Dynamical Systems and Control

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Lecture – 17
Stability of Linear Autonomous Systems - III

Hello friends, welcome to this lecture, so if you recall, in previous few lecture, we have defined the term stability of a solution of a nonlinear system on a system and we have discuss certain example to illustrate the concept of a stability and in previous lecture, we have say, find out method to find out the stability of the linear system.

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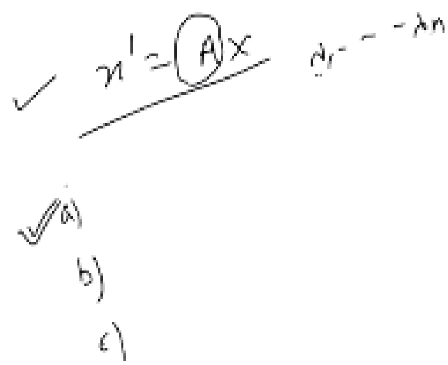

This is x dash = Ax and we have seen that the stability of any solution is equivalent to a stability of zero solution and a stability of zero solution can be determined in complete manner provide with the help of the eigenvalues of coefficient matrix A, so we have discuss the following theorem that if all the eigenvalues of A, say lambda 1 to lambda n have negative real part, then zero solution is going to be stable solution.

In fact, it is more than a stable solution, it is asymptotically a stable solution and that part we have discussed in previous lecture and in the second part, we have seen that if at least one of these eigenvalues have positive real part, then zero solution is going to be unstable solution and hence all the solution of x dash = Ax is going to be unstable solution and in the c part, if one of the; if all the solutions; all the eigenvalues have non-positive real part.

And corresponding to eigenvalues with zero real part, if AM = GM, algebraic multiplicity = geometric multiplicity, then all the solutions are stable solution and if algebraic multiplicity of any eigenvalue with zero real part is strictly greater than geometric multiplicity corresponding to that eigenvalue, then any solution of x dash = Ax is going to be unstable solution. So, we have discuss the part one in previous lecture.

Now, we continue with the second part that here we are assuming that at least one of the eigenvalues of A has a positive real part, so let us consider here.

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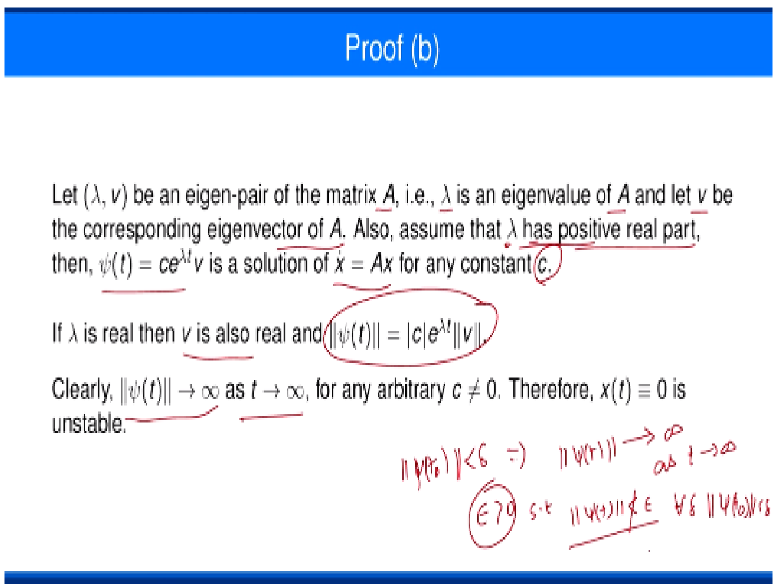

Let lambda v be an Eigen pair of the matrix A that means that lambda is an eigenvalue of A and v be the corresponding eigenvector of A, so let us consider one Eigen pair of matrix A and also assume that lambda has positive real part, right, so it means that here we are taking only one say, eigenvalue which has positive real part, then we can simply find out the solution as psi t = c e to the power lambda t * v is a solution of x; of x dash = Ax for any say constant c.

And now, we may have 2 different cases, one case is that lambda is purely real or lambda is a complex eigenvalue, so if lambda is real, then we can show that v is also real and hence we can write down the norm of psi t = modulus of c e to the power lambda t * norm of v here. Now, the modulus of e to the power lambda t is nothing but e to the power lambda itself, and here we can see that as t is tending to infinity, then psi t is tending to infinity.

So, it means that in it may happen that initially, your psi of t0 is < delta but this; but we have already shown here that psi of t is tending to infinity, as t tending to say infinity, it means that I cannot find out any epsilon > 0 such that norm of psi t is < epsilon for any delta such that norm of psi t0 is < delta. So, it means that whatever epsilon you will choose, since this norm of t is in increasing function of t.

We can always find out some time t such that norm of psi t is not less than this epsilon, so it means that your psi t is not a stable solution, now this is the case corresponding to lambda real.

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If $\lambda = \alpha + i\beta$ is complex, then $v = v^1 + iv^2$ is also complex. In this case

$$\begin{aligned} e^{(\alpha+i\beta)t(v^1+iv^2)} &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(v^1 + iv^2) \\ &= e^{\alpha t}[(v^1\cos\beta t - v^2\sin\beta t) \\ &\quad + i(v^1\sin\beta t + v^2\cos\beta t)] \end{aligned}$$

is a complex valued solution of $\dot{x} = Ax$.

$$e^{\alpha t}[v^1\cos\beta t - v^2\sin\beta t]$$

$$e^{\alpha t}[v^1\sin\beta t + v^2\cos\beta t]$$

Now, consider the case when lambda is a complex eigenvalue, so corresponding to complex eigenvalue, we have a complex eigenvector, let us say v1 + iv2 and then we write down the solution like e to the power alpha + eta t * v1 + iv2, this implies that e to the power alpha t cos beta t + i sin beta t and we are writing v1 + iv2. Now, when you simplify and we can take out the real and imaginary part, I can write down e to the power alpha t v1 cos beta t – v2 sin beta t + I times v1 sin beta t + v2 cos beta t.

So, this is your real part, so real solution I can write it e to the power alpha t and v1 cos of beta t – v2 sin of beta t, so this is also a solution of x dash = Ax and the imaginary part is e to the power alpha t and v1 sin beta t + v2 cos of beta t, so out of this complex solution, we have got 2 real valued solution of x dash = Ax.

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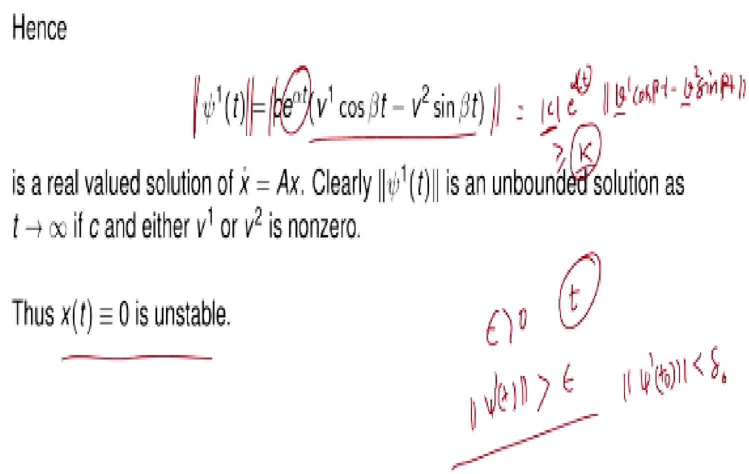

So, let us take one real value solution, let call it psi1 t, psi1 t is c e to the power alpha t v1 cos beta t – v2 sin beta t. Now, look at the; so this is the real value solution of x dash, now we can look at the norm of this and if you look at the norm of this, then it is what; and this is bounded by say, norm of v1 + v2, but this quantity this, so it is < = e to the power alpha t modulus of c and this quantity is bonded by you can say, norm of v1 + norm of v2.

So, we can simply say that so here we can calculate the norm of psi 1 t which is given as norm of c e to the power alpha t v1 cos beta t – v2 sin beta t and this I can write it modulus of c e to the power alpha t and norm of v1 cos beta t – v2 sin beta t and here if we we simply say that if seen an either of one of v1 and v2 are non-zero, then corresponding to any constant, I can find out a t large enough such that this quantity is bigger than the given constant K.

So, it means that I can always; you take any epsilon here > 0, I can find out a t suitably large such that psi 1 t, norm of psi 1 t is bigger than this epsilon, so it means that even though your psi 1t0 is < your delta0 for any delta but I can always find out a suitable large t such that norm of psi

1t is bigger than epsilon, so it means that this quantity cannot be made a small for all t > = t0 and hence we can say that your zero solution is unstable.

And hence every solution of x dash = Ax is unstable, so this simply says that if at least one of eigenvalues with the positive real part exist, then all the solutions of x dash = Ax is going to be unstable solution.

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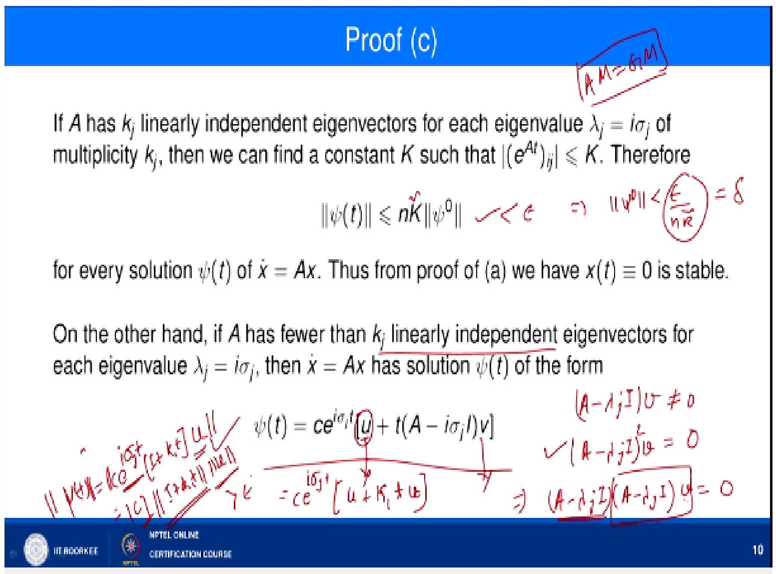

So here now we try to discuss the part c, so here we are assuming that the some of the all the eigenvalues 1; we are assuming 2 thing; first that all the eigenvalues are having non-positive real part, second thing is that some of the eigenvalues are having 0 real part, right, it means that not all the eigenvalues have negative real part, so in part A, where we have assume that all the eigenvalues are negative real part.

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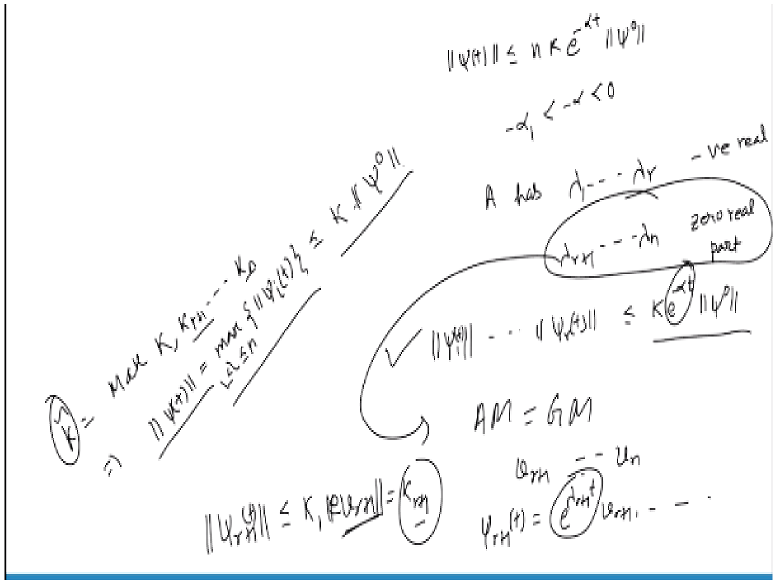

And then we have shown this thing that norm of psi t is < = n K e to the power –alpha t norm of psi 0 and this we have shown in the first part but here I am assuming that all the eigenvalues are having negative real part and hence we can choose your - alpha such that – alpha is bigger than – alpha 1, where –alpha is one is the largest zero; largest negative real part of eigenvalues but here since some of the eigenvalues are having say zero real part.

So, I cannot choose –alpha which is say < 0, so here I have to use some other say methods, so now let us assume that A has some eigenvalues A lambda 1 to say lambda r which are having say negative real part and lambda 1, r +1 to say lambda n, they are having say zero real part right, so we already know that the corresponding to lambda 1 to lambda r, if we final out the solution say psi 1 t to say norm of psi r t, we know that this is less than or equal to say K e to the power – alpha t norm of psi of 0.

So, corresponding to these we know that each one is satisfying this condition but corresponding to this, we need to know whether this will; they will satisfy this or not, so if you consider lambda r +1 to lambda n and here we are assuming that AM = GM means, algebraic multiplicity = geometric multiplicity, it means that they are having the eigenvectors and there is no generalised Eigenvector exist.

So, it means that corresponding to lambda r +1 to lambda n, we are having say vr + 1 to say vn has corresponding eigenvectors and we can find out the solution say psi r + 1t as e to the power lambda r +1t vr + 1 and so on, so it means that corresponding to this, I can simply write the norm of psi r +1 t is < =; I can take the same constant K here or some other constant K1 e to the power say simply; this is simply this has zero real part.

So, lambda r +1 t is basically complex value, so norm of modulus of e to the power lambda r +1t is basically bounded by 1 and we can simply say that it is norm of vr +1 and if we include these 2; we can say that psi r + 1 is < = we can denote this quantity as K r + 1, similarly this I can do for all r +1 to n and we can choose, so it means that up to say 1 to r, it is bounded by this and r + 1 to +n, it is bounded by Kr+1, Kr+2 up to Kn.

So, now we can choose the maximum, so we can choose K as maximum of this earlier let me to choose some K tilde, K tilde is maximum of K, Kr +1 to Kn, right, so this implies for this K tilde, we can simply say that psi of t norm of psi of t which is nothing but maximum of norm of psi it, i is from 1 to n, then this is < = K, K times norm of psi 0, right because up to r +; up to 1 to r, it is bounded by K e to the power –alpha t norm of psi 0.

Now, this quantity is < 1, so I can say that norm of psi 1 to psi rt, it is < K, norm of psi of 0 and for r + 1 to n, it is bounded by K r +1 where Kr +1 is nothing but K1 times norm of vr + 1, right, so we can say since these are constant eigenvectors, so I can find out the norm of vr +1 and we include this with K1 to write Kr +1, so it means up to Kr +1 to Kn, we have are some other values and then choose K tilde as maximum of K, Kr +1 to Kn.

So, we can say that norm of psi t which is nothing but maximum of norm of psi it is now bounded by K times norm of psi 0, which is; which we are writing it here, so if it is bounded by K times norm of psi 0, I can simply say that it is bounded by n times K time norm of psi of 0 where K gives you the sorry, here it is K tilde, so here we can simply say that every quantity is < K times norm of psi 0.

So, I can simply say that norm of psi t is < nK tilde norm of psi0, for every solution psi t of x dash = Ax, so now we have shown that they exist a constant K tilde, such that norm of psi t is < = nK tilde norm of psi0. Now, we want to show that this quantity is < epsilon, so make this quantity < epsilon, so this implies that norm of psi 0 < epsilon/nK tilde and we can choose this quantity as delta, right.

So, here we can simply say that this K tilde is a positive quantity, so epsilon/nK tilde is define and we can take this quantity as delta, so it means that whenever norm of psi0 is < delta, this implies the norm of psi t is < epsilon and hence we can say that zero solution is in a stable solution and hence we can say that every solution of x dash = Ax is also a stable solution, okay, now let us, so here we have assumed that Am = GM so that so here we have use this condition.

So that all the eigenvalues say corresponding to r +1 to n, they are all say simple eigenvectors means constant eigenvectors, there is no existence of generalised eigenvectors exist and hence we can show that norm of psi t is < = nK tilde norm of psi0, now if we assume that A has fewer than Kj linearly independent eigenvector, where this Kj is the algebraic multiplicity of the eigenvalue lambda j.

So, it means that here we may have solution psi t of the form, psi t = c e to the power i sigma jt u + t times A- I sigma ji * v, so here it may be something more but I am writing it like this that the simplest form of the solution may be like this, psi t +; psi t = c e to the power i sigma + jt u + t times A – i sigma j iv, where u is the eigenvector and v is the generalised eigenvector, right. So, here we simply say we wanted to show that this solution is going to be a unstable solution.

For that we can simply say that since v is a solution of what; v is solution for this A- lambda ji, v is non-zero and A- lambda ji v, sorry square = 0, now, this I can write it as A- lambda ji * A- lambda ji operating on v = 0, so I can say that this quantity A- lambda ji v is a solution of A- lambda ji = 0. So, it means that this I can consider this that we already know that it has only one say one linearly independent eigenvectors.

It means that A- lambda ji v is a multiple of eigenvector u, so I can write this as c e to the power say i sigma jt u + some constant times, let me use some other constant let us say that K1 t u and that is all, so we may have, so what we have done here that if algebraic multiplicity is less than is strictly > geometric multiplicity, then these solution of the form like this, psi t = c e to the power i sigma jt u + t times A- i Sigma ji v.

And where v is the generalised eigenvector and u is a eigenvector and we know that by definition v is a solution of A- lambda ji whole square v = 0 and this I can rewrite as A- lambda ji * A- lambda ji v * 0, so it means that this A- lambda ji v is a constant multiple of the eigenvector u, right. So, it means that in place of A- lambda ji v, I can write it K times, K1 times this u, right, so K1 times u if we write it, then we have psi t can be written as c e to the power i sigma j + u + K1 t times this u.

Now, I can take this u out and we simply say that we have psi t = c e to the power i sigma jt, this is what, this I can write it i + K1 t and u here, now if you take the norm here since u is a constant vector, norm of this e to the power i sigma jt is 1 and norm of c, so it is modulus of c and norm of i + K1 t and norm of u, now this is a constant value, this is the constant value but this is a polynomial function of t.

So, as t tending to infinity, the normal psi t is also turning to infinity, so it means that psi t is cannot be bounded by any some given any epsilon, so we can say that for any given epsilon, you can choose the large enough such that this quantity is bigger than that epsilon, so it means that psi t is not a a stable solution, so here we simply say that zero solution is not a stable and hence any solution of x dash = Ax is not a stable solution.

Now here, we have discussed one more definition which says that how we define asymptotically stable?

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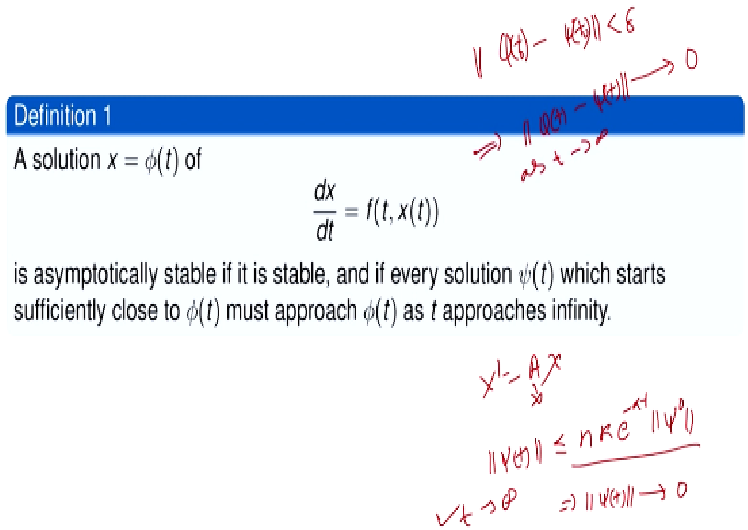

A solution x = phi t of dx/dt = f(t, xt) is asymptotically stable and if every solution psi t which start sufficiently close to phi t must approach phi t as t approaches infinity. Why we are discussing this definition; it says that you consider one solution phi t and it is a stable solution first thing, now we want to show that this is a say, asymptotically stable, what do you mean by asymptotically stable?

So for that if you take any other solution psi t which initially very close to this, it means that norm of psi t0 –psi t0 < delta, now this implies that norm of phi t –psi t, now this will be tending to 0 as t tending to infinity. So, it means that if phi t has this kind of property, it satisfy the following two properties. First thing that it is a stable solution, second thing is that as t tending to infinity norm of phi t – psi t is tending to 0.

For every psi t which is which start near to phi t0, then we call phi t is an asymptotic stable solution, why we are considering here? If you remember in case of x dash = Ax where A has all the eigenvalues with zero, negative real part, we have obtained this thing that norm of psi t is < = nK e to the power –alpha t norm of psi 0. Now, here if you take t tending to infinity then this quantity is tending to 0.

So, it means that norm of psi t is tending to 0 as t tending to infinity, so it means that if all the eigenvalues of the coefficient matrix A have negative real part then the zero solution is asymptotically stable solution and hence every solution of x dash = Ax is asymptotically stable solution, okay so that is what we have discussed here. Now, let us consider certain example based on this.

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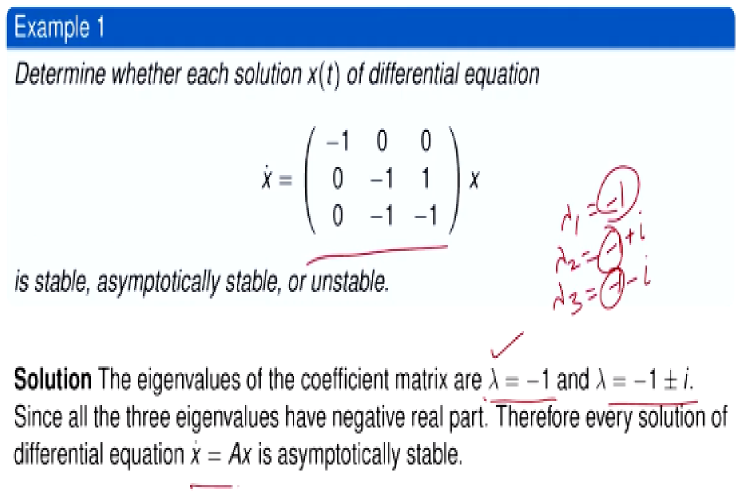

So, first example is that determine whether each solution xt of differential equation x dash = Ax where A is given as -1 0 0 0 -1 1 0 -1 -1 is asymptotically stable or unstable, so here we want to see whether each solution is stable or otherwise, so here as we have discussed in previous theorem, this whole thing can be determined if we know the eigenvalue of the coefficient matrix A.

So, here the coefficient matrix A is given by this -1 0 0 0 -1 1 0 -1 -1, so we first calculate the eigenvalues of the coefficient matrix and it is coming out to be lambda = -1 and lambda = -1 + -i, so this part I am leaving it to you, how to find out the eigenvalue of the coefficient matrix, so if look at carefully then lambda 1 = -1, lambda 2 = -1+ i and lambda 3 = -1 –i, so if you look at the real part of each eigenvalues it is -1 here, it is -1 here.

And so it means that every eigenvalue have real part which is strictly negative, so it means that here we simply say that zero solution is not only stable solution but asymptotically stable solution and hence we can say that every solution of the differential equation x dash = Ax where A is this matrix, then it has to be asymptotically stable, so here I just want to point out one more thing that if all the eigenvalues are strictly or having strictly negative real part, then it is asymptotically stable.

If all the eigenvalues are having non- positive real part and corresponding to 0 real part if algebraic multiplicity = geometric multiplicity, then solution is stable solution, in that case we may not get the asymptotic stability but all the solutions are stable solution, so please recall if eigenvalues are having non-positive real part and corresponding to 0 real part, if AM = GM that is algebraic multiplicity = geometric multiplicity, then every solution is a stable solution.

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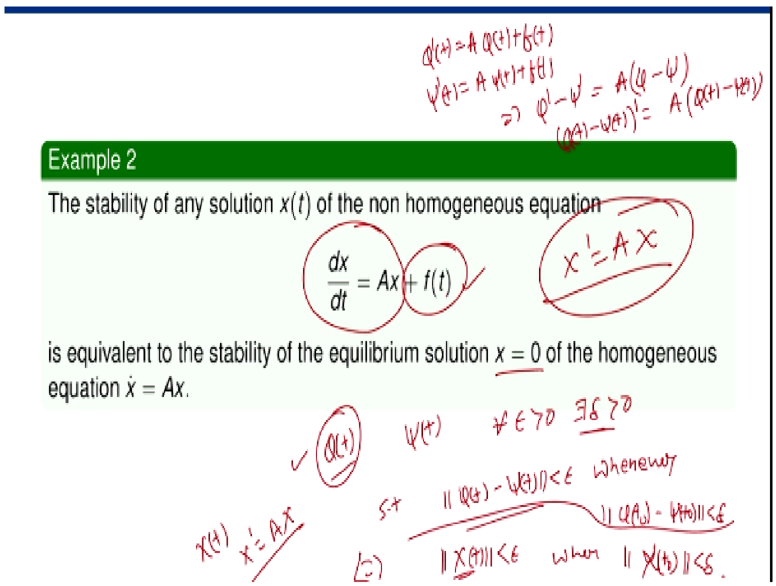

And if at least one of the eigenvalues are having positive real part then all the solutions are having say are unstable solutions, right so that is what we have discuss from this theorem, now we have one more very important observation from this system that stability of any solution xt of the nonhomogeneous system dx/dt = Ax + ft is equivalent to the stability of the equilibrium solution x = 0 of the homogeneous equation x dash = Ax.

So that is very, very important observation in the sense that even if we have any function ft, non-homogeneous part some force function but we have no change in the stability behaviour that if this linear system is stable then whatever force you want to apply on the system, it will remain a stable solution and if dx/dt = Ax is having unstable solution, then ft will not change any the stability thing.

So, ft has no role in determining the stability behaviour of dx/dt = Ax + ft, so this is an autonomous systems, so autonomous system; stability of autonomous system is totally depending on the eigenvalues of the coefficient matrix, it will not depend on the say, forcing function ft. So, let us just observe this result here, so let us say that we want to have a solution say, phi t and we want to check whether it is a stable solution or not.

So, to check the stable solution of; stability of this phi t, we consider another solution psi t of dx/dt = Ax + ft and we want to check that for every epsilon > 0, they exist a delta > 0, can we find out a delta > 0 such that norm of phi t – psi t is < epsilon whenever norm of phi t0 – psi t0 is < delta, so that what we want to check but here just observe this thing that if phi t is a solution of dx/dt = Ax + ft and psi t is also a solution of this.

Then phi t – psi t will be a solution of the linear part that is dx/dt = Ax, so how we can simply say that phi dash t = A phi t + ft and psi dash t = A psi t + ft, then we can simply say that phi dash – psi dash = A phi – psi and that is all, so it means that this phi t – psi t is a solution of phi t – psi t dash = A phi t – psi t, so phi t - psit is a solution of x dash = A of x, right, so it means that the stability of phi t is equivalent to show that that the zero solution is; a zero solution of x dash = Ax is stable.

Because I can look at here that call this the difference phi t – psi t as say your xt, then I can simply say that this is nothing but and this can be written as xt norm of xt is < epsilon whenever norm of x of t0 is < delta, so if you look at where xt is what; where xt is the arbitrary solution of x dash = A of x, so it means that this statement is equivalent to the statement that let xt be an arbitrary solution of x dash = Ax.

Then for every of epsilon > 0, we want to find out delta such that norm of xt < epsilon, whenever norm of xt0 is < delta, so it means that these two statements are equivalent in fact, this also implies this, right because if xt is an arbitrary solution and phi t is fixed, then I can say that phi t – psi t, we can say that the difference phi t and xt which is nothing but psi t is an arbitrary solution.

So, it means that these 2 are both ways okay, clear, okay, so it means that the stability of phi t is equivalent to show that the zero solution of x dash = Ax is a stable solution, so it means that stability of any solution dx/dt = Ax + ft is equivalent to the stability of the zero solution of the homogeneous equation x dash = Ax. So, with this observation we end our lecture here and we will continue our discussion in next lecture also. So, thank you very much for listening us, thank you.