Dynamical Systems and Control

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Lecture – 02
Formulation of Dynamical System - II

Hello friends. Welcome to this lecture. In this lecture, we focus our discussion on formulation of some physical system. So to start with, we start with very simple model that is population model.

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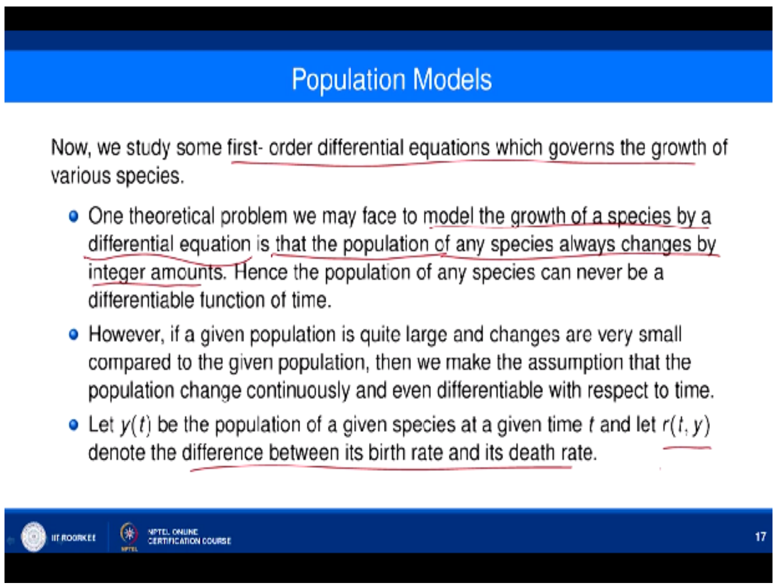

So in population model, we study some first order difference equation which governs the growth of various species. But the practical problem in doing so is the following that one theoretical problem we may face to model the growth of a species by a difference equation is that the population of any species always changes by an integer value, that it gives you say +50, -50, +10 or -10.

So here if we consider that the change is only not in a continuous process but given in an integer amount and hence the population of any species can never be a differentiable function of time. So how to get rid of this particular theoretical problem? So to handle this, we may consider this assumption that if a given population is quite large and changes are very small compared to the population which is given, then we make the assumption that the population change continuously and even differentiable with respect to time t.

So here whatever population model we are providing, here we are assuming that we are dealing with a population which is quite large and the change is compared to the population is quite less. So in that case, we can assume that the population, change in population is basically a continuous function, not only continuous function, it is differentiable function of time t. So let us assume that yt be the population of a given species at a given time t.

And let rty represents the difference between its birth rate and its death rate. So rty is the difference between the birth rate and its death rate. So using this, let us assume one more condition that the population is isolated. So it means that given a domain, only that population we are considering. No other interference is allowed. So it means that population is isolated and it has no interference from the outside or it is not going outside.

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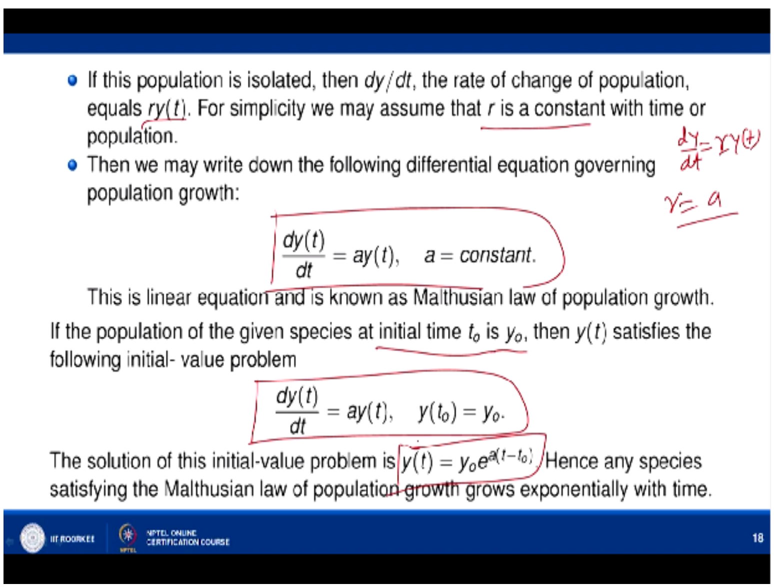

So in that case the dy/dt, the rate of change of population=ryt. So it means that here you can say that dy/dt is given as ry of t here. And for simplicity, let us assume that this r which is, we are assuming and the rate of change of population, the difference between birth rate and death rate, I am assuming that it is not depending on time t and the population at a given time t. So it means we are assuming that this rate is independent of time t basically or population.

So here I am assuming that r is a constant. So I am considering a very simple example where this r is a constant value. You may consider, in a later stage, that if r is a variable quantity, then how we can handle this particular population model. So if we assume these conditions, first condition is that population is a differentiable function of time t. Second, that population is say independent or isolated from other, all other world.

And third is that here the rate of change of say population is independent of time t. So in that case, your population I can write this as dy/dt=ayt where a is a constant and I am assuming that r=a. So in this, the simplest differential equation governing the population growth is given by dy/dt=ay where a is a constant value.

So this is a linear differential equation. You can check that it is a linear differential equation and it is known as Malthusian law of population growth. And if the population of the given species at initial time t0 is given that is y0, then yt satisfies the following initial value problem, that is dy/dt=ayt with a condition given at the initial point that is t0 say that yt0=y0 and we can simply solve this problem. This is a simple separable variable. And you can solve this problem and your solution is given by yt=y0e to the power a t-t0. So your solution is given by this.

So here, I am assuming these constant, our solution is given by y0=e to the power at-t0. And hence any species satisfying this particular law of population growth grow exponentially with time t. So it means that as t tending to infinity, your population is growing very, say in an exponential way and it is going to be very large. But in general, this may not be a very, kind of convincing population growth model because when, if you look at the domain, domain is limited.

The source of food, living and all these are limited. Then if the population goes beyond a certain limit, then the species may compete each other for living and food and all other things. So it means that when, here I am assuming that they are having no competition among the population. And it is growing without any kind of problem. But even the population is say large, then they have to compete each other with the available choice of say food and living space and all that. Then the growth will not follow this exponential growth model.

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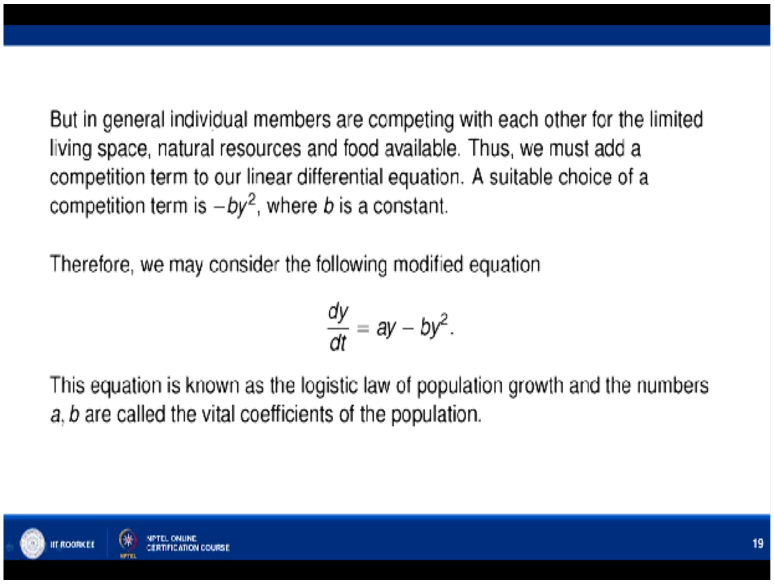

So here we are assuming that in general, inducible members are competing with each other for the limited living space, natural resources and food available. So it means that when the competition is allowed, then this growth should not follow the previous available model. So here we have to add one more competition term which is proportional to the interaction between the inmates of the population.

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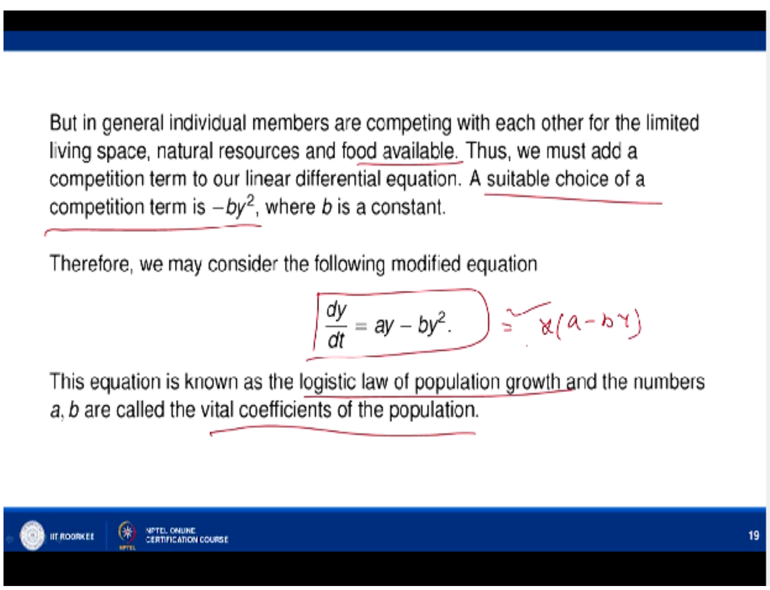

So here we say that the suitable choice of a competition term is given as =by square. y square means when 2 members are competing each other, then there is a decrease in the growth and b represents the constant which is known as vital coefficient and b is a positive thing, - is putting because competition basically reduces the population of the given species. So therefore, if we include this competition term, then we may consider the following modified differential equation, that is dy/dt=ay=by square.

If you say that if b which is due to the competition of the inmate of the population, then if b=0, it means that there is no competition among the inmates of the population, then it is nothing but dy/dt=ay. But or you can say that if y is small or we can say the population is small population, then this y square term is small and we can ignore that. But if y is large, then we cannot ignore the term y square.

And we say that this is also an important term affecting the population of the species. So this particular model is known as the Logistic law of population growth. So here we say this is known as Logistic law of population growth and the numbers a and b are called the vital coefficients of the population. And now we try to see how this population growth model actually behave.

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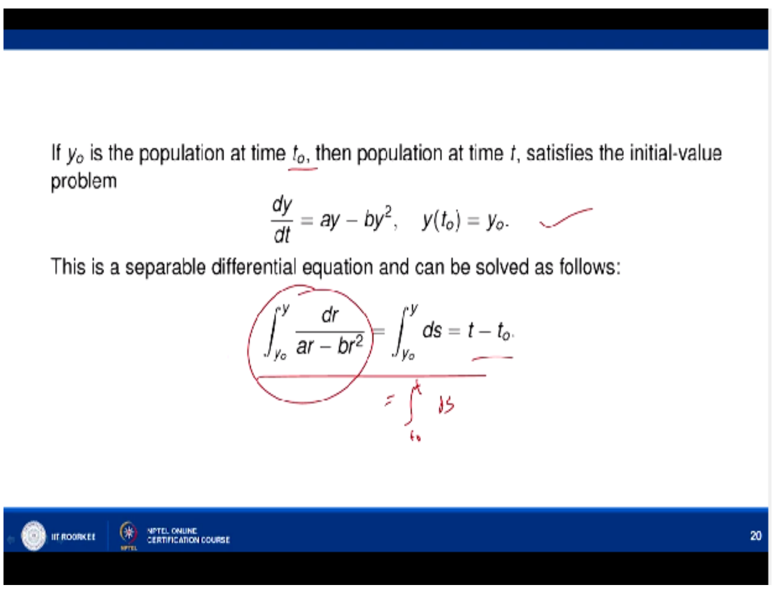

So here if you look at, this is a simple problem. You can simplify this as ya-by and this is a variable separable form and you can simply solve. So now if y0 is the population at time t0, so initial condition is given, then population at time t satisfy the following initial value problem that is dy/dt=ay-by square, yt0=y0. Now we can integrate and we can find out the value of yt, y at a given time t, that is yt.

And this we can simplify dy/ay-by square=dt. If we integrate between y0 to y, so y0 to y, dr/ar-br square=y0 to yds. Here this corresponding to y0, the time is t0. So you can simply say that this is nothing but t0 to td of s. So this is nothing but t-t0 and to solve this, we use the, we can simplify this.

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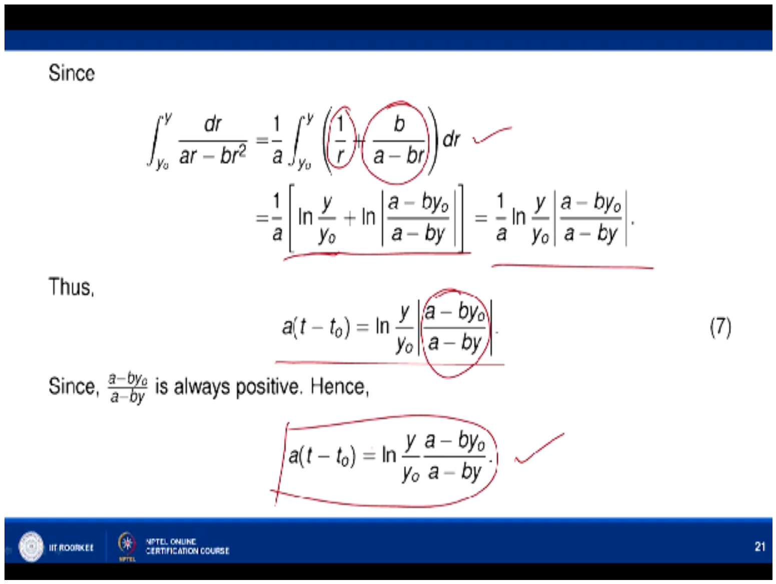

And we can have the following thing that y0 to ydr/ar-br square=1/ayo to y1/r+b/1-brdr. Here you can use partial fraction and you can write it like this. So here using partial fraction, we can write y0 to ydr/ar-br square as this thing. And now we can integrate in a very easy manner. So 1/a, and this is nothing but ln of r and when putting the limit y0 to y, you can write it ln y of y0, + here again you can use the logarithmic function and you can write this as ln of a-by0/a-by.

So this after putting the limit also. And when you simplify, you can write it 1/a ln of y/y0 and here modulus of a-by0/a-by. And if we simplify further, you have at-t0=ln of y/y0 modulus of a-by0/a-by. You can easily verify that the term available here remain positive. So using this, you can write this as a of t-t0=ln y/y0 a-by0/a-by and with this, we can simplify this expression in terms of y here.

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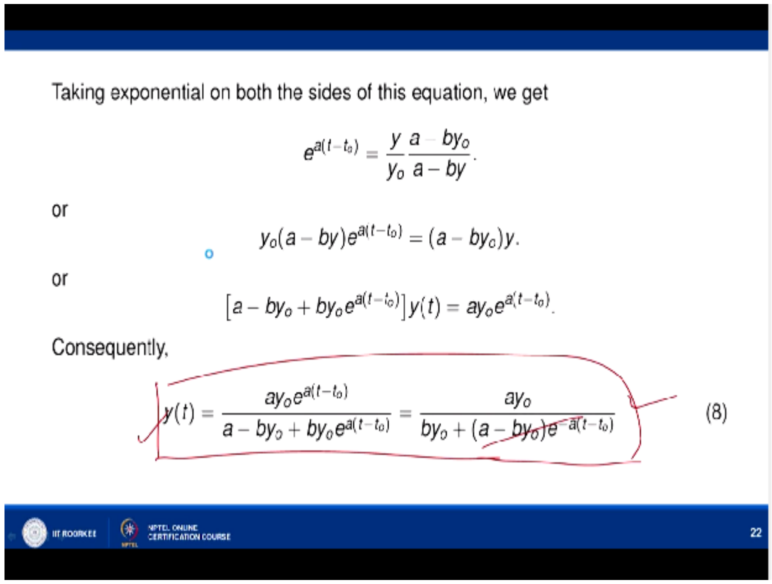

And we can simplify further by taking the antilog here. So or taking exponential on both the sides, we have e to the power at-t0=y/y0 a-by0/a-by and after simplify, we can write down yt=ay0e to the power at-t0/a-by0+by0e to the power at-t0. You divide by e to the power at-t0 and you can get the expression for yt as ay0/by0+a-by0*e to the power -at-t0 here. And if you look at this expression for the population at time t, then we are interested to know that what happens if the time t is tending to infinity.

It means that when t is very large, what should be the value of this yt.

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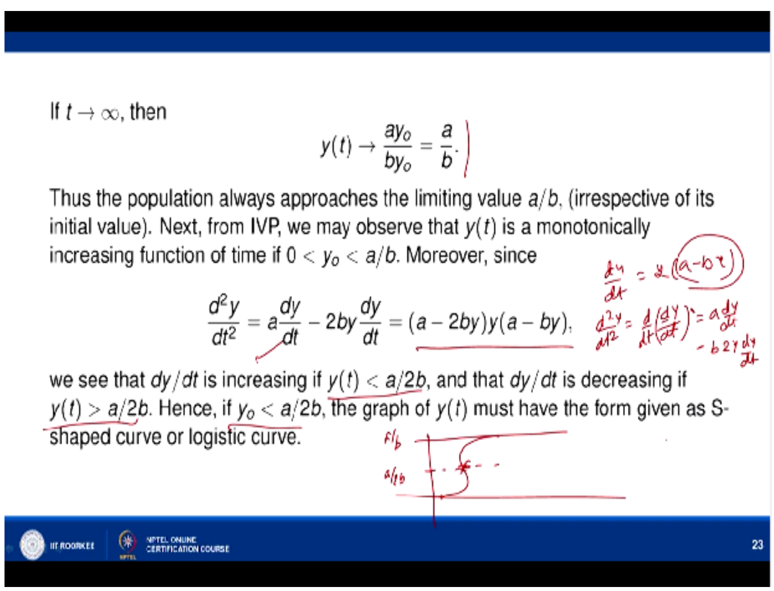

So when you take t tending to infinity, then your yt, the value of yt is tending to ay0/by0.

Because you look at this term as t is tending to infinity, this term is tending to 0. So this term is tending to 0, so what is left here is ay0/by0 and y0 is non-0, so we can say that at time t tending to infinity, means at asymptotic value of yt is nothing but a/b. So it means that the population always approaches the limiting value a/b.

Whatever be the initial value, but population is approaching to the limiting value a/b. And next we can observe from the given initial value problem, that is dy/dt=ya-by, that here y is monotonically increasing if y0<a/b. In fact, if you look at this quantity is positive, then dy/dt is always positive. So it means that your population is monotonically increasing if y0, initial condition is basically lying between 0 to a, b.

And also we can verify that if we take the d2y/dt, so d2y/dt square=d/dt of dy/dt and that is nothing but ayady/dt-b2ydy/dt and if you simplify, you have this thing that ady/dt-2bydy/dt and if you put the value of dy/dt and you can write down d2y/dt square as a-2by y*a-by. So it means that this dy/dt is increasing if yt<a/2b, and it is decreasing that is dy/dt is decreasing if yt is bigger than a/2b.

And hence, it means what? That if your population<a/2b, then the graph of yt must have the following form. You look at this. You have this asymptotic value that is a/b here and here value a/2b. So here it is behaving like this kind of graph. So it means that it is increasing, so here this is the initial value. It is increasing up to the value that is a/2b. As soon as till initial point to a/2b value, your dy/dt is increasing.

So it means that population is increasing. And once it has reached up to a/2b values, then dy/dt starts decreasing and ultimately we have the graph satisfying this kind of figure that is known as S-shaped figure or known as logistic curve here. So it means that is why this curve is known as logistic curve and that is why the equation given in, governing this model that is dy/dt=ay-by square, yt0=y0, is known as logistic model of population growth here.

So it says that initially population starts increasing till it reaches to the half of the value that is a/2b value. Then we say that then this competition is a kind of non-ignorable. You cannot ignore the competition and because of this competition, your population starts, the rate of population starts decreasing and it is tending to the population that is the value a/b. So it means that initially your rate is positive and after coming to the half value, your rate is coming out to be negative and it is reaching up to the value a/b here.

So that is one model known as population model. We have discussed the simplest model that is exponential growth model when there is no competition term. And then we discussed a model including the competition term. And we discussed that the curve of the population is coming out to be the S-shaped curve or logistic curve. Next, we go to the next model that is the spread of technological innovation.

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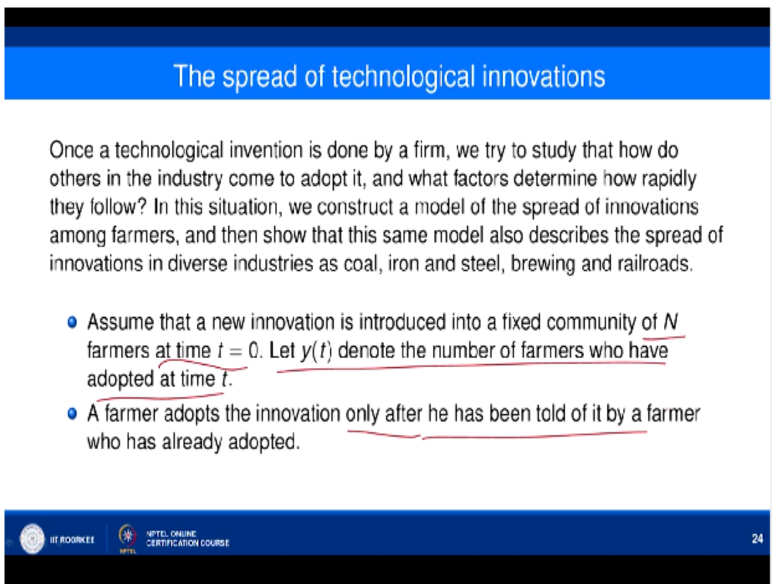

So basically this problem is like this that once there is a new invention in say market, then how this invention is reaching to the population for which this is invented. So here we have the following thing that once a technological invention is done by a firm, we try to study that how do other in the industry come to adopt it and what factors determine how rapidly they follow? Means given an invention, we need to know that how this innovation is reached to the say its customers.

So in this situation, we construct a model of the spread of innovations among farmers. So we take a particular case when we are considering that farmer is your customers and we have an invention and we try to see that how it is covering the entire population of farmer in a particular area. And once we understood this model, then we can show that the same model can also describe the spread of innovation in diverse industries such as coal industries, iron, steel and other related industries.

So assume, first let us understand this particular model. So assume that a new innovation is introduced into a fixed community of N farmers. So we know that population of farmers is given by the N numbers and at time t=0, so when we are starting, providing the innovation. So at time t=0, we have N farmers. And let yt denote the number of farmers who have adopted the innovation at time t. Now a farmer adopt the innovation only after he has been told of it by a farmer who has already adopted.

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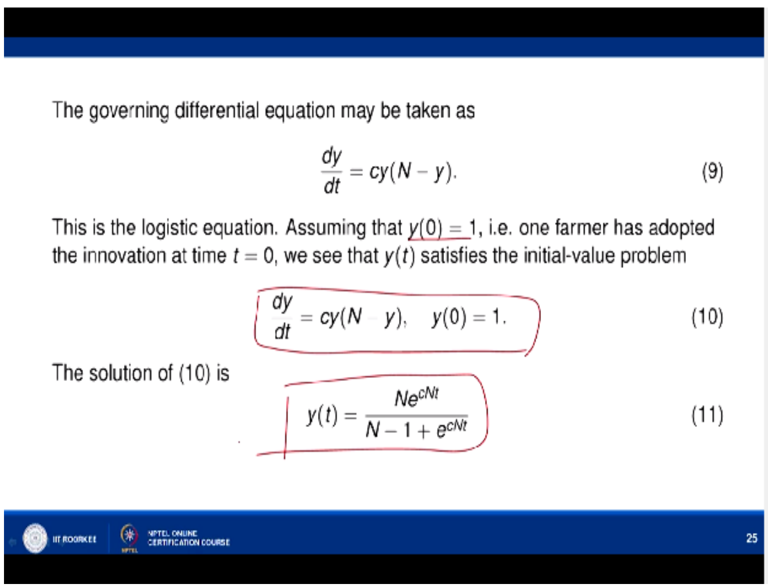

So it means that the governing differential equation may be taken as the change in the number of the farmers who have already adopted is proportional to the farmer who have already adopted and the population which is left out, which is not adopting the innovation. So here we can say that dy/dt is proportional to y, the number of farmers who have already adopted N-y, it means the number of the farmers who have not adopted till that time t.

So we can say that c is the proportionality constant and we can write dy/dt=cyN-y as a governing differential equation for this particular situation. If you look at, it is nothing but a similar case we have already discussed that is the case of population growth model given in logistic equation. So this is the logistic equation which we have already discussed. Now along with this, we say some initial condition that is we will say that y0=1 is the initial condition.

So it means that at the beginning, we are assuming that there is a farmer at time t=0 who has already adopted the innovation. So it means that yt will satisfy the following initial value problem, that is dy/dt=cyN-y, y0 at 1. So I am not solving it again and I assume that you please follow the solution procedure and you can write down that the solution is given as yt=Ne to the power cNt/N-1 e to the power cNt.

And as we have discussed earlier that since it is following the logistic equation, so it means that the number of people who are going to adopt the innovation is increasing, the rate is increasing till we reach the half population that is N/2 and after that, rate is decreasing till it is going to the complete, covering the complete population.

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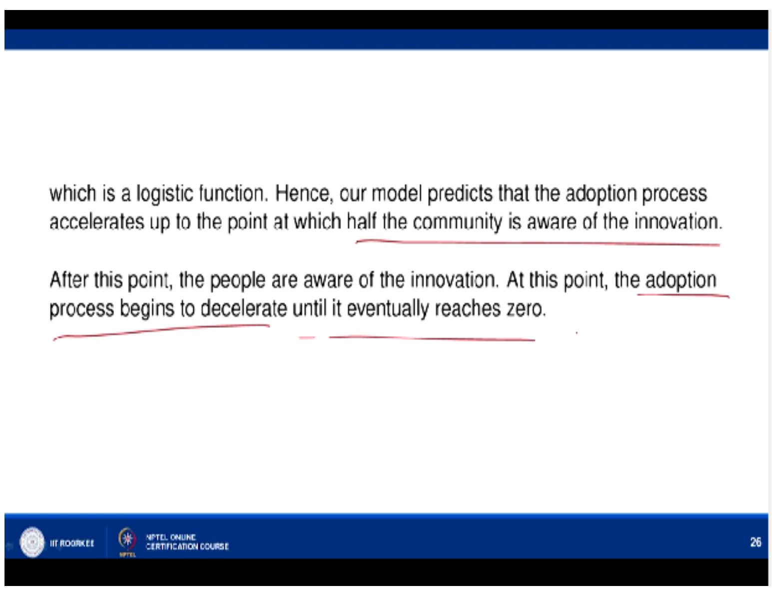

So here as we have pointed out, our model predicts that the adoption process accelerate up to the point at which half the community is aware of the innovation. And after this point, the adoption process begin to decrease until it eventually reaches to 0. So if you look at this equation, so here you say that dy/dt=0 when y=0 or y=N. So it means that when y is whole of N, then dy/dt=0. So it means that there is no change in the population.

In fact, it says that if rate will increase in the beginning till the half population reaches. After the half population, it will decrease. The rate of change in population will decrease and it will goes eventually to the 0 value. And at 0 value, your population is coming out to be whole of N. Now consider one more problem which is very interesting problem that is atomic waste disposal problem.

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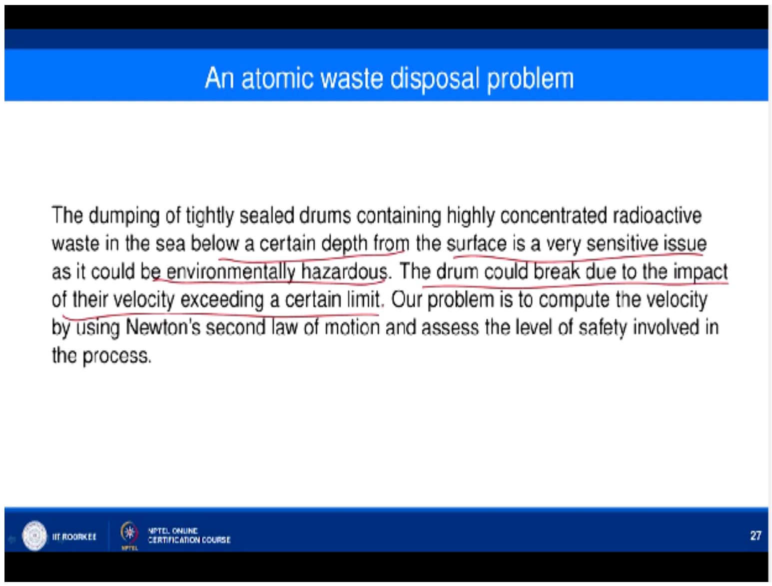

So in this problem, since now days this nuclear related things are coming up. People are, countries are focusing more on this nuclear related problems and nuclear related developments. And in the process, they are doing certain experiments and as a by-product, they have lot of atomic waste and it is very dangerous to dispose them off. So what many countries adopt this procedure that they keep the atomic waste in a tightly sealed drum.

And then they dispose it in a say very deep in a sea. Quite a usual trick they are adopting and this particular problem is based on that procedure and it was seen in that in a particular situation, when you put your atomic waste in that drum and when you leave it in sea, from sea level, then it reaches to the bottom of the, that particular depth. Then there is a possibility that drum will crack.

And if it is cracked due to some procedure, then there is a highly dangerous that the atomic waste is now disposed to, exposed to the inmates of ocean or sea. Then it is not safe to use this procedure. So this problem is based on this procedure. So the dumping of tightly sealed drums containing highly concentrated radioactive waste in the sea below a certain depth from the surface is a very sensitive issue as it could be environmentally hazardous.

The drum could break due to the impact of their velocity exceeding a certain limit. So I am considering only one very particular example. In that example, it was assumed that your drum will break down if it reaches to the surface of the sea beyond the 40 feet per second. If its velocity is more than 40 feet per second at that particular time of impact, then it may break. So now we try to see whether making a suitable model, what should be the velocity of the drum when it is hitting the sea surface.

So now let yt denote the position at time t of the object, here the drum is the object measured from the sea surface indicating y=0 as sea surface as a positive quantity.

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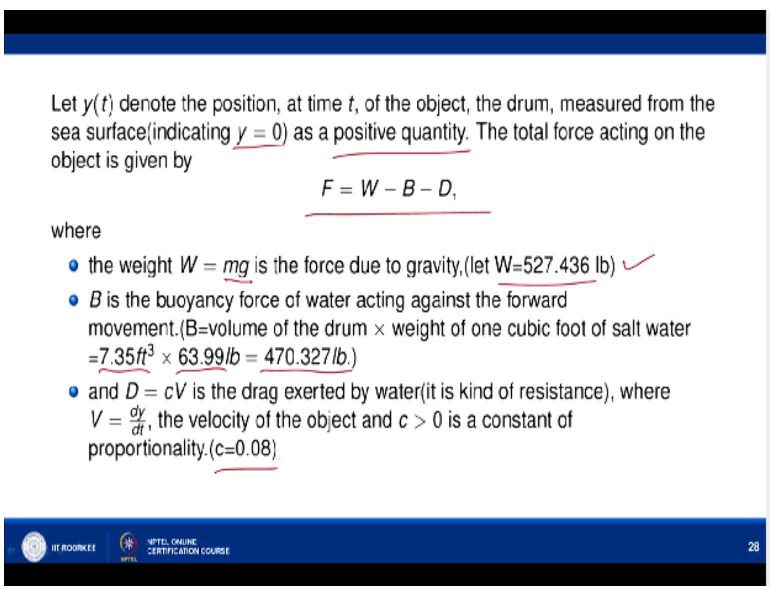

So yt denote the position at time t. So at sea level, it is y=0 and in depth, it is going to be increasing. So y is increasing in a downward. So the total force acting on the object is given by F=W-B-D. What is W, B, D? Let us try to understand. F is the force. Now W is what? W is the weight of the drum due to gravity that is mg. Here I am assuming a particular example. Here I am assuming that W is given by 527.436 pounds.

Here I am assuming a particular model. That is why I am taking a particular value for this W also. Now B is the buoyancy force of water acting against the forward movement that is, and it is given by volume of the drum*weight of the 1 cubic foot of salt water. So here let us assume that the volume of the drum is given by 7.35 cubic feet and the weight of 1 cubic foot of salt water is given in that particular time.

It is 63.99 and in this way your B is coming out to be 470.327. So basically this force is acting against the forward movement. Forward movement means it is opposing the movement of drum which is coming down. And the D which is known as the drag force and it is the force acted by water to, again opposing the movement of the drum. And it is kind of a resistance which is developed by water. So here your V is given by dy/dt where y is the position at time t and noted as velocity of the object and c is some constant of proportion.

And here I am assuming for particular thing that c=0.08. And I am taking these values from an example where it is assumed that there is a drum and if that drum is hitting the ground beyond 40 feet per second speed, then it may break down. So we are considering that by looking at this model, what should be the impact speed at the time of hitting the ground. So we form the differential equation.

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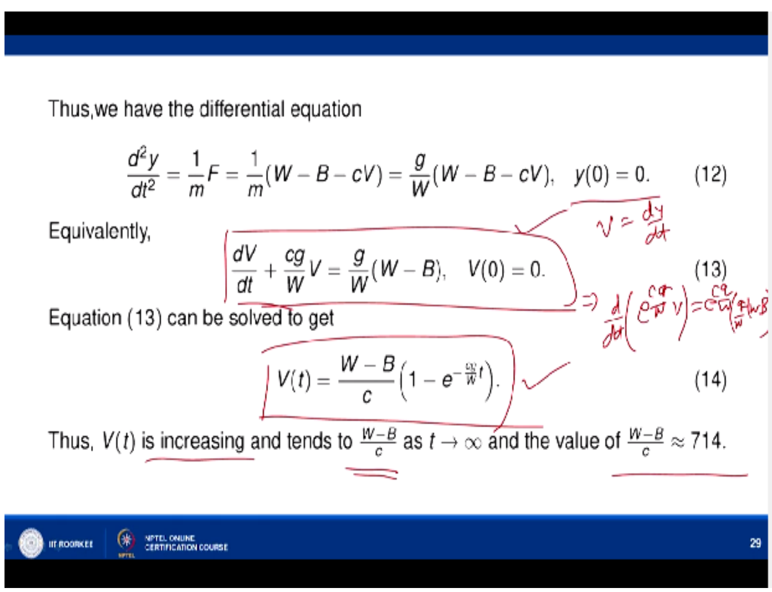

So here your F is given by m*a where a is the acceleration of the drum. So this we are writing as d2y/dt square=1/m*F where F is given by W-B-cV. So we simply write down the value of F and it is given by 1/mW-BcV which is given as g/W. Here m I am writing as W/g. So g/WW-B-cV and y0=0. So that means that at the beginning point, your drum is on the sea level.

So if we simplify this in terms of V that is dy/dt, then I can write this equation number 12 in this following initial value problem that is dV/dt+cg/WV=g/Ww-B, V0=0. So this is the simple linear differential equation in terms of velocity. And we can solve this like this. How we can solve, it is simple. You can write this as d/dt of e to the power cg/WV=e to the power cg/W*g/WW-B. And then integrate and you can write it like this.

So here, after integrating, we have the solution given in terms, this Vt=W-B/c*1-e to the power ecg/Wt. Now here, Vt is increasing function. We can say that as t is tending to infinity, this term is tending to 0, n is small and Vt is increasing. And its terminal velocity as time t tending to infinity, it reaches up to W-B/c. And we already know the value of W. We know the value of B and know the value of c and we can calculate that this value is coming out to be 714 approximately.

And it is quite high compared to 40 feet per second. So it means that they come to know that it is quite, say, high and drum has to leak down when it is hitting the ground. So they have, say, certain objection on this. They are saying that here this velocity we are getting when t is tending to infinity. So it means that here we are assuming that this drum keeps on moving for infinal many times. So it means that this may not predict the correct situation.

So they say that let us modify it a little bit. Rather than assuming that velocity is a function of time t, let us assume that velocity is a function of position.

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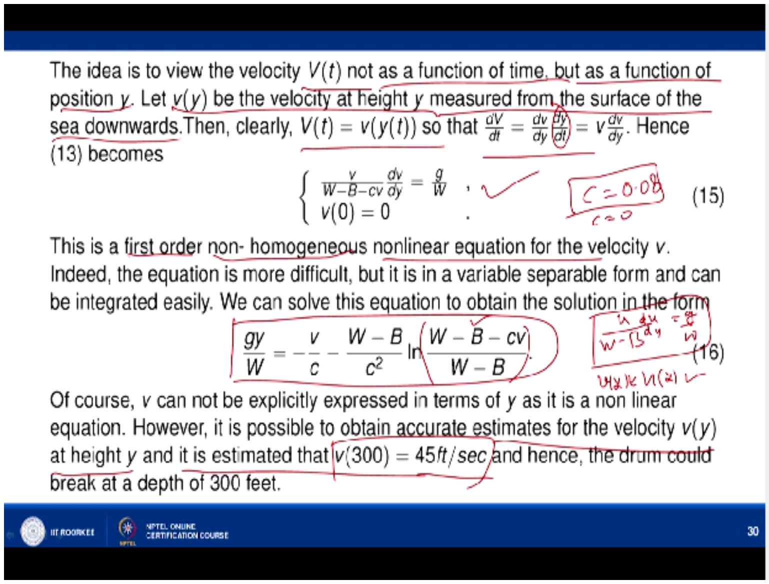

That is that velocity Vt not as a function of time, but as a function of position y. The idea is that in this case when we assume that velocity is a function of position y, we try to see that at what position, it is safe to dump your drum. So let us relook the same problem, the entire problem. Now looking at V as a function of y, rather than V as a function of t. So let Vy be the velocity at height y measured from the surface of the sea downwards.

Then clearly, Vt can be written as vyt and we can simplify, dy/dt=dv/dy dy/dt. Now dy/dt is V, so we can write dy/dt=Vdy/dy and we can rewrite our equation number 13, that is this equation. And it is, looking at v/W-B-cv dv/dy=g/W and V0=0. So now this is the initial value problem. Now rather than it is in equation number 13, it is a linear differential equation in terms of V. Now this equation number 15 is not a linear differential equation, rather than it is a first order non-homogenous nonlinear equation for the velocity V.

Now since it is a nonlinear equation, but still it can be solved because it is given in a variable separable form and you can simplify and you can integrate and you can get your solution as this, gy/W=-v/c-W-B/c square ln of W-B-cv/W-B. You can prove that this quantity is always a positive, that is why we can define the ln of W-B-cv/W-B. So but here, your y is, or B given in terms of y is an implicit form.

So it is quite difficult to express v as a function of y. However, it is possible to obtain accurate estimate for the velocity vy at height y and they estimated that if y is 300, then velocity, if drum is reached up to 300 feet, it is 45 feet per second. And hence, we can say that the drum could break at the depth of 300 feet. So it means that if it is say dumped in a place where the depth is less than 300 feet, then it may not break.

But if it is around 300 feet, then it may break. So that it may create problem. So how we can get this approximate value? So one way is to use available computers which is very accurate. But if you want to get a good approximate, you simply say that if you look at c, c is 0.08 now. So we say that if this is negligible, so if we use that c=0, then we can get the following form, udu/dy/W-B=g/W.

So here I am replacing u just to say that it is an approximate value of equation number 15. So now it can be solved very easily and you can get approximation of this 15 in this way. And you can get a good approximation of v in terms of Vy approximated by u of y. And from this, you can get a value which is near about 45.7 feet per second. So I am not going into detail of this. And we can actually prove more that here this approximation is quite a good approximation.

And this simply says that if at 300 feet, if drum is dropped, then the depth of 300, its velocity of the drum is 45 feet per second and it may break at the depth of 300 feet. So it is not safe. So we have to take some more measure to stop this. So now we will move 1 more example which is a very classical example of real world problem that is pendulum problem.

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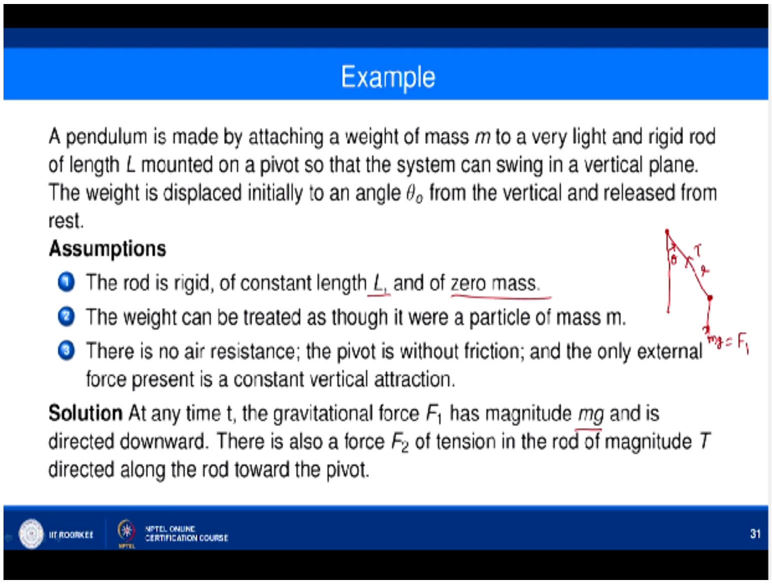

So here we have a point which is given as pivot. So a pendulum is made by attaching a weight of mass m to a very light and rigid rod of length L mounted on a pivot so that the system can swing in a vertical plane. So here we have, this is a position. Now here we say that this is a rod of length L and here we have a say weight of mass m is attached here. So downward, it is having the weight due to gravity and here it is the theta angle measured in this plane and I am looking at t is the tension here.

Now we try to look at the, we find out the say moment of this pendulum. So here we are, before considering this, let us assume some assumption. The weight is displaced initially to an angle theta 0 from the vertical and released from rest. So assumptions are following. The rod is rigid, constant length of L and is of 0 mass. Second, the weight can be treated as though it were a particle of mass m. So we simply treat this as a particle mass.

And there is no air resistance, the pivot is without friction and the only external force present is a constant vertical attraction. So at any time t, the gravitational force F1 has magnitude mg which we are writing here as this. And is directed downward. There is also a force F2 of tension in the rod of magnitude T directed along the rod towards the pivot, that is this F2 here.

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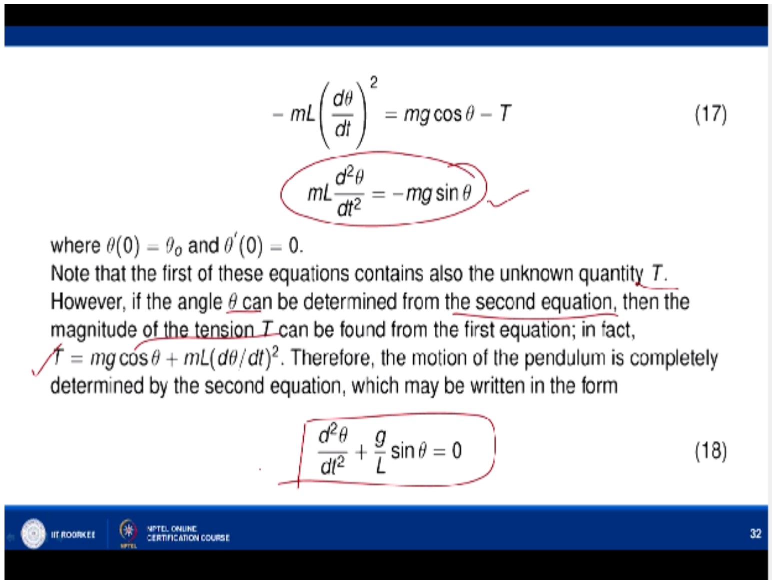

Now we can find out the governing equation and you can write governing equation as -mLd theta/dt whole square=mg cos theta-T, that you can verify here. Here you can see this is your theta. So here it is mg cos theta and here this is mg sin theta. So here we say that T-mg cos theta. So mg cos theta-T is given by -mL d theta/dt whole square, that is if we look at this direction. This direction if you look at, then here we have T-mg cos theta=md theta/dt whole square.

And next is mLd2 theta/dt square=-mg sin theta, where theta at 0, initial point, it is given as theta 0 and theta dash=0. And if you look at the last equation which will govern the motion of this pendulum. So note that the first of these equation contains also the unknown quantity T. However, if the angle theta can be determined from the second equation, then the magnitude of the tension T can be found.

And in fact, it is given by this. Therefore, the motion of the pendulum is completely determined by the second equation which may be written as d2 theta/dt square=g/L sin theta=0. And if you look at, this is very old pendulum problem and it is very difficult to solve this nonlinear pendulum problem. So it is in the assumption when there is a no say resistance is done due to air.

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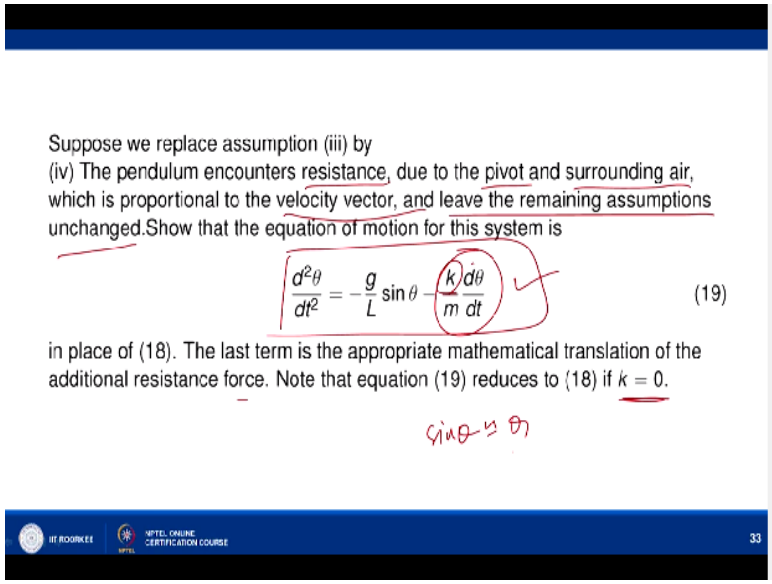

But if we assume that the pendulum encounters some kind of resistance due to the pivot or surrounding air, then this resistance will be proportional to the velocity vector and leave the remaining assumption as it is and we can write down our equation as d2 theta/dt square==g/L sin theta-this k/m d theta/dt. Here, this k is the resistance constant which is acting on the motion of the pendulum and d theta/dt is the angular velocity which we are considering.

So here equation number 19 represents the motion of the pendulum d2 theta/dt square=-g/L sin theta-k/m d theta/dt. The last term is the appropriate mathematical translation of the additional resistance force and this equation reduced to the previous equation if k=0. And believe me, it is quite difficult problem and in this present form, we cannot solve. And to solve this, we assume certain condition that sin theta is the deviation from the vertical position is very small.

So sin theta is approximated by theta and then we can solve this problem 18 or 19. I am not going in to detail of this, solution of 18 and 19 because it is quite classical problem and we have already solved it many times in our previous classes. So with this, I end my lecture. So in this lecture we have discussed certain physical system and we reformulated the physical system in terms of differential equation. And discussed few things about these systems. So we will continue in next lecture. Thank you very much for listening. Thank you.