Dynamical Systems and Control
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Lecture - 19
Stability of Weakly Non Linear Systems - II

Hello friends. Welcome to this lecture. In this lecture, we will continue our study of stability of nonlinear system. So if you recall in previous class, we are discussing the following system of differential equation that is x dash=Ax+gx where x is n x 1 vector and is n x n matrix and g is the n x 1 matrix function.

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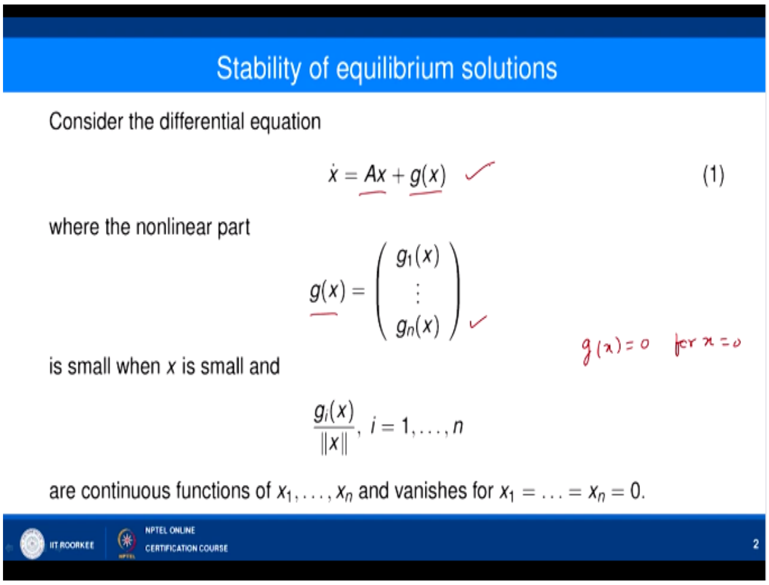

It is given as gx=g1 x to gn x and here we are assuming that the nonlinearity given by the function g of x say is small compared to the linear part that is A of x. So here we have assumed that gx is a function such that gx/norm of x is (()) (01:20) gi x/norm of x are continuous function of x1 to xn and vanishes for x1 to xn=0. So here by putting this condition that g of x=0 for x=0 simply says that x=0 is an equilibrium solution of the system 1.

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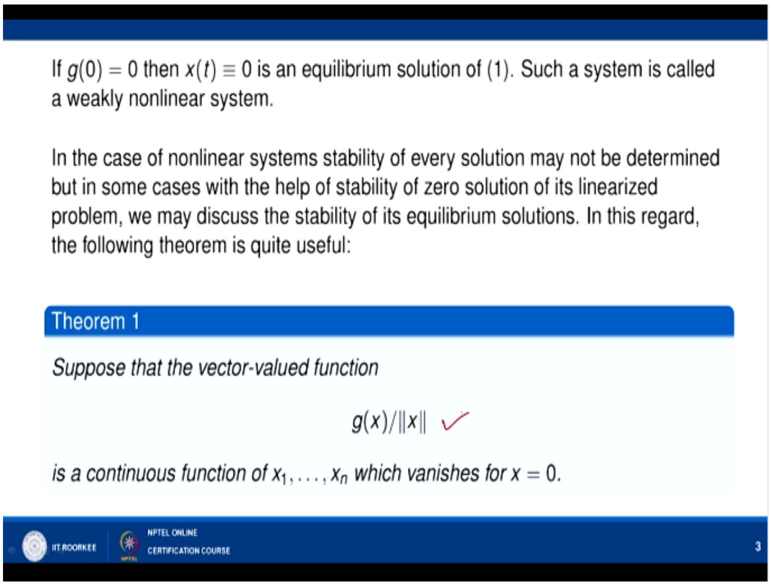

And here and we call this kind of a system as weakly nonlinear system and we have discussed, we have started proving the theorem number 1 which says that suppose the vector-valued function gx/norm of x is a continuous function of x1 to xn which vanishes for x=0.

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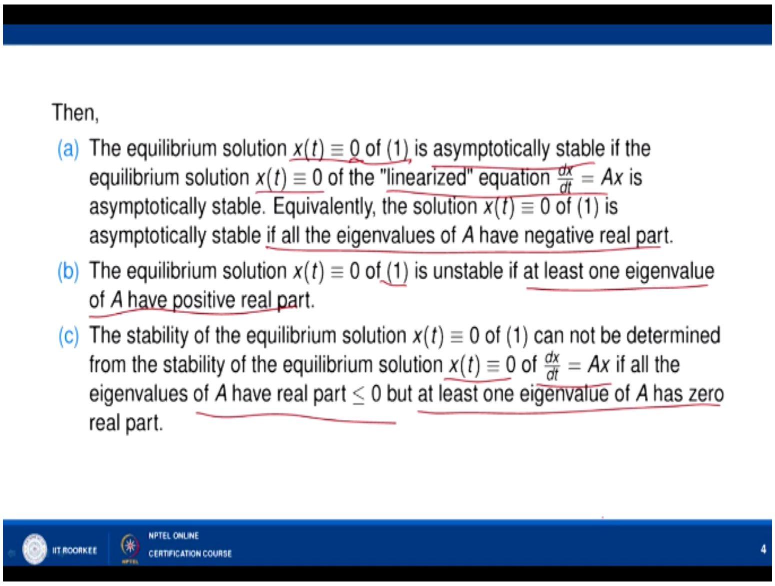

Then, the part a that the equilibrium solution xt identically=0 of 1 is asymptotically stable if the equilibrium solution of xt identically=0 of the linearized equation that is dx/dt=Ax is asymptotically stable and we have already seen that the system of linear equation dx/dt=Ax every solution of this will be asymptotically stable provided that all the eigenvalues of A have negative real part.

So it means we can say that we summarizes these two fact then we can say that the equilibrium solution xt identically=0 of 1 is asymptotically stable if all the eigenvalues of A have negative real part. So and the part b says that the equilibrium solution xt identically=0 of 1 is unstable if at least one eigenvalue of A have positive real part, so this is the b part.

And c part is that the stability of the equilibrium solution xt identically=0 of 1 cannot be determined from the stability of the equilibrium solution xt=0 of dx/dt if all the eigenvalues of A have real part<=0 but at least one eigenvalue of A has zero real part and in last lecture we have discussed say two examples based on the part c.

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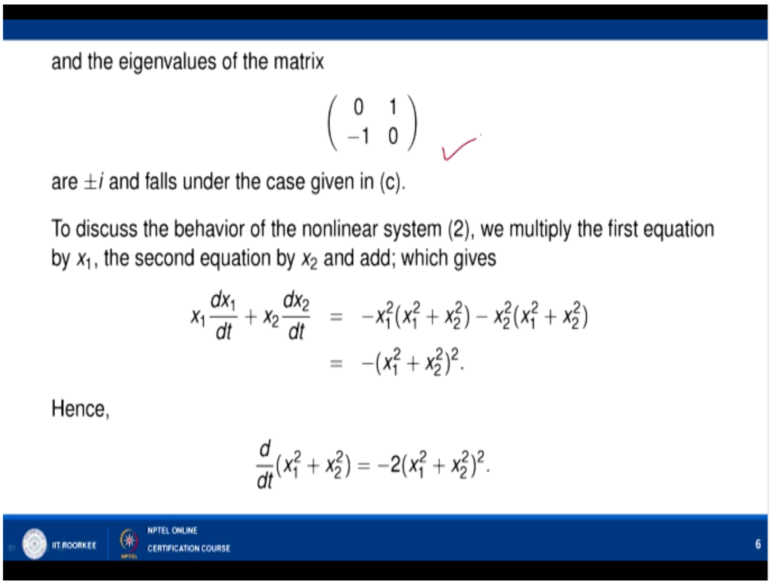

In fact, we started with part c and we have shown that in this part when dx1/dt=x2-x1*x1 square+x2 square, dx2/dt=-x1 –x2 x1 square+x2 square. In this case, your coefficient matrix of the linearized system has zero real part.

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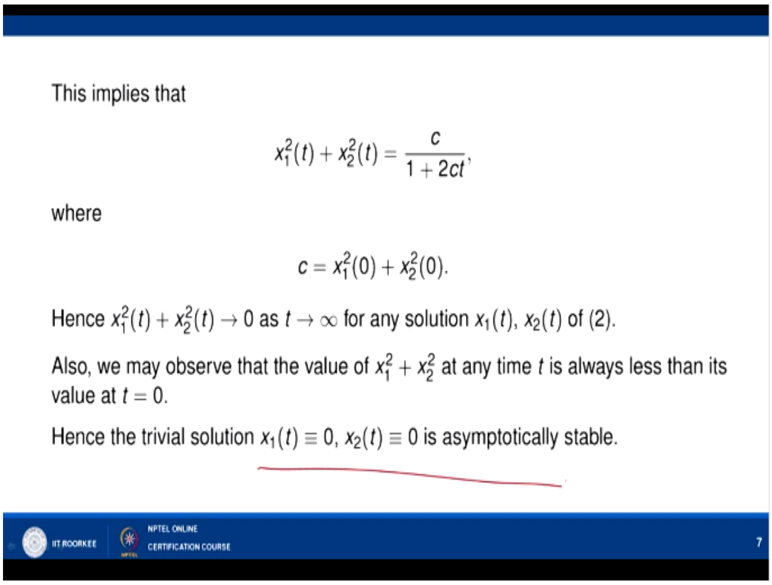

And here we have seen that the solution is asymptotically stable.

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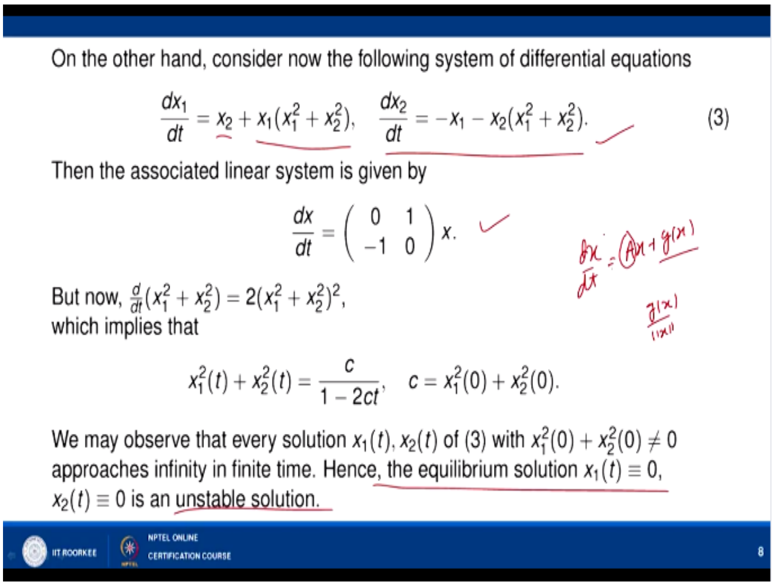

But if we just perturb our system of differential equation like this that dx1/dt=x2+x1*x1 square+x2 square and dx2/dt=-x1 –x2 x1 square+x2 square. If you look carefully, then the associated linear system is same and the eigenvalues of A is having zero real part and in previous question we have shown that the zero solution is asymptotically stable solution but in this case your zero solution, the equilibrium solution x1 t=0 and x2 t=0 is an unstable solution.

So it means that if you are discussing dx/dt=Ax+g of x where x satisfying the condition that g0=0 and gx/norm of x is a continuous function of x and it vanishes as x=0. Then, we cannot conclude anything if eigenvalues of A, all the eigenvalues of A are nonpositive and at least one of the eigenvalues has zero real part. So in that case, we cannot conclude anything. Now let us prove the result for the first two cases.

That is that eigenvalues of A are having negative real part for all the eigenvalues and if at least one of the eigenvalues have positive real part. So let us prove in the case of 1 and 2.

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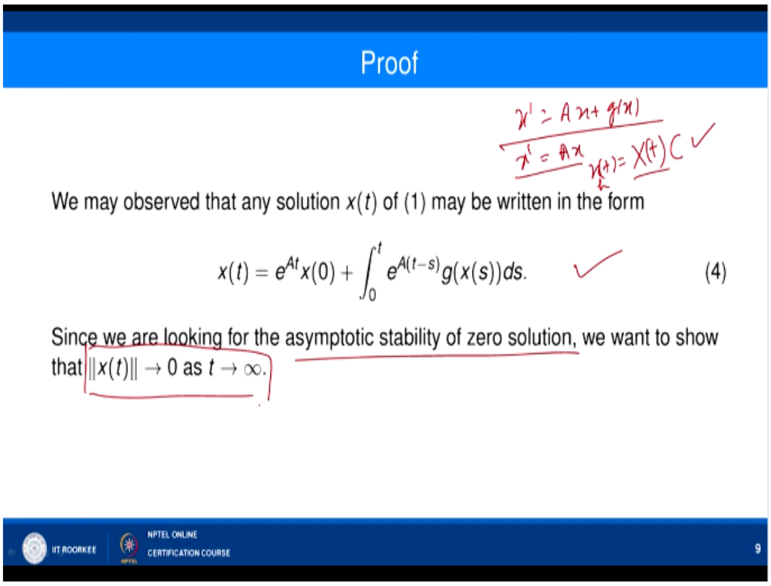

So we may observe that any solution xt of 1, here 1 is this x dash=Ax+g of x, here we can say that this any solution of this maybe written in the following form that xt=e to power At x of 0+0 to t e to power A t-s g of x s d of s and here we have used the variation of parameter method. In fact, we know that we can write down the solution of x dash=Ax as say this is the fundamental matrix xt*that constant function c.

So your xt homogenous part can be written as xt*c and then by varying the c we can find out the solution of this and this is I think we have already discussed in one of the lecture. So we can say that here taking this fundamental matrix solution xt as e to power At we can write down the solution of x dash=Ax+gx as follows that is it is xt=e to power At*x at 0+0t e to power A t-s g of x s ds.

And here since we are looking for the asymptotic stability of zero solution, it means that we are looking at that this zero solution is a stable solution and any other solution which start in the neighborhood of zero solution will tend to zero as t tending to infinity. So this we can prove provided that norm of xt is tending to 0 as t tending to infinity can be proved. So our focus is to prove that norm of xt is tending to 0 as t tending to infinity okay.

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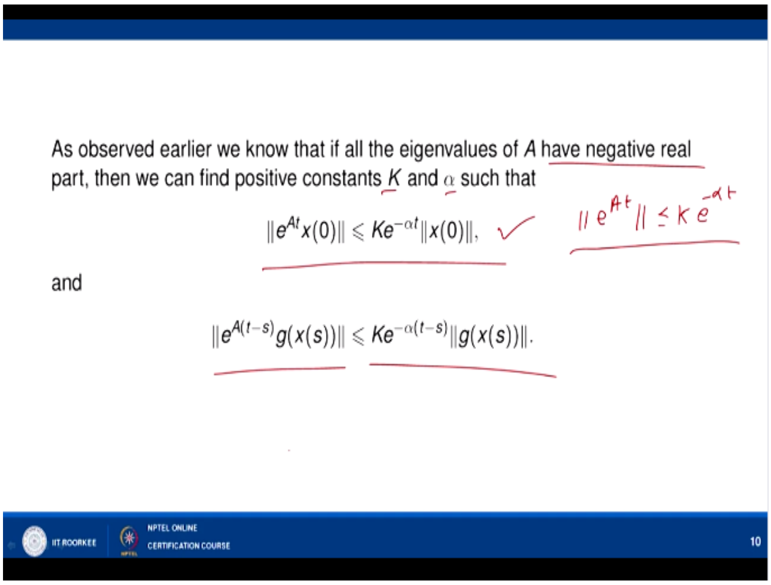

Now as we have observed in the case that if all the eigenvalues of A have negative real part, then we can find out a constant k and alpha in a way such that this e to power norm of e to power At is bounded by Ke to power –alpha t, so that we have already discussed in previous few lectures. Now here using this let us find out the upper bound of norm of e to power At*x of 0 which is nothing but<=norm of e to power At*norm of x0.

Norm of e to power At is bounded by Ke to power –alpha t. So we can say that norm of e to power At*x0 is<=Ke to power –alpha t norm of x0. Now once we have this now let us look at because here if we have two terms, one term is here, another term is inside. So this can be bounded by Ke to power –alpha t norm of x0. Now look at the integrant basically e to power A t-s g of x s.

Here in a similar way we can say that norm of e to power A t-s g of x s is<=Ke to power –alpha t-s norm of g of x s. So here we have shown we have found the upper bound here. Now still it is not sufficient because it is quite difficult to say simplify this expression 4.

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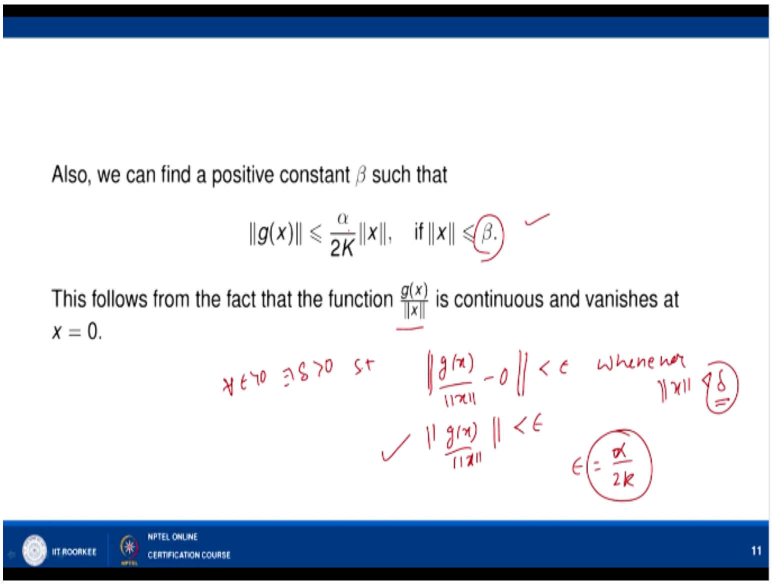

So here we also use the following thing that we can find a positive constant beta such that norm of gx is<=some constant*norm of x and if norm of x is<beta and this follows from the fact that this gx/norm of x is continuous and vanishes at x=0. So how we can obtain this if you look at the continuity of gx/norm of x. So we can simply say that norm of g of x/norm of x-0 because it is vanishing at 0 is<epsilon whenever, so continuity says that for every epsilon>0 there exists a delta>0.

Such that this quantity is<epsilon whenever norm of x is<delta and this is what this I can write it gx/norm of x is<epsilon. So now for this particular problem let us choose epsilon as alpha/2k. We will see that why this alpha/2k I have taken. You can take any constant, later on we can fix that. Now this delta let us say call it beta here.

So we say that since gx/norm of x is a continuous function and continuous function for norm of x and continuous function of x and it is tending to 0 as x is tending to 0. So with the help of this, we can say that norm of gx is<=alpha/2k*norm of x provided that norm of x is<=beta here. So corresponding to alpha/2k we can find out delta and let us call that delta as beta.

So it means that using continuity, we can write that norm of gx is<=alpha/2k*norm of x if norm of x is<=some beta. Now that beta is depending on this alpha/2k here.

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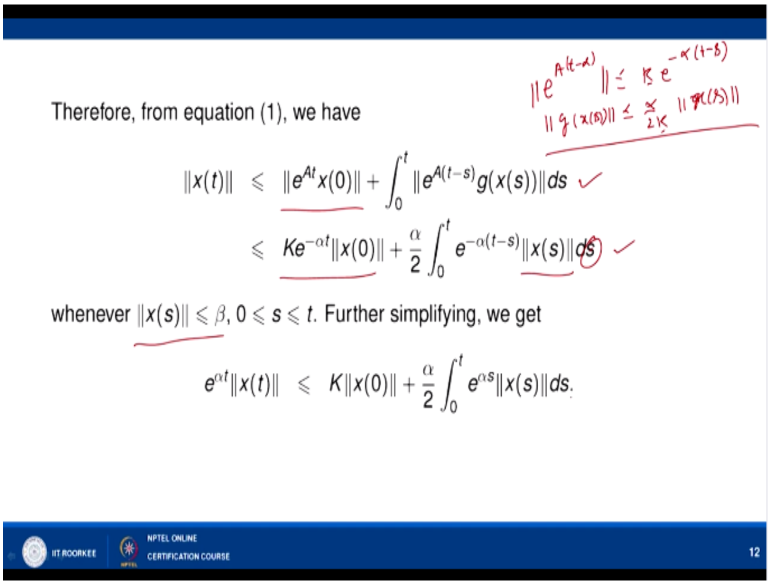

Now once you have all these things then we can look at our solution that is norm of xt<=norm of e to power At x0+0 to t norm of e to power A t-s g of x s ds. Now this quantity is<Ke to power –alpha t*norm of x0+0 to t. Now this is<=e to power –alpha t-s. Now this norm of g of x of s is<alpha/2*norm of x. Now that one K like here it is e to power A t-s is bounded by Ke to power –alpha t-s and norm of g of x of s is bounded by alpha/2k norm of x of s.

So this K will be canceled out here and you will get alpha/2 0 to t e to power –alpha t-s norm of x s d of s. Now here this inequality will be true provided that norm of x is<=beta where s is lying between 0 to t here because s is lying between 0 to t here. So if you further simplify in fact we can multiply on both the side by e to power alpha t and then we can have e to power alpha t norm of xt is<=K times norm of x0+alpha/2 0 to t e to power alpha s norm of x s ds here.

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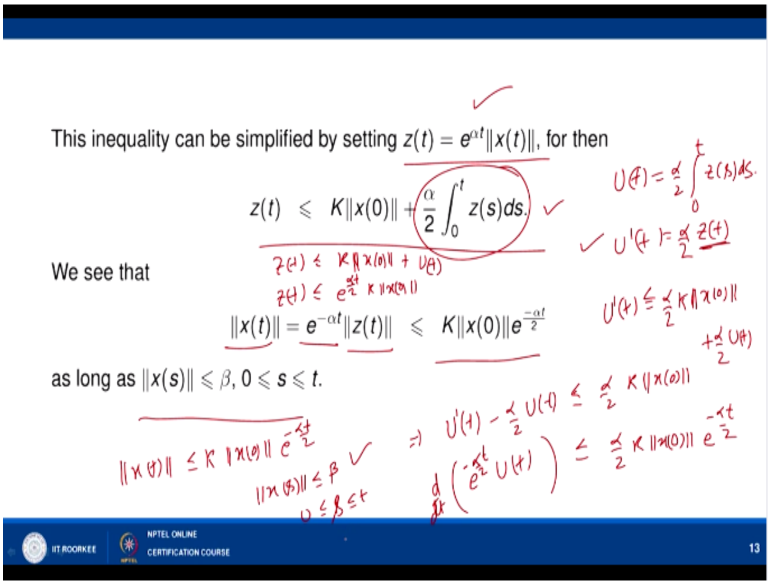

So this we can further simplify by taking zt=e to power alpha t*norm of xt and then we can write it zt is<=K times norm of x0+alpha/2 0 to t z of s ds here. So here we want to find out the bound on z of t here. So here we have two ways, either you use directly the Gronwall's inequality if you know, otherwise we can solve this by taking let us say ut as say alpha/2 0 to t z of s d of s here.

So the only thing is that we cannot solve for zt by just differentiating it because differentiation of the inequality may not be preserved here. So to solve this, we let us assume that ut=alpha/2 0 to t z of s d of s. Then, we can write down u dash t is=alpha/2 here and it is z of t here. So we can say that we already know that zt is<=this quantity, so we can write u dash t=alpha/2 zt is<this quantity, so it is K of norm of x0 and +alpha/2.

Now this quantity is what, this quantity is your ut, so let me write it here. So this is what u dash t-alpha/2 u of t is I think this is<=this is<=so this is<=alpha/2 k times norm of x of 0 here. So what we have done here, we just assumed this quantity as u of t and then we have evaluated the derivative of this that this u dash t=alpha/2 zt. Now zt we know that the upper bound of zt is what, zt is bounded by K times norm of x of 0+this is ut here.

So now using the bound of zt I can write down u dash t is<=alpha/2 K times norm of x0+alpha/2 u of t. So we can write down u dash t-alpha/2 ut is<=alpha/2 K times norm of x0. Now here this we can solve in terms of ut here. For that we simply multiply by e to power alpha t/2 here. If you multiply here, then I can write this as d/dt of this is what let us say –sign here, then this will give you what e to power –alpha t/2 ut.

If you differentiate this what you will get, this ut and differentiation of this will be what, e to power –alpha t/2*-alpha/2 and if you take this as e to power –alpha t and ut now this is<=alpha/2 K times norm of x0*e to power –alpha t/2. So what we have done we multiplied by e to power –alpha t/2 then this left hand side is reduced to d/dt of e to power –alpha t/2 ut and right hand side we have written alpha/2 K times norm of x0 e to power –alpha t/2 here.

Now let us so we are solving this d/dt of e to power –alpha t/2 u of t which is<=alpha/2 K times norm of x0 e to power –alpha t/2 and this we can solve as follows.

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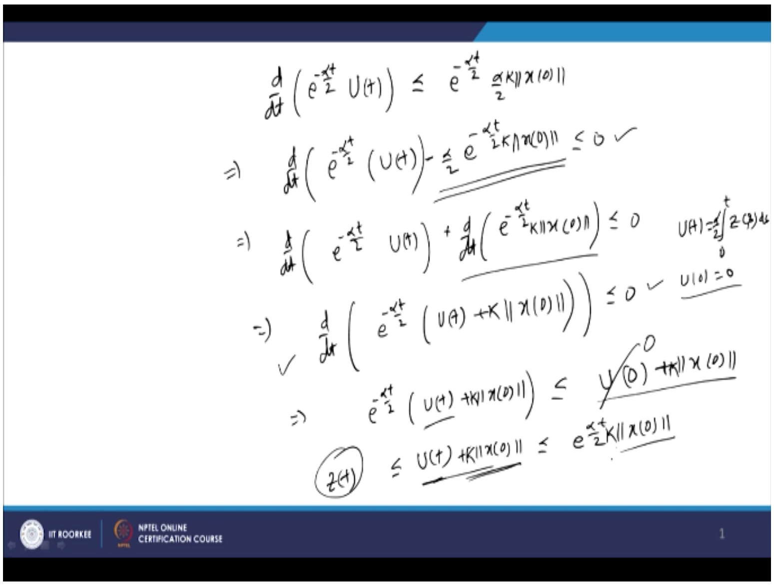

We can write down d/dt of e to power –alpha t/2 u of t<=e to power –alpha t/2 alpha/2 K times norm of x0 here. Now this we can take it in left hand side and we can write d/dt e to power –alpha t/2 ut-alpha/2 e to power –alpha t/2*K times norm of x0 is<=0 here. Now this part I can write as d/dt of e to power –alpha t/2 K times norm of x0. So using this we can write down this as d/dt e to power –alpha t/2 ut+K times norm of x0<=0 here.

Now we simply integrate with respect to t from 0 to t, so we have e to power –alpha t/2 ut+K times norm of x0<=putting the value 0 here. Then, e to power –alpha 0/2 is simply 1 that is u0+K times norm of x0. Now what is u0 here, so u0 you can simply find out using this that ut=alpha/2 0 to t zs ds. Now if you put t=0 here, then this integral will vanish, this whole thing will vanish and you can get the value of u0=0 here.

So we can get that e to power –alpha t/2 ut+K of x0 is given by this is simply 0 K times norm of x0. So here we can get the value of ut that ut+K times norm of x0 is<=e to power alpha t/2 K times norm of x0 and if you look at this is the bound of zt. So zt is bounded by e to power alpha t/2 K times norm of x0. So that is what we have obtained from zt here, that zt is bounded by e to power alpha t/2 K times norm of x0.

So we can say that from this inequality if we solve we can get that zt is<=e to power alpha t/2 K times norm of x0 here. So that is what we have achieved here e to power alpha t/2 K times norm of x0. Now what is xt here, so from this we can find out xt is e to power –alpha t*zt here. So norm of xt is=e to power –alpha t*norm of zt and that we can write it here that K times norm of x0 e to power –alpha t/2 you will get provided that norm of xs is<=beta but s is lying between 0 to t. So we can say that from this we can write it here.

That if so what we have achieved here let me write it here that norm of xt is<=K times norm of x0 e to power –alpha t/2 provided that norm of xs is<=beta where s is lying between 0 to t. Now here now our claim is that this will be automatically true if we assume that x of 0 is<some quantity. Now here we can say that e to power –alpha t/2 is basically<1, so we can say that norm of xt is<=K times norm of x0.

So if we choose our x0 suitably then your x of s will be<beta for s lying between 0 to t here.

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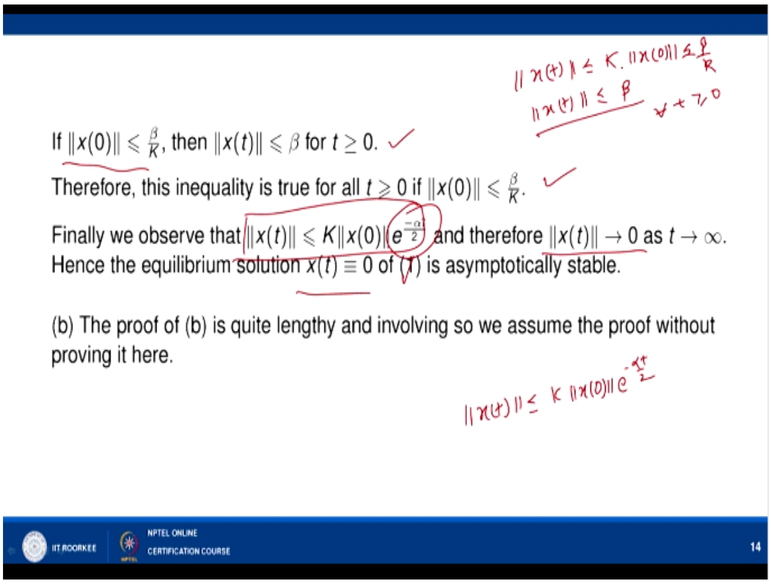

So it means that if I choose that norm of x0 is<beta/K, then we can say that norm of xt is<=here we have written it is K times x of 0, so it is K times norm of x of 0 here. So we simply say that if x0 is<beta/K then you can say that norm of xt is<=beta here beta times norm of yeah beta. So if norm of x0 is<beta/K, then your norm of xt is<=beta for all t>=0 here, so it means that the previous this inequality is always true for all t>=0 here.

So it means that the restriction that norm of x is<=beta can be realized can be met if we assume that norm of x0 is<=beta/K because in this K norm of xt is always<=beta for t>=0. So this condition is now met automatically that norm of x is<=beta. So this if you say that norm of x0 is<=beta/K then norm of x of t is<=K times norm of x0*e to power K times norm of x0 e to power-alpha t/2.

So here this inequality is always true for all t>=0 if norm of x0 is<beta/K. So finally we observed that norm of xt is<=K times norm of x0 e to power –alpha t/2 and then we can say that norm of xt is tending to 0 as t tending to infinity and hence we can say that the equilibrium solution xt=0 is asymptotically stable solution. So this says that it is stable solution.

And then because of e to power –alpha t/2 term is here as t tending to infinity this term will vanish to will tend to 0 very fast and we can say that xt will tend to 0 as t tending to infinity and hence we can say that xt identically=0 is an asymptotically stable solution. So what we have proved that if eigenvalues of A, if all the eigenvalues of A have negative real part then the solution zero solution of x dash=Ax+gx is asymptotically stable solution.

Now here regarding the proof of b, this proof is quite lengthy and involving. So here we can assume the proof without proving it here and so it means in this way we can say that we have done, we have considered all the cases, case a, case b and case c here. So now with this, we have completed the theorem. Now let us consider the application here. So in this proof, we have just discussed the stability of equilibrium solution which is identically=0.

Now how we can utilize this theorem to discuss the stability of nonzero equilibrium solution. That we are going to discuss in next few slides. So now we say that this theorem 1 is also useful in determining the stability of equilibrium solution of arbitrary autonomous differential equation.

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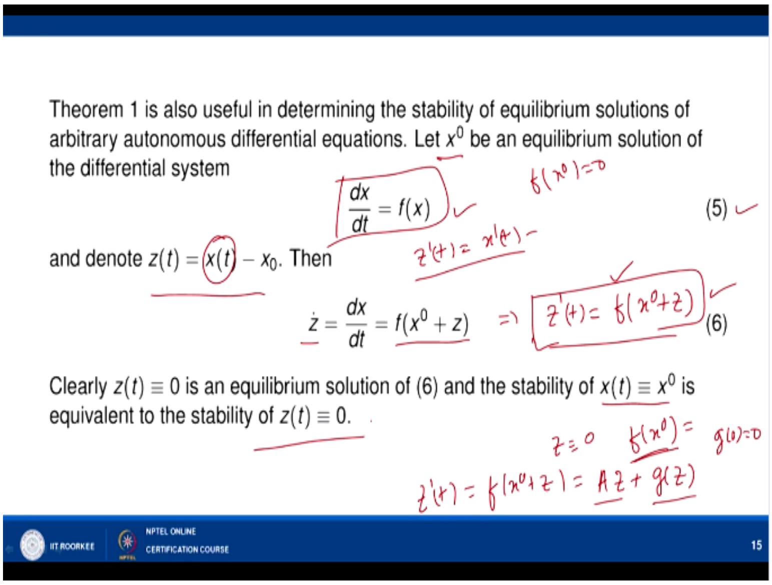

So let x0 be an equilibrium solution of the differential equation dx/dt=f of x. Now this x0 may not be a 0, if it is a 0 we have the result we can get it but if x0 is nonzero let us see how we can say utilize our theorem 1 to discuss the stability of equilibrium solution here. So now we simply shift our origin and we simply write zt as xt-x0 here. So using this now I can write z dash t is=x dash t- since x0 is equilibrium solution, so x0 dash is simply 0.

So z dash t=x dash t. Now z dash t=we can simply write dx/dt and dx/dt is already given as f of x. Now x can be replaced as zt+x0, so we can say that now we can have z dash t=f of x0+z here. Now what is the change in equation number 5 and this equation number 6 here, that here your equilibrium solution of 5 is x0, it means that f of x0 is=0 here but if you look at the system 6 here if you say that z=0 is an equilibrium solution here because when you put z identically=0 then it is nothing but f of x0 and we know that it is 0.

So it means that we are able to convert a problem where we have a nonzero equilibrium solution to a problem where we have 0 as an equilibrium solution here. So this is now converted into z dash t=f of x0+z. Now we can say that here this can be converted into a form which you have discussed earlier that is z dash t, we want that if we impose certain condition on f here, then this f of x0+z may be written as A of z+some g of z where Az is a linear part in z and gz is something that g of 0=0.

And gz/z is a continuous function of 0 and vanishes at z=0. So we need to find out a condition on f such that we may write it like this. So here let me write it here that clearly zt=0 is an equilibrium solution of 6 and the stability of xt identically=x0 is equivalent to the stability of zt identically=0 here.

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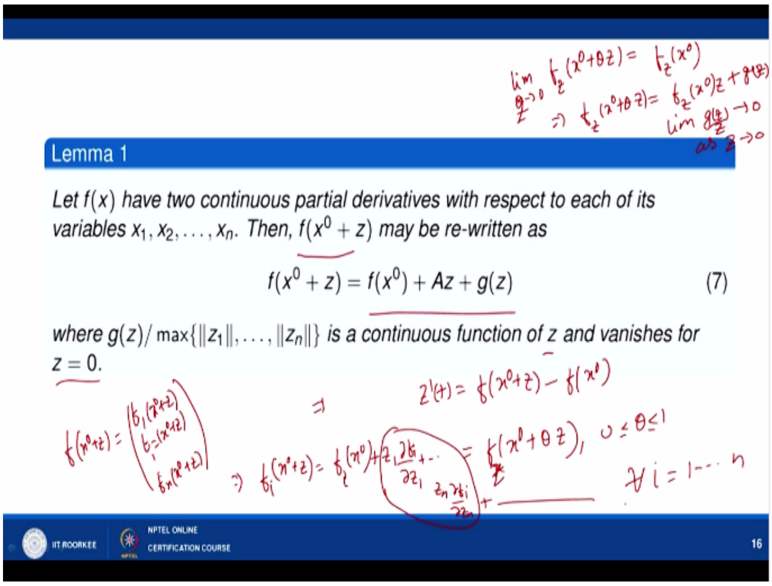

So here we have the following lemma by which we can say that we can convert, we can write down the expression f of x0+z as f of x0+Az+gz here where gz satisfy the property mentioned in the previous theorem that is gz/norm of z is a continuous function of z and vanishes for z=0 here. So the condition we are putting on f is the following that let fx have two continuous partial derivatives with respect to each of its variable x1 to xn.

Then, f of x0+z may be rewritten as f of x0+z=f of x0+Az+g of z here. Now I am giving you these are the sufficient condition we impose on f here. Here we may consider only that this thing that first-order partial derivative exist continuous and second-order partial derivative simply exists, then also we can write down f of x0+z=f of x0+Az+gz here. The following lemma this lemma has several proof.

We are just using one result we simply here we have written z dash t=f of x0+z here. Now we already know that this f of x0 is=0 here. So I can write this as f of x0 here right. Now since f of x0+z-f of x0. Then, using your mean value theorem I can write this as f of x0+some theta z here where theta is lying between 0 and 1 here and so here now so this can be written as z dash t=f of x0+theta z.

Now here we are assuming sorry it is with respect to the derivative with respect to x here. Now sorry it is a function of z. So fz x0+theta z it is given here. Now we have assumed that this has continuous partial derivative, so it means that this implies that let limit theta tending to 0, your fz x0+theta z is nothing but z of x0 here, sorry it is z tending to 0. So limit z tending to 0 fz x0+theta z is=fz x0 or I can write here fz x0+theta z=fz x0*z+some gz here.

Now this implies that gz is tending to 0. So this implies that limit gz is tending to 0 as gz/z is tending to 0 as z is tending to 0 here. So this is one way to look at the say proof of this. Another way to write down the proof is using Taylor's theorem. Then, since f is what, f is f x0+z is nothing but we can write it f1 say x0+z f2 x0+z here and fn x0+z here. Now for each fi we can write down say fi x0+z as your fi x0+now here we can write down z1 dou fi/dou z1+1 zn dou fi/dou zn+the high order term.

Now this is nothing but okay and similarly we can write down for each i, i=1 to n here. So it means that I can write down let me write it here.

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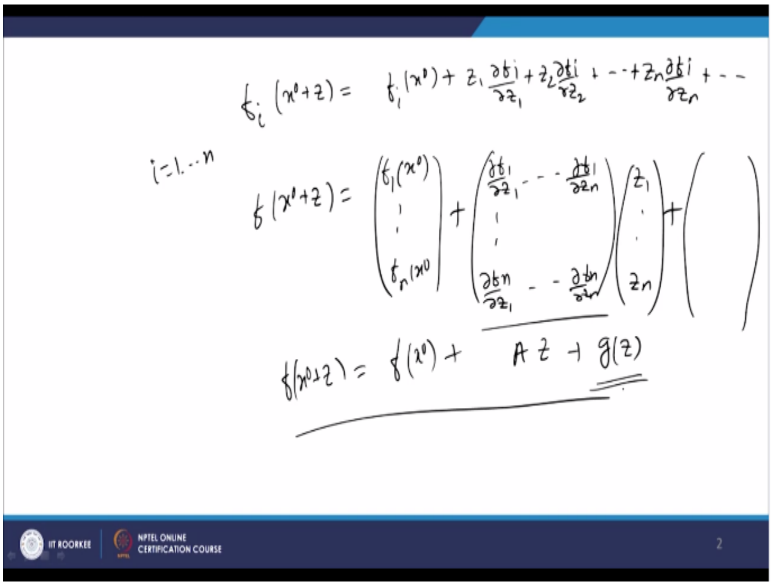

That for each i fi x0+z is given as fi x0+your z1 dou fi/dou z1+z2 dou fi/dou z2 and so on zn dou fi/dou zn+high order term. So this I can write it for each i, i=1 to say n. So we can simply say that in particular I can write f of x0+z as this is f1 x0 to say fn x0+now this let me write it here. This is what dou f1/dou z1 to dou f1/dou zn and to dou fn/dou z1 to dou fn/dou zn*z1 to zn here+high order term.

Now high order term means involving the say squared terms of z1, z2 and zn's and multiple of this. So it means that here your gz will be containing say polynomials in terms of zi's of degree more than 2 here. So this means that this is your f of x0 here, this we denote as A and this is z+whatever left is your gz here. Now gz contains say polynomial terms in terms of z1 to zn of degree 2 or more here.

So this is what we have written here x0+z I can write it like this. So that is what is given here that if fx have two second-order continuous special derivative with respect to each of its variable x1 to xn then f of x0+z may be written as f of x0+z=f of x0+Az+gz here. So here we are not assuming anything on any condition on x0 but if we assume that f of x0=0 then we can do it by same mean value theorem also and the continuity properties here.

But here in general if fx contains the, fx has the second order continuous partial derivative then we can rewrite f of x0+z as f of x0+Az+gz here. Now since f of x0 is 0 then I can write this the previous problem z dash t=f x0+z dash Az+gz where gz contains the term in terms of z1 to zn with power 2 or higher. So here we can rewrite we can apply our theorem 1 for general nonlinear system also where we want to check the stability of an equilibrium solution x0 here.

So with this we finish our lecture here and next lecture we will discuss some example based on this observation and discuss the stability of an arbitrary equilibrium solution and that we discuss in next lecture. Thank you very much for listening us. Thank you.