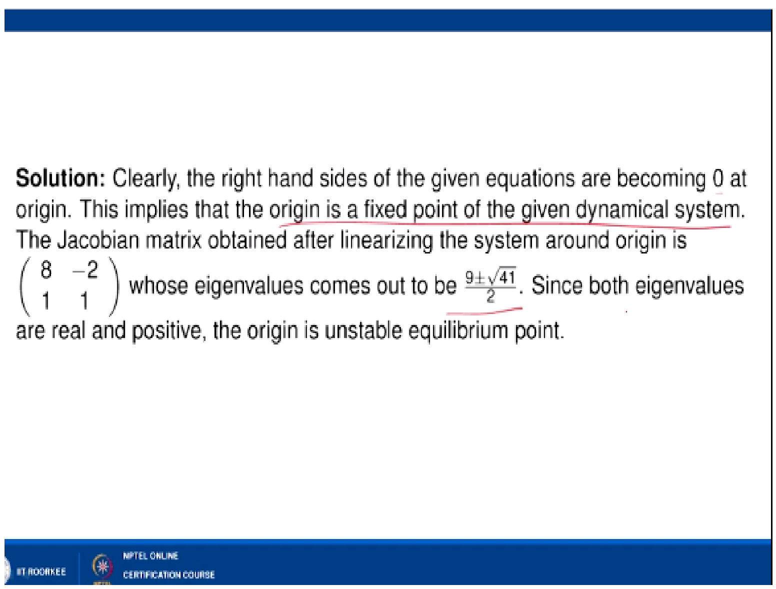

Now, so here we can simply say, let us look at the solution here. Clearly the right hand sides of the given equations are becoming 0 at origin. So this implies that the origin is a fixed point of the given dynamical system. And the Jacobean matrix we have pointed out this 8 -2 1 1 that we have just 8 -2 1 1 so we have obtained. And here we are not calculated the eigenvalues just for by observation we have seen that both the eigenvalues lambda 1+ lambda 2 = 9 and lambda 1*lambda = 10.

So it means that both the eigenvalues are having either positive or negative sign. But this is since sum is positive so it has to be positive. So eigenvalues are comes out to be 9+-root 41/2. And since both the eigenvalues are real and positive we simply say that origin is unstable equilibrium point.

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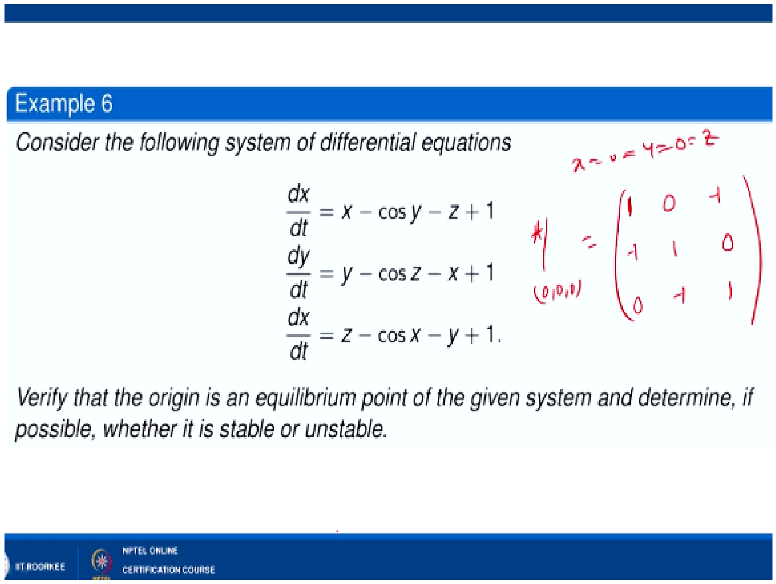

Okay, now consider one more problem that is consider the following system of differential equations dx/dt = x-cos y-z+1. So now this time it is 3*3 system. dy/dt=y-cos z-x+1; dx/dt=z-cos x-y+1 and we want to verify that the origin is an equilibrium point of the given system and determine, if possible whether it is stable or unstable. So verification is quite easy. When you put x=0=y=0=z then you can say that right hand side is coming out to be 0 here.

And to check the stability at 0 0 0 you look at the Jacobean at 0 0. So to find out Jacobean what is here, it is x, so it is 1-; if you differentiate this it is sin y and sin y at 0 it is 0 here. And this is -1 here. And similarly here you can calculate this is -1 1 and 0 here and similarly you can calculate it is 0 -1 and 1 here. And you can check the say eigenvalues of this matrix and you can get the information about the stability or unstability of equilibrium solution z=0 here.

So here we are not doing any translation work. Here we are just calculating the Jacobean of the matrix at the point at the equilibrium point and just looking at the eigenvalues of the matrix we are getting the information about the stability or unstability of the non-linear system here. So with this we finish our discussion and in next lecture, we will continue our discussion. Thank you very much for listening. Thank you.