Dynamical Systems and Control
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Lecture - 18
Stability of Weakly Non Linear Systems - I

Hello friends. Welcome to this lecture and this lecture we continue our study of stability of a given solution. So if you recall in previous few lectures, we have discussed this x dash=A of x here.

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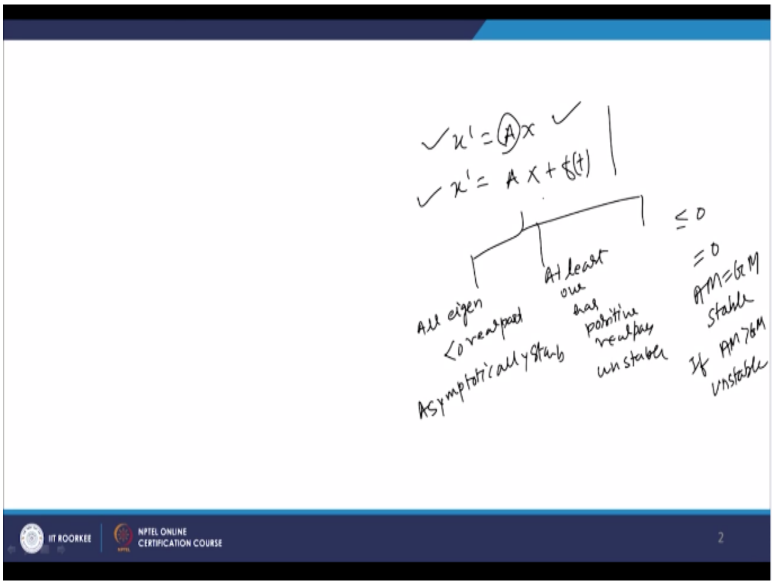

So it means that the stability of any solution will depend on the behaviour or the eigenvalues of the coefficient matrix A and then at the end we have discussed that the stability of x dash=Ax+f of t is equivalent to the stability of zero solution of x dash=A of x. So I can write down in 3 parts. So if all the eigenvalues, all eigenvalues have negative real part, then it is asymptotically stable.

And if at least one has positive real part, then unstable and if say all the eigenvalues have<=zero real part and corresponding to zero real part if we have Am=Gm then it is stable. If Am is strictly>Gm then unstable. This we have discussed in previous lecture. Now let us move little bit further and now we consider the nonlinear system of differential equations.

If you consider both are linear differential equation, this is a linear differential equation and this is a nonhomogeneous linear system of differential equation.

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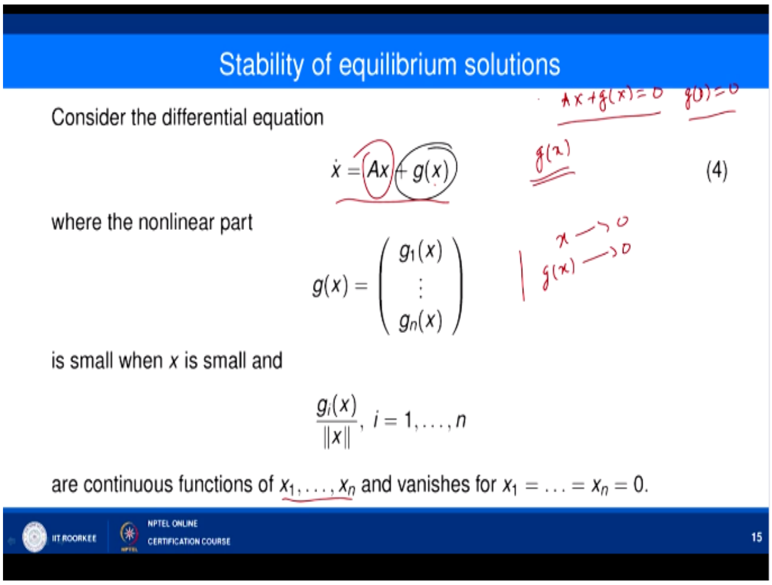

Now let us consider say the differential equation x dash=Ax+g of x where the nonlinear part g of x=g1 x to gn x is small when x is small and we are assuming that gi x/norm of x where i is from 1 to n are continuous function of x1 to xn and vanishes for x1 to xn=0. So if you look at what is the difference between this and the previous one, the difference is this part g of x. Now here we assume that let us assume that gx is weak in the sense of corresponding to linear part.

It means that here we are trying to say that the linear part is dominated over the nonlinear part. So here we simply assume that if x is small, then gx is also small. So it means that we are trying to show that if norm of x, x is tending to 0 if you want to show that gx is tending to 0, then we simply say that the neighborhood of x=0 this linear part that is A of x is dominating over this nonlinear part g of x.

So for that this is the philosophy of this result which we are going to stabilize. Now here how we can say that that as x tending to 0 gx is also tending to 0 and for this we are assuming that gi x/norm of x are continuous function of x1 to xn. So here we are saying that continuous function of x1 to xn means if these are tending to 0 then gx/norm of x is also tending to 0 and it vanishes when x1 to xn's are all 0 okay.

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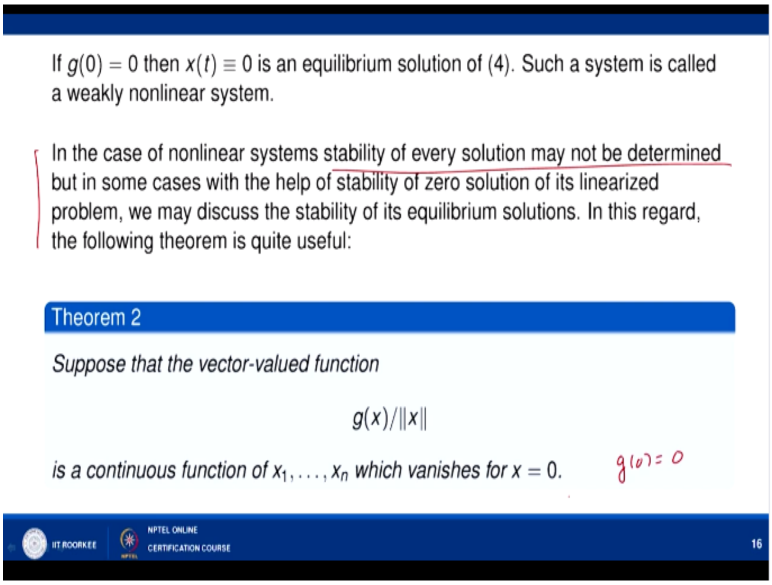

Then, here we also assume one more thing that is g of 0=0. Then, if g of 0 is=0 then we can simply say that xt=0 is an equilibrium solution of 4. In fact, if you look at here, zero equilibrium solution we can obtain by putting Ax+g of x=0. So it may have say several equilibrium solutions but we know that if g of 0=0, then at least one equilibrium solution is a zero solution.

So it means that if g0=0, then zero solution is one of the equilibrium solution of x dash=Ax+g of x. The key point here is to note down that in case of nonlinear system, stability of every solution may not be determined, that is the difficult part we can consider that. In case of linear system, we can determine the stability of every solution but in case of nonlinear system we cannot check the stability of every solution but rather than you can check the stability of an equilibrium solution.

So here rather than checking the stability of any equilibrium solution, we are checking the stability of zero solution. The idea is that if we have some nonzero equilibrium solution, then we can just shift that equilibrium point near to zero and we can say that without loss of generality it is sufficient to find out the stability of equilibrium solution xt identically=0.

We will see that how the stability of some other equilibrium solution can be determined with the help of the stability of zero solution or equilibrium solution that is zero solution. So first thing we are looking here the stability of equilibrium solution xt identically=0 here. So here we observe this thing. This is very important thing that in the case of nonlinear system, stability of every solution may not be determined.

But in some cases with the help of stability of zero solution of its linear problem associated linear problem, we may discuss the stability of its equilibrium solution. In this regard, the following theorem is quite useful and this is the theorem which we are talking here. So here first we are assuming that the vector-valued function gx/norm of x is a continuous function of x1 to xn and which vanishes for x=0. And also we are assuming that g0=0, so that I can assume that zero is an equilibrium solution of this.

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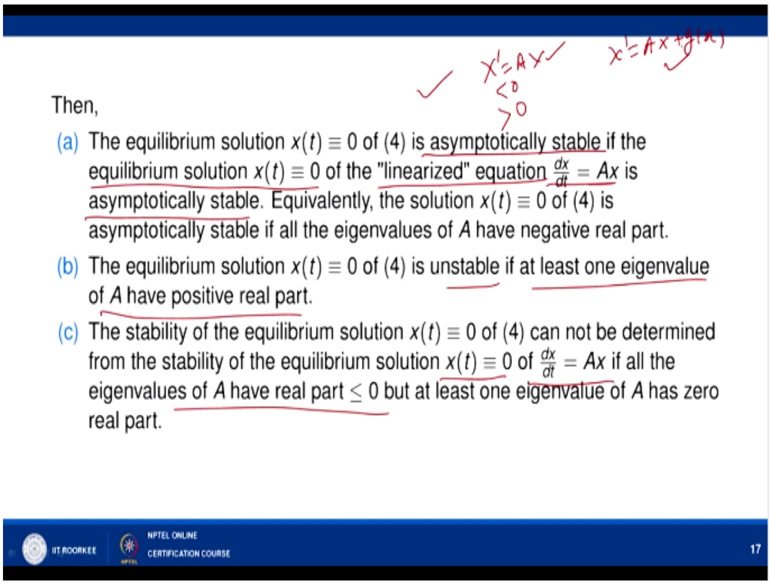

Then, we have 3 parts, first part says that the equilibrium solution xt identically=0 of 4 is asymptotically stable if equilibrium solution xt identically=0 of the linearized equation that is dx/dt=Ax is asymptotically stable. So it means that zero solution of this system 4 x dash=Ax+g of x is asymptotically stable if the zero solution of the corresponding linear part that is x dash=Ax is asymptotically stable.

And if you want to look at this statement in terms of eigenvalues, then we can say that the zero solution of x dash=Ax+gx is asymptotically stable if all the eigenvalues of A have negative real part. So that is our part 1. Now part b says that the equilibrium solution zero solution is unstable if at least one eigenvalues of A have positive real part right.

So if at least one of the eigenvalues of A has positive real part, then linear system will have all the solutions are unstable solutions as well as this x dash=Ax+gx will also have unstable equilibrium solution xt=0 here. Now in c part, the stability of the equilibrium solution xt identically=0 of 4 cannot be determined from the stability of the equilibrium solution xt identically=0 for dx/dt=Ax if all the eigenvalues of A have real part<=0 but at least one eigenvalue of A has zero real part.

So this we need to understand in the sense that if you look at the part a and part b is similar to the part a and part b of the previous theorem that all the eigenvalues are having negative real part, then in linear equation x dash=Ax and x dash=Ax+gx here every solution is asymptotically stable, here equilibrium solution is asymptotically stable solution in the case of all the negative real part.

Now if we have at least one eigenvalue with positive real part, then every solution of x dash=Ax will be unstable solution and here we can say that the equilibrium solution of x dash=Ax+gx is unstable solution but in part c when that real part of the eigenvalues maybe 0 then this here we have obtained that here we have a condition that Am=Gm then we have a stable solution.

But in this case x dash=Ax+gx may not be determined as we have determined for x dash=Ax. In fact, here we simply say that in that case when eigenvalues of A some of the eigenvalues of A have zero real part, then it will depend on gx whether equilibrium solution is unstable equilibrium solution or stable equilibrium solution here.

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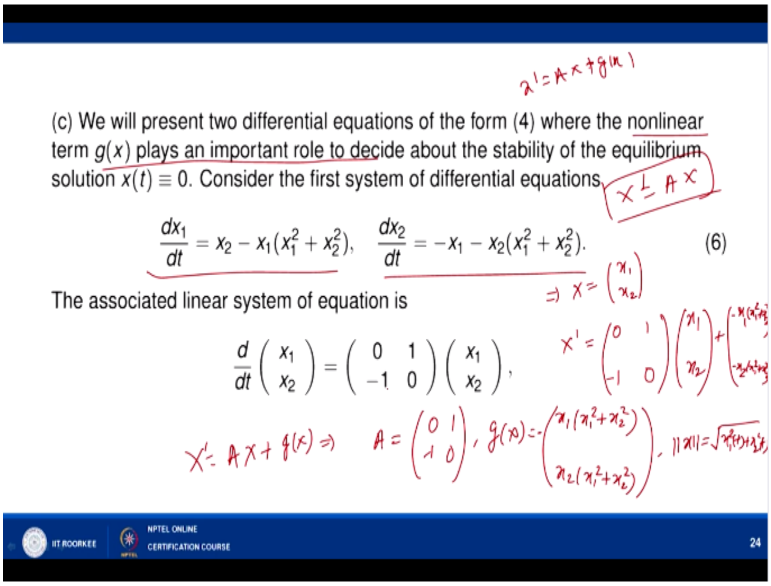

So we will prove part a and part b little bit later but let us first focus on the part c. It means that when some of the eigenvalues of A have zero real part, then how this gx will play a very important role. So it means that here you want to consider this part c that we will present two different differential equations of the form x dash=Ax+g of x where the nonlinear term gx plays an important role to decide about the stability of the equilibrium solution xt identically=0.

So first let us consider this system, dx1/dt=x2-x1 x1 square+x2 square and the second equation is given as dx2/dt=-x1 -x2 times x1 square+x2 square and if you write down I can write down this like this that if we assume x as x1 and x2, then I can write down this x dash=now I can write it here x1 x2, so it is x2- so it is 0 and 1 and here we have –of x1 times x1 square+x2 square.

And the second equation is –x1 so that is -1 and 0 and here it is –x2 x1 square+x2 square right. So it means that here I can write this as x dash=A of x+g of x. So here your A is basically what 0 1 -1 0 and g of x is given as say – I can take it – out x1 x1 square+x2 square and x2 x1 square+x2 square. So this is the thing. Now if we look at the norm of x which is say under root of x1 square t+x2 square t right. Then, I can look at the gx/norm of x, so let me write it here.

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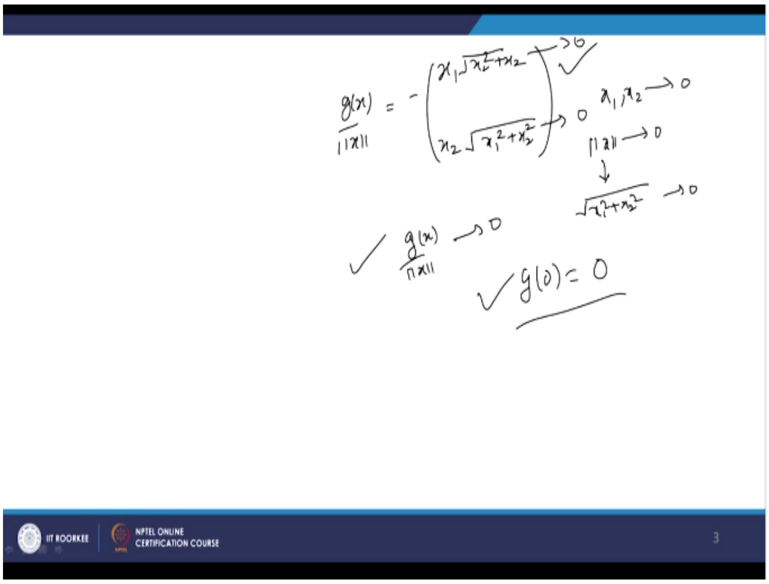

So gx/norm of x will be what, I can simply write it is as – say x1 and here it is under root x1 square+x2 square and here it is x2 under root x1 square+x2 square right, so gx/norm of x. Now can we say that it is a continuous function of x, so as x1 and x2 tending to 0 or I can say that that norm of x is tending to 0 where norm of x we are denoting as under root x1 square+x2 square is tending to 0, then we can prove that gx/norm of x is tending to 0 right.

So it means that the condition look at g0, so if you look at g of 0 then g of 0 is nothing but 0 here, so it satisfies both the condition that g0=0 and gx/norm of x is a continuous function of x and as x1 x2 tending to 0, then norm of x is tending to 0 and we can say that this term is tending to 0 and this will also tending to 0. So gx/norm of x is a continuous function of x g0=0 and gx/norm of x vanishes when x1=x2=0.

So it satisfies all the condition of the theorem and now we want to say that that here stability may not be determined with the help of stability of the linear system that is x dash=A of x. So if you look at the linear system x dash=Ax where A is what, A is having 0 1 -1 0, so if you look at what are the eigenvalues of matrix A.

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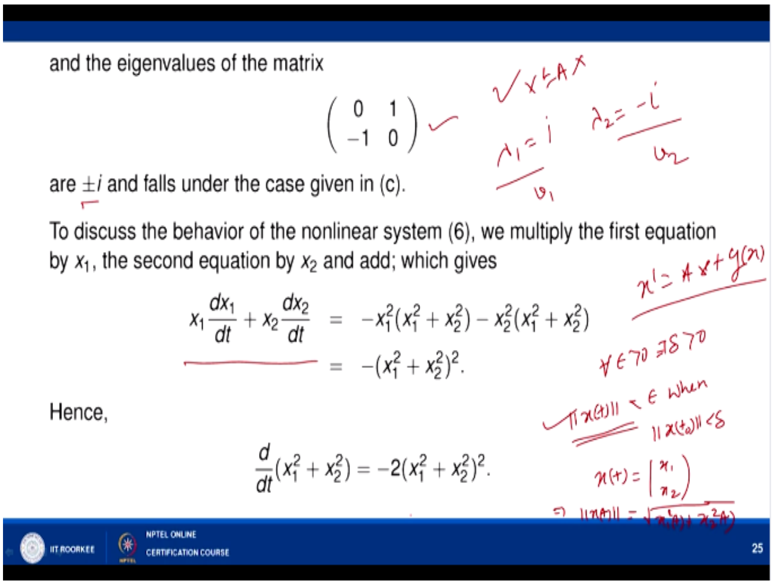

And here we can simply say that the eigenvalues of the matrix A are +-i right. So here the thing will depend on the eigenvectors of matrix A whether it will have two linearly independent eigenvector or not. So let me so you can say that the stability of linear system will be depending on the number of linearly independent solution of say 0 1 and -1 0. Now here you can say that one eigenvalue is 1 say i and another one is lambda 2=-i.

And these two are distinct and hence there is a guarantee that we have two linearly independent eigenvectors. So it means that the linear system x dash=Ax has all the solutions which are stable solution. So it means that here zero solution is an unstable solution and hence all the solutions of x dash=Ax is stable solution but now look at our linear and nonlinear system that is x dash=Ax+gx.

Now we want to look at the stability of x dash=Ax+g of x. So how to look at the stability of zero solution, then we recall this condition that for every epsilon>0 there exists delta>0 says that norm of x of t is<epsilon whenever norm of x t0 is<delta. So here we have to calculate the norm of x of t where x of t is basically what, x of t is we are writing it as x1 and x2 and here we are taking norm of xt as say two norm of x that is under root x1 square t+x2 square t.

So we want to find out the bound on x of t, the behaviour of norm of x of t. For that, we look at the differential equation and calculate this x1 dx1/dt+x2 dx2/dt. So we multiply the first equation by x1 and second equation by x2 and simply add. So if you look at the first equation is what, so first equation is this so x1 x2 –x1 square+x2 square. So that is what we have written here that sorry let me look at here let me calculate this quantity.

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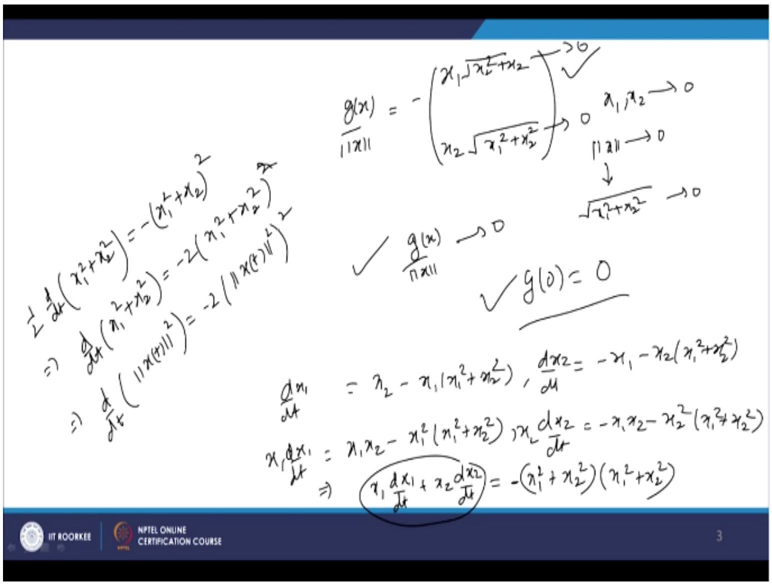

It is what x2-x1 x1 square+x2 square it is here dx1/dt and dx2/dt will be what dx2/dt is=-x1 -x2 x1 square+x2 square. So here we multiply by x1, so we have x1 dx1/dt which is nothing but x1 x2 –x1 square x1 square+x2 square and here it is x2 dx2/dt is=what –x1 x2 –x2 square x1 square+x2 square. So when you add these two, we have x1 dx1/dt+x2 dx2/dt=-x1 x2 will be cancel out, –x1 square+x2 square you can take it out and we have x1 square+x2 square again.

So here what you will get x1 dx1/dt+x2 dx2/dt=-of x1 square+x2 square whole square. So I can write it this as now this also I can write it, it is nothing but 1/2 d/dt of x1 square+x2 square=-of x1 square+x2 square whole square. So I can write this as d/dt of x1 square+x2 square=-2 times x1 square+x2 square. So basically I can write it here as d/dt of norm of x of t square=-2 times norm of xt whole square whole square.

So basically it is a differential equation in terms of norm of xt. So we can find out the solution here and we can simply say that this is nothing but d/dt of x1 square+x2 square is=-2 times x1 square+x2 square whole square.

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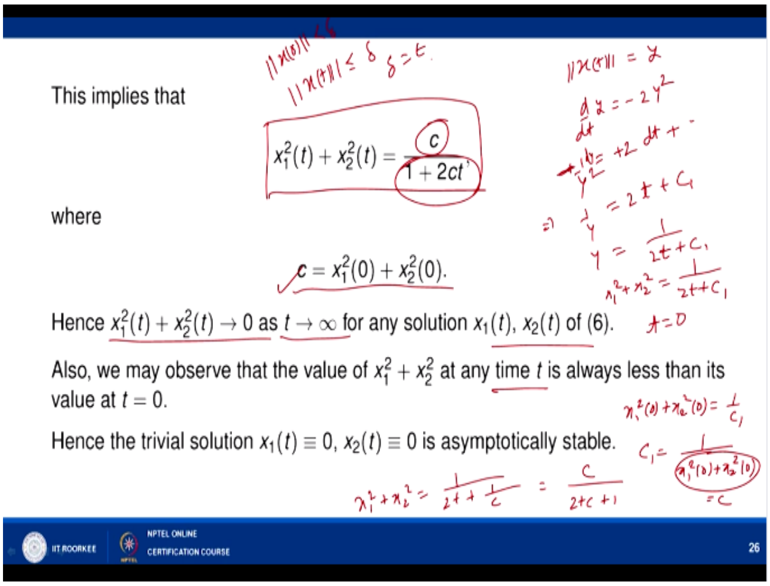

And we can calculate the solution here we can assume that norm of xt, let us say that it is simply y. Then, it is given as d/dt of y=-2 times y square and we know how to solve this equation and we can simply say that it is -1/y, so it is -1/let me write it here. So this I can write it, so here I can write it here as -1/y=-2 dt, so – we can cancel out, +dy/dt. So it is what, it is –1/y square with – sign.

So this I can write it 1/y=2t+some c. So I can write it y as 1/2t+c and I can simplify this and we can write it our solution y as 1/2t+c. So what we have obtained here that is 1/y=2t+c, let us call it c1, y=1/2t+c1. Now calculate this value of c1. What is y here? y is x1 square+x2 square=1/2t+c1. Now let us assume that we need to calculate this value of c1, so at t=0 what should be the value, here it is x1 square 0+x2 square 0=1/c1.

So if we write down the value of 1/c1, so c1 is=1/x1 square 0+x2 square 0 here. Then, I can plug in the value here and what is the value here, x1 square+x2 square=1/2t+c1 let me write it here, 1/this value let us call this as c, so this is c, so 1/c here. So this I can simply say that it is

2tc+1*c. So if we use this, then we can have this result that x1 square t+x2 square t=c times 1+2ct and where c is given as x1 square 0+x2 square 0 here.

So what we have done? Here we have find out the expression for norm of xt whole square. So we know that if xt is stable then norm of xt will also be stable or xt is tending to 0 then norm of xt will also tending to 0. Since in denominator we have this 1+2ct and c is positive here, so it means that this is never 0, so denominator is never 0 and in fact it is a positive value and as t tending to infinity then this value is always tending to say smaller than the previous value.

So we can say that this x1 square t+x2 square t is tending to 0 as t tending to infinity for any solution x1 and x2 t of 6. In fact, here we have not obtained an explicit solution x1 and x2 rather than we have found the say bound of or the expression of norm of x square t. So here we have shown that whatever be the solution in x1 and x2, then x1 square+x2 square t is tending to 0 as t tending to infinity.

And also we may observe this thing that at any time t the value of x1 square t+x2 square t is always<the value of c. Now what is c here? C1 is x1 square 0+x2 square 0, so it means that if norm of x0 is<delta then norm of xt is certainly<this delta right. So it means that if I assume this delta as epsilon, then we can say that norm of xt is<epsilon whenever norm of x0 is<say delta. So here we simply say that our trivial solution x1 t identically=0 and x2 t=0 is asymptotically stable solution.

So here the linear part and nonlinear part both are matching together, so it means that here linear solution when the linear part x dash=Ax solution is stable but the nonlinear part have solution which is asymptotically stable. Now look at the next thing, next system of differential equation.

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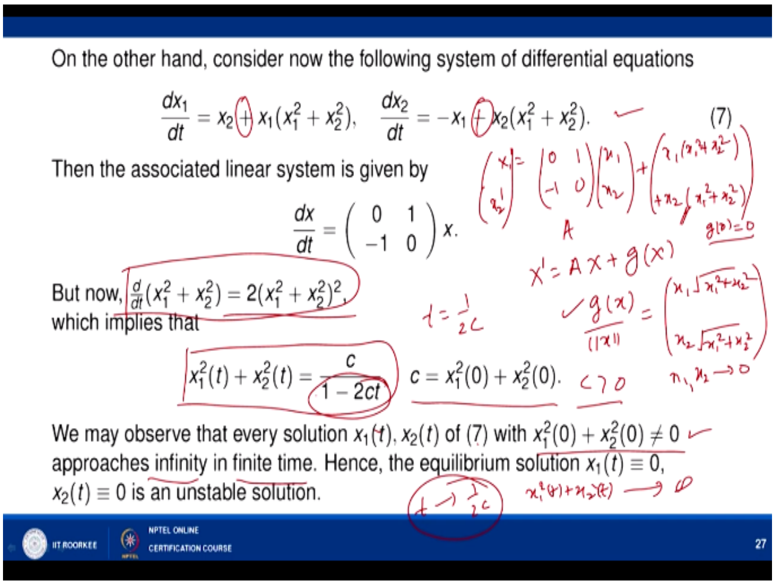

Dx1/dt=x2+x1 x1 square+x2 square and dx2/dt=-x1 -x2 x1 square+x2 square right. So here also we can write down the corresponding system of linear equation. So we can simply write x1 dash=so x2 dash here, so let me write it here. So this is what 0 1 and here it is -1 0 x1 x2+x1 x1 square+x2 square, here it is –x2 x1 square+x2 square right. So here there is a small mistake here, it is +here.

So second equation is dx2/dt=-x1 +x2 times x1 square+x2 square okay. So here this is + here, so this is this thing. So here if you look at your A is same as the previous one. If you look at A is 0 1 -1 0, so it is 0 1 -1 0. So here we have shown that the linear system is same right but look at the nonlinear system, so nonlinear system is what, x dash=Ax+g of x. Now g of x/norm of x will be what? g of x is given by this.

So gx/norm of x where norm of x is under root x1 square+x2 square is basically x1 x1 square+x2 square and x2 x1 square+x2 square. So one thing is very much clear that here g of 0 is 0 means you put x1=x2=0 then we can say that g0=0 okay and second thing is that gx/norm of x is a continuous function of say x here. So as x1, x2 is tending to 0, then gx/norm of x is tending to 0 here and it will vanish when x1 and x2 are all 0 here.

So now we do the similar thing and we just calculate the say d/dt of x1 square+x2 square and we want to show that it is given as 2 times x1 square+x2 square whole square. For that you simply look at here.

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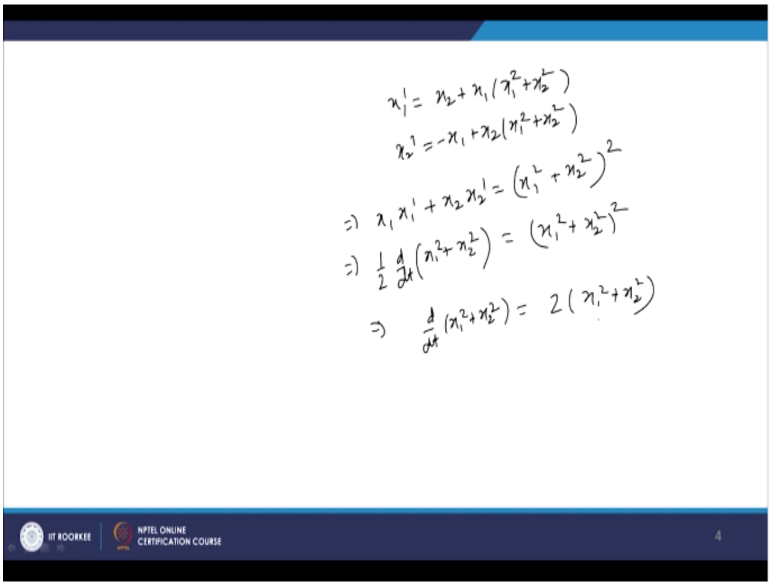

Let me write it here, say x1 dash is=x2+x1 square+x2 square. I hope it is okay. Now x2 dash is=-of x1+x2 x1 square+x2 square. So we can write down x1 x1 dash+x2 x2 dash is=these two will cancel out, we will have x1 square+x2 square whole square and then I can write this as 1/2 d/dt of x1 square+x2 square, then it is x1 square+x2 square whole square. Then, this is nothing but d/dt of x1 square+x2 square=2 times x1 square+x2 square.

So if you look at there is only one difference that here in previous problem, it is all negative. If you look at here, here in previous problem, it is negative here and in this problem it is both positive things. This is the only change we have considered and when we do this kind of change, we have this equation dx/dt=d/dt of x1 square+x2 square=2 times x1 square+x2 square=0.

And as we have solved earlier, we can solve it here also and we have x1 square t+x2 square t=c/1-2ct and the expression of c is x1 square 0+x2 square 0. So we can say that c is positive right. So now if c is positive, then at time t=1/2c right. Then, at time t=1/2c, the denominator is tending to in fact denominator will be 0. So it means that as t tending to 1/2c your expression of x1 square t+x2 square t is going to be large and large.

And we can say that as t tending to 1/2c your x1 x1 square t+x2 square t is tending to infinity. So here we are assuming this condition first of all that x1 and x2 of 7 with x1 square 0+x2 square 0 is nonzero. So it means that we are considering any solution which is other than your trivial solution. Then, we can say that every solution of x1 and x2 of the system 7 will tend to infinity in finite time itself.

So as t tending to 1/2c, your solution will be tending to infinity and hence we can say that your solution, equilibrium solution x1 t identically=0, x2 t identically=0 is an unstable solution right. So here if you look at in both the equation number 6 and 7 your linear part is same that is dx/dt=Ax and A is the same matrix that is 0 1 -1 0 but in previous problem we are getting the zero solution is an asymptotic stable solution.

But in this problem we are seeing that zero solution is an unstable solution. So it means that in case when the coefficient matrix have at least one eigenvalue with zero real part then the stability of equilibrium solution of x dash=Ax+gx may not be determined using the given theorem. That will depend on the behaviour of g of x and this theorem is not capable to give any conclusion in this regard.

Maybe there are certain results available but here we may not discuss the same result here. So we simply say that when coefficient matrix have eigenvalue the zero real part, then this theorem will not give any conclusion right and everything will depend on the further analysis of the nonlinear part gx okay. So with this I finish our lecture. In this lecture, we have discussed only the c part of the theorem.

And in next lecture, we will consider the remaining part, so will continue in next class. Thank you very much for listening. Thank you.