Dynamical Systems and Control
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Lecture – 16
Stability of Linear Autonomous Systems - II

Hello friends, welcome to this lecture, in this lecture, we will continue our study of stability of non-linear system, if you recall in previous class, we have define; what do we mean by stability of a solution of a nonlinear system.

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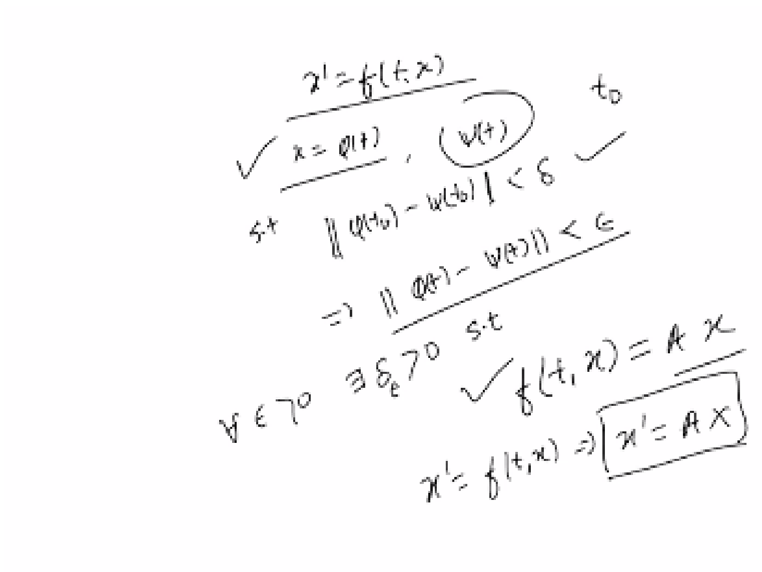

For example, if you have a say x dash = f tx and we have a solution like x = phi t, then we say that this phi t is a stable solution provided that if you take any other solution psi t of x dah = f tx such that initially, let us say the initial point is t0, t0 is a initial point where this differential equation is start functioning or t0 is some initial point such that this phi t0 – psi t0 is < some very small quantity call it delta.

And so, if we take a any other solution psi t, such that this norm, this is basically norm; norm of phi t0 - psi t0 is < delta implies that that norm of phi t – psi t is < some epsilon, so it means that for, I can call this as for every epsilon > 0, they exists a delta > 0 of course, it is depending on epsilon such that this quantity is true means norm of phi t – psi t is < epsilon, whenever the they are near to say, whenever norm of phi t0 – psi t0 is < delta.

So, if for every psi t, we have this condition true, then we say that x = phi t is a stable solution of x dash = f tx, so this is how we have defined the definition, defined the stability of a given solution now, let us focus on a particular case of this, when let us assume that f tx is simplify and we can write it like A of x, so it is an autonomous system, so now x dash = in place of f tx, we are considering a particular special case of this that is x dash = Ax.

It means that we are considering that f tx is given by A of x and then we try to get some condition on say, coefficient matrix A such that the every solution of x is stable or unstable or say half stable, something like that, so let us try to find out the condition on the coefficient matrix A such that we can talk about the solution of this linear system x dash = A of x.

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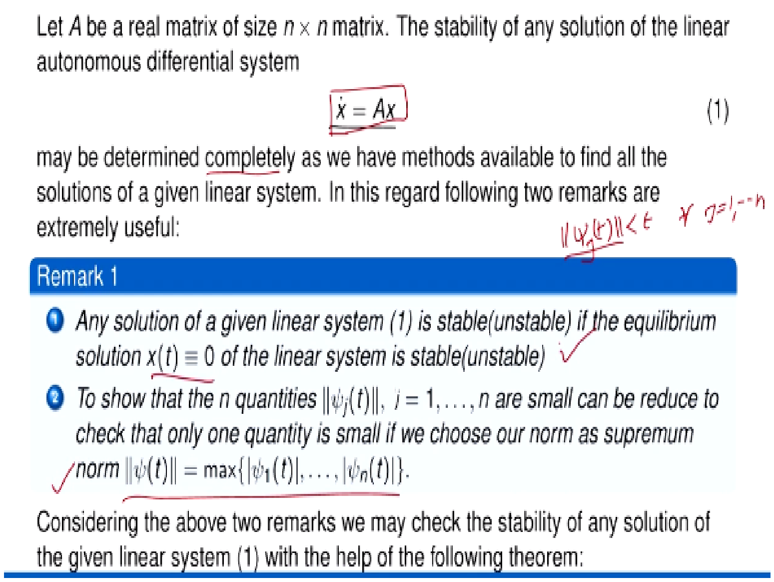

So, now let us move to this, so let A be a real matrix of size n cross n matrix, the stability of any solution of the linear autonomous differential equation x dash = A of x and this may be defined by this may be determined completely as we have methods available to find out all the solution and we have already discussed the method to find out the complete solution of x dash = Ax and once we have complete solution of x dash = Ax, we can find out the behaviour of its solutions.

So, it means that we have methods available to find out all the solution of a given linear system and in this we can discuss the stability of any solution of x dash = Ax in complete manner, so in this regard we have following 2 remarks and first remark is that any solution of a given linear system x dash = Ax is stable or unstable if the equilibrium solution xt identically = 0 of the linear system is stable.

So, it means that what this remark wants is that if xt identically = 0 is a stable then, any solution is a stable and vice versa means, if any solution of a given linear system is a stable, then zero solution is also stable, so it means that here the stability behaviour is equivalent whether it is any solution or a zero solution, so it means that zero solutions will give you the stability behaviour of any solution.

So, if zero solution of linear system is a stable, then any solution of linear system is a stable and if zero solution of linear system is unstable, then any solution of the linear system is also unstable, so this is the first and very, very important remark. Now, second remark is that since here we have to calculate n quantities, because we are talking about that this phi t – psi t is mere, so it means that we have an and we are assuming that phi t and psi t are column vector function of size n cross 1.

So, it means that we have to check that n quantities like psi j t i = 1 to n are small if we are taking, considering the stability of zero solution, then we have to check that norm of psi t is < epsilon and this implies and; this implies that every component likes psi j t is < epsilon for every j = 1 to n. So, rather than checking for n quantities, psi j t, we have to; we will use not we have to; we are using one different form of norm.

Here, we are taking this norm which is known as maximum norm, so here we simply take norm of psi t = maximum of modulus of psi 1 t up to modulus of psi n t, so here we are using the supremum of say, maximum of each psi it and then we call this as norm of psi t, so if we are using this, then we can stabilise our result only in terms of this norm and if it is true for this norm, then it means that each psi it is < epsilon.

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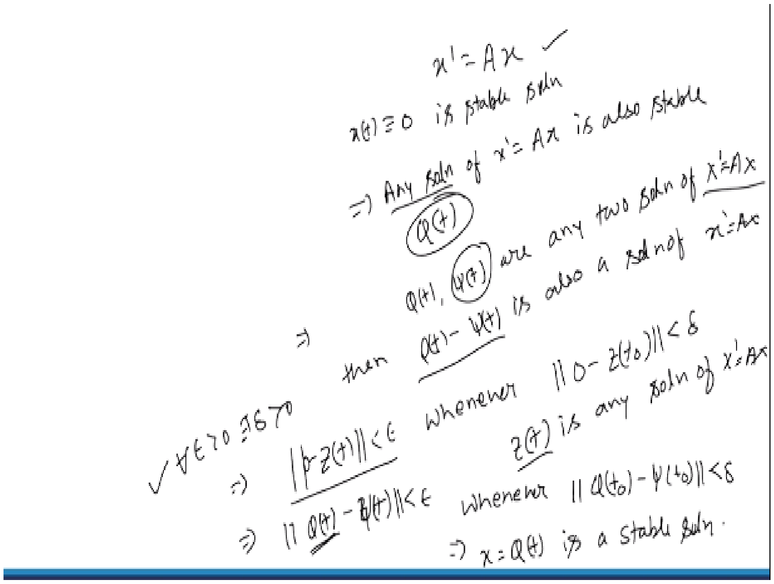

And hence we can have the required result, so first let us check the first one the let us verify this remark 1 and it says that you look at here, we simply say that we have a system x dash = A of x and we say that if xt identically = 0 is stable solution, this implies that any solution of x dash = A of x is also stable, so let us take in particular let us say any solution let us take, phi t as one of the solution and we want to say that if xt identically = 0 is stable.

Then we want to check that phi t which is the solution of x dash = A of x is also stable, now we already know that in case of homogeneous linear system, we already know that if phi t and psi t are any two solutions of x dash = A of x, then phi t – psi t is also a solution of x dash = Ax that is what we already know that even this solution set forms a vector space, right, so for x dash = A of x.

So, it means that if phi t and psi t are any two solution then their difference is also a solution of x dash = A of x. Now, let us say that zero solution is a stable means, we simply say that if you take any other solution, let us call it zt, so 0 – zt is < epsilon, whenever it is for every epsilon > 0, they exist a delta > 0, such that this 0 -; norm of 0 – zt is < epsilon, whenever 0 – zt0 is < delta, where zt is any solution of x dash = A of x.

Now, since zt is any solution and we already know that if phi t is a fixed solution of x dash = Ax for which we want to find out the stability criteria and let us say that psi t is any solution so, fix phi t and psi t is any solution, so it means that is psi t is an arbitrary solution of x dash = A of x and phi t is a fixed solution x dash = A of x, then we can say that phi t – psi t is also an arbitrary solution of x dash = A of x.

So, in particular, I can take the expression zt as phi t – psi t, so it means that this I can rewrite as 0 – zt is nothing but zt, so I will rewrite phi t – psi t is < epsilon, whenever norm of phi t0 – psi t0 is < delta, so it means that for every epsilon > 0 they exists a delta > 0, such that norm of phi t – psi t is < epsilon, whenever norm of phi t0 – psi t0 is < delta, if we look at this carefully and if we take phi t as the fixed solution for which we are searching for the which; for which we are looking at the stability behaviour.

Then this simply says that for every epsilon > 0, they exist a delta > 0 such that and norm of phi t – psi t, where psi t is any solution of x dash = A of x is < epsilon, whenever norm of phi t0 – psi t0 is < delta, so we can say that in this way phi t; x = phi t is a stable solution, so this we are getting when we are assuming that 0 is a stable solution. So, this implies that if 0 is a stable solution then phi t, where phi t is any fix solution of x dash = Ax is also a stable solution.

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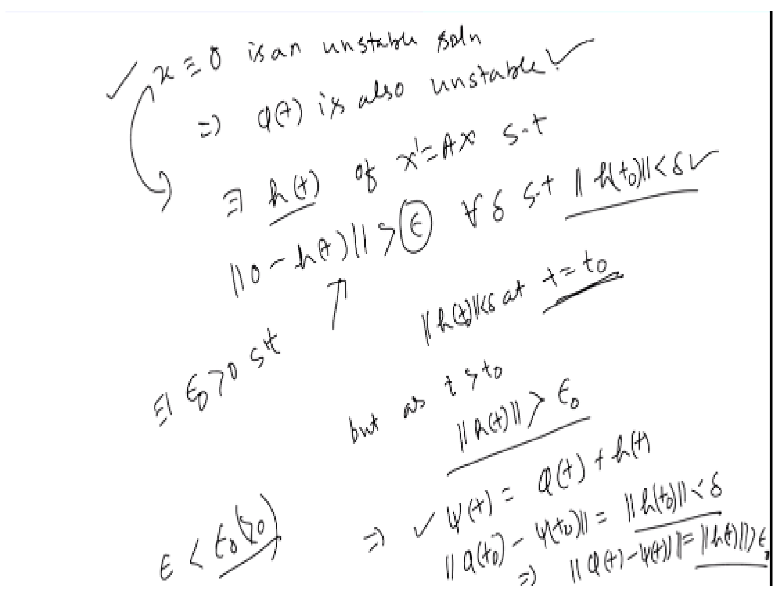

Now, we want to prove the other way round, we simply say that if 0 is unstable, if x identically = 0 is an unstable solution, then we want to prove that any solution let us say phi t is also unstable, so how we can look at; so if x = 0 is unstable solution, it means that they exist say, a solution ht of x dash = A of x, such that the 0 – h of t, so for every epsilon > 0 they exist a delta, I can find out that we have a solution ht such that norm of 0 – ht si > epsilon for every delta such that h of t0 is < delta.

So, it means that for every delta, so for you can always find out one epsilon such that norm of 0 - ht is > epsilon, whatever delta I will choose, so it means that I can always, so it means that they exist an epsilon > 0 such that norm of ht is > epsilon for every delta such that norm of ht0 is < delta, so it means that or we can write it like this that h of t is one such solution such that it is near to 0 at t= t0.

So, at t = t0, this norm of ht is small, ht0 is < delta but as t is > say t0, then that norm of ht is going to be bigger than some preassigned number, so it means that we can say that ht is now moving away to this 0, right, so that is, what do you mean by unstable solution, now we want to show that this phi t is also unstable, for that we simply consider one solution psi t as phi t + this ht, which we have taken here.

So, if we take psi t and since ht is one such solution and phi t is already we know, then psi t is one solution of x dash= Ax such that that norm of phi t0 – psi t0 which is nothing but h of t0, we already know that this is < delta but this also implies that norm of phi t – psi t which is nothing but norm of h of t and it is bigger than some epsilon, let us fix that epsilon. So, it means that if we take, let us call this as epsilon 0 here.

So, it means that they exist a epsilon0 > 0 such that norm of ht0 is < delta is true but norm of ht is always bigger than this number epsilon0, so it means that if I take epsilon< epsilon0 then the previous condition does not hold true, it means that for epsilon which is < epsilon0, I cannot find out any delta such that norm of zt is < epsilon, whenever norm of zt0 is < delta because this norm of ht is always bigger than this epsilon0, is that okay.

So, it means that for the same epsilon 0, we can simply say that we can find out a solution psi t such that psi t which is given as phi t + h of t, it means that for this epsilon which is < this epsilon0, I cannot find out any delta such that norm of ht0 is < delta, implies that norm of ht is <

epsilon, right because this norm of ht is always bigger than this epsilon0, so it means that if zero solution is unstable solution, then this implies that phi t is also unstable solution.

And this is very, very important result in the sense that rather than checking the stability of any solution, we will just check the stability of only zero solution, right so it means that if we simply check the zero solution is stable, then we say that any solution is unstable; stable and if a zero solution is unstable, then we simply say that any solution is unstable, so keeping this thing in mind let us consider the following theorem, which gives you a complete description of stability of zero solution or we can say the stability of any solution.

Because the stability of zero solution is equivalent to stability of any solution of x dash = A of x, so now let us look at the theorem.

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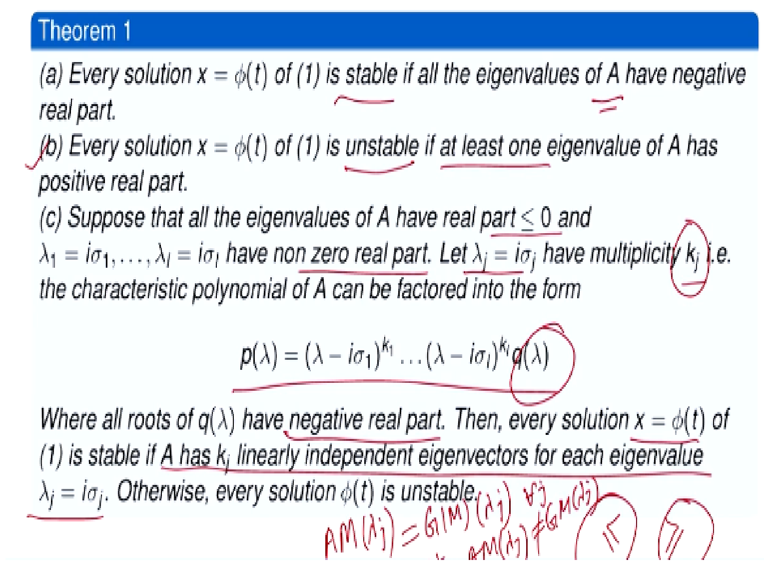

It says that every solution x = phi t of 1 is stable, if all the eigenvalues of A have negative real part and so, it means that if you calculate eigenvalues of A and it may be real, it may be complex but in any case, the negative; real part whether that is purely real or a complex value, if the real part for all the eigenvalues are negative, we simply say that solution is a stable, zero solution stable and hence any solution is a stable.

Second part of this theorem says that every solution x = phi t of 1 is unstable, if at least one eigenvalues of A has positive real part, so it means that if any of the eigenvalues of A have one positive real part, a positive real part then the solutions are all un stable, so all solutions are unstable solution, so the first part deals with the negative real part, if all this eigenvalues are having negative real part, it is done.

We have concluded that every solutions are stable solution, now the second part says that if at least one of the eigenvalues have positive real part then, all the solutions are unstable solution and the part C says that if we have the some eigenvalues with 0 real part, then what we need to consider; so C part says that suppose that all the eigenvalues of A have real part < = 0 because if it is any of the eigenvalue have positive real part, it is all unstable.

So, there is no question arise when we have one Eigen value with positive real part, then we have one certain answer that it is unstable solution but if we have say consider that all the eigenvalues of A have nonnegative, sorry non-positive real part and let us say that we consider few eigenvalues with 0 real part, so let us say that lambda 1 is i sigma 1 and lambda l = i sigma l, these are some eigenvalues with nonzero real part, sorry, zero real part.

So, let lambda j = i sigma j have multiplicity kj so, in the case when real part is 0, then we have to look at the algebraic multiplicity or geometric multiplicity, so if we say that if algebraic multiplicity is kj, it means that lambda j is repeated root of characteristic situation of multiplicity kj and the characteristic polynomial of A can be factored into this form that p lambda = lambda – i sigma 1 to power k1 * lambda – i sigma l kl q lambda, where q lambda have all the roots having negative real part, right.

Then, in this case every solution x = phi t is stable, if A has kj linearly independent eigenvector for each eigenvalue lambda j = i sigma j, otherwise every solution is unstable, so please try to understand here, in first part all the eigenvalues are negative right, so having negative real part, in second it is positive, at least one with positive real part. Now, consider equality one side, let us say equality is here.

Because if we have equality here, then we know that all solution are unstable, so there is a possibility here that solution may be stable, may not be stable, so here we consider the case when equality appears, so it means that they are eigenvalues for which we have zero real part okay, so now, let us say that your lambda 1 to lambda l are eigenvalues with 0 real part, then we have to look at the algebraic multiplicity of say lambda j and geometric multiplicity of lambda j, right.

If it is equal for each j, right then we say that solutions are stable solution, otherwise solutions are unstable, so it means that if for all j, algebraic multiplicity of lambda j = geometric multiplicity of lambda j, then the solution is stable and if they exist 1j such that this is not equal, it means that Am lambda j is != GM of lambda j, then all the solutions are unstable solutions, so this is the content of this theorem.

In fact, here a stability of any solution x dash = Ax is now purely depending on the behaviour of eigenvalues of the coefficient matrix A, so that is; that gives a very, very important result in terms of the eigenvalues of the coefficient matrix. Now, let us try to prove this, so first let us prove the first part A that all the eigenvalues have negative real part.

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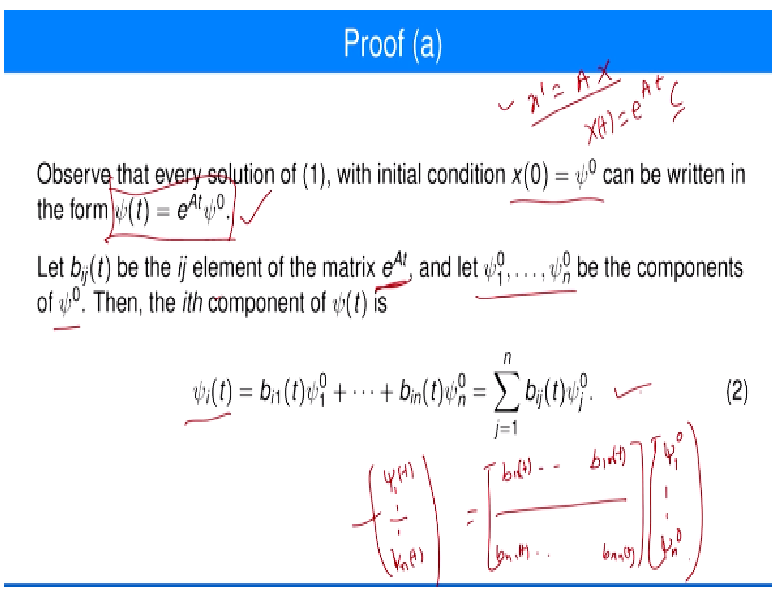

So, first thing we need to observe that every solution of 1 with initial condition x of 0 = psi 0 can be written as psi of t = e to the power At * psi of 0 and this we can done in the sense that when we have say solution; differential equation x dash = A of x, we already know that e to the power

At will serve as a fundamental matrix solution and any solution of this equation x dash = Ax can written as e to the power At * c let us call this as xt.

So, xt can be written as e to the power At * c, now depending on the initial condition we can fix this constant matrix that is c, so now we let us consider the initial condition x of 0 = psi of 0, then we can denote this particular form of the solution is given by psi t, psi t is given by e to the power At psi of 0. Now, any solution with the condition x of 0 = psi 0 is having this form that psi t = e to the power At psi 0.

Now, we want to look at how the stability of this psi t is depending on the eigenvalues; behaviour of the eigenvalues of A; matrix A, so let bij t be the ij element of the matrix, e At, so we have e AT, e At is like matrix of say solutions, it means that every column of e to the power At is a solution of x dash = A of x, not only this that each column, I mean in fact x1 to xn's are linearly independent solution of x dash = Ax, okay.

Now, let us say that psi1 0 t0 psi n 0 be the component of psi0, now we want to look at this psi t = e to the power At psi 0 in terms of component, okay, so here let me write it, this is the matrix bij t and psi 0 I am writing psi1 0 to psi n 0, so it is something like this, b11 t to b1n t and here it is bn1 t to bnn t and here we have psi1 0 to psi n 0, okay and this is your psi 1t to psi nt, so it means that if we looking at the ith component then I can write out psi it as ith row multiply by this.

It means that I can write psi it as bi1 t * psi1 0 + bi2 t * psi2 0 and so on bin t psi n 0 or I can write it like this, j = 1 to n bij t psi j0.

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Now as we have assume that all the eigenvalues of A have negative real part and let $-\alpha_1$ be the largest of the real part of the eigenvalues of A. It may be shown easily that for every number $-\alpha$, with $-\alpha_1 < -\alpha < 0$, we have a number K such that

$$|b_{ij}(t)| \leqslant Ke^{-\alpha t}, \; t \geqslant 0.$$

Therefore,

$$|\psi_i(t)| \leqslant \sum_{j=1}^{n} Ke^{-\alpha t}|\psi_j^0| = Ke^{-\alpha t}\sum_{j=1}^{n}|\psi_j^0|$$

for some positive constants K and α. Now, $|\psi_j^0| \leqslant \|\psi^0\|$.

Now, we already assume that all the eigenvalues of A have negative real part and hence we can find out the largest of the real part of the eigenvalues of A, so it means that suppose, negative real part means, suppose we have say, lambda1 – lambda 1 – lambda 2 and so on - lambda n, here I am assuming that lambda I are positive, so we have these, you just arrange in this, so somewhere here it is 0 and some way here, we have lambda n and some way we have lambda; 1 lambda 1.

So, let us look at the nearest to 0 that is call it – alpha 1, denoted like – alpha 1, so what; whoever be it may be anything, so let us say that – alpha 1 is the largest of the real part of the eigenvalues of A, so it means that it may be real or it may be complex, just look at the real part of the eigenvalues of A and let us say this - lambda 1 denote the real part of the eigenvalues lambda 1 and let me uses some other notation.

Let us use lambda 1 tilde, so these are the real part, we are looking at the real part, so let us say that –alpha 1 be the largest of the real part of the eigenvalues of A, right or in terms of if you want to look at in terms of the absolute value, then look at the absolute value here, lambda 1 tilde and lambda n tilde then this alpha 1 will be the smallest absolute value corresponding to this smallest absolute value of the real part of the eigenvalues of A.

So, now corresponding to this – alpha 1, let us choose one more number say alpha such that – alpha1 is < - alpha < 0, right, so first thing is we have all the eigenvalues arrange it in this manner, arrange the real part of eigenvalues in this manner and choose the greatest largest real negative real part as – alpha 1 and corresponding to this – alpha 1, find out one more number - alpha such that – alpha 1 is < - alpha < 0.

So, here you choose somewhere here, your – alpha, right, so it means that now corresponding to this – alpha, our claim is that I can find out a number K such that bij t is < = K e to the power – alpha t, for t is > = 0 that is very, very important observation in the sense. Now, if you look at that e to the power At is a matrix of this form, so it means that the first column is b11 t, b2 1t and so on up to bn 1t.

Now, it may happen that we have eigenvalues like we have n eigenvectors, then this is nothing but some constant time some e to the power some lambda 1 or lambda 1t but it may happen that we may not have a, n linear independent Eigen vectors then first or any of the column this of e to the power At, may be say of this form like qt times e to the power lambda it, let me write it like this, qy t lambda; e to the power lambda it, where this q it team is a polynomial in terms of t and whose degree is at most n -1.

It cannot be more than n -1, so it means that the at most the degree of the polynomial is n -1 and * e to the power lambda it and it is, it will achieve the degree n - 1 in the case when we have only say, eigenvalues lambda i up to say repeated n times and for example, if you have say identity matrix, 1 1 1, so in this case we have only one root that this 1 and it is repeated n times, in this case here it is diagnosable, it is quite obvious.

But if you look at the Jordan block corresponding to 1 1 of size n, then we can; we have only one Eigen vector, rest are all generalised Eigen vector and the solution will look like this.

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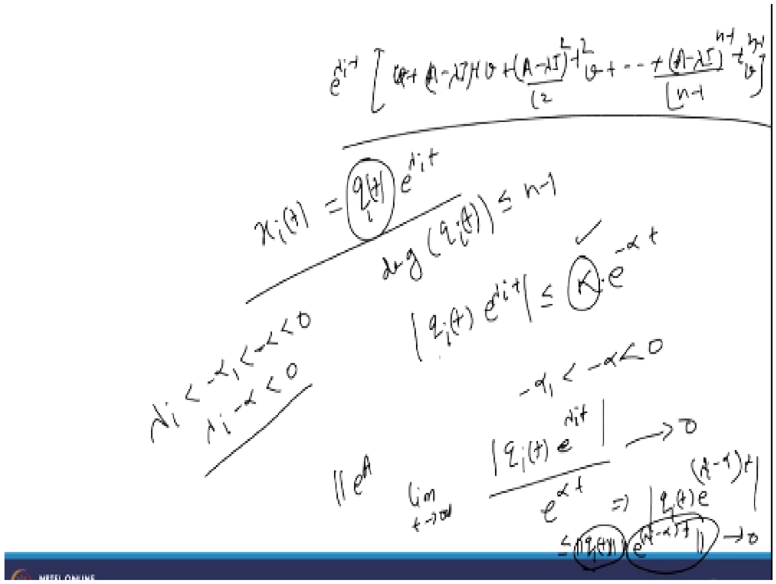

Let me write it here, solution will look like what; solution corresponding to is looking like e to the power lambda it and here it is u that is Eigen vector + A - lambda i t * v, I can write it v here, no problem, + A - lambda i square t square upon factorial 2, v and so on up to A - lambda i power n -1 t to the power n -1 upon factorial n -1 v, right and this I am calling as, we can arrange this and I can write this as qi t e to the power lambda it, we arrange; we rearrange in this manner.

And we can write it like qit e to the power lambda it, I can write it like this, so it means that each column, let me write it here xi t is of this form and maximum degree of this polynomial is going to be maximum n -1, so degree of q it is < = n – 1, okay so if it is happening and then we can say that q it e to the power lambda it and it is less than I can write it K e to the power – alpha t, so our claim is that since it is true for see this alpha, - alpha is bigger than – alpha 1.

So, my claim is that I can always write norm of e to the power At in fact, I can write it like this that norm of b ijt is < K e to the power – alpha t, so how I can write it here, my claim is that I can find out 1K such that q it e to the power lambda it modulus of this norm of this is < K e to the power –alpha t, so how we can say; this I can achieve like this that q it e to the power lambda it divided by e to the power alpha t, right and take limit t tending to 0.

So, this I can write it here is equal to norm q it e to the power, this will go, lambda i – alpha t and it is tending to 0, sorry, t tending to infinity, now if you look at what is lambda i – alpha; lambda

I's are; if we are looking at the norm here, then it is < = norm of q it * norm of e to the power, in fact it is modulus of e to the power lambda i – alpha t, right.

Now, here look at the sign of lambda i – alpha so, lambda i is < - alpha 1 and it is < - alpha < 0, so lambda i - alpha will be negative, right, so it means that if it is negative and st tending to infinity, this will go to 0 and since exponential function is dominating function over the polynomial, I will say that this limit will tend to 0 as t tending to infinity. So, if it is tending to infinity, I can always find out a function K such that norm of b it is < K e to the power –alpha t that is what we have written here.

That b ijt is < = K e to the power –alpha t, t > = 0, this we can always do this, let us again work on this and this is true, so you can consider this as an homework and let us relook it. Therefore, norm of psi it is < = j = 1 to n, now b ijt is bounded by K e to the power –alpha t, so we are writing this as K e to the power –alpha t norm of psi j 0 and this is K e to the power – alpha t summation j = 1 to n psi j 0.

For some positive constant K and alpha, now we already know that this psi j 0 is < = norm of psi 0, hear norm of psi 0 is basically maximum of psi j 0, j is from 1 to n, so each one is < = maximum of psi j 0, which I am denoting as psi j 0, so this is < = psi; each one is < = norm of psi 0.

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Hence,

$$\begin{aligned}\|\psi_i(t)\| &= \max\{|\psi_1(t)|, \ldots, |\psi_n(t)|\} \\ &\leqslant nKe^{-\alpha t}\|\psi^0\|\end{aligned}$$

Let $\epsilon > 0$ be given. Now we may choose $\delta(\epsilon) = \epsilon/nK$, so that $\|\psi_i(t)\| < \epsilon$ if $\|\psi^0\| < \delta(\epsilon)$ and $t \geqslant 0$, as

$$\|\psi_i(t)\| \leqslant nKe^{-\alpha t}\|\psi^0\| < nK\epsilon/nK = \epsilon. \tag{3}$$

Thus, the equilibrium solution $x(t) \equiv 0$ is stable.

So, in place of this I can write it here that norm of psi it = maximum of psi 1t to psi n t and we already know that for each psi it, it is < n times norm of psi0 0 K e to the power –alpha t and it is true for each I, so in particular the maximum will also be < n K e to the power alpha t norm of psi 0, right. So, it means that psi it is < this, now e to the power –alpha t is always <1, right, so I write this that norm of psi it is < = nK * o norm of psi 0.

So, now you can choose delta, since delta we have to choose, so given any epsilon, we have to choose delta such that norm of psi it is < epsilon, whenever psi it0 is < your delta, so now choose delta as epsilon/nK right, so that norm of psi it is < epsilon, if norm of psi 0 is < delta epsilon at t0 at t is > = 0 here, t0 I am choosing as 0 here, so it means that norm of psi it is < epsilon, whenever norm of psi t0, here it is 0 is < delta.

And hence we can say that the equilibrium solution x of t = identically = 0 is stable, so we have shown here that if all the eigenvalues have negative real part, then I can find out – alpha in a way such that norm of b ijt is < K e to the power –alpha t and with this we try to find out the bond of norm of psi it and hence we can show this thing that norm of psi it is < epsilon, whenever norm of psi 0is < delta.

And this shows that the zero solution is a stable solution and if zero solution is a stable, we have already shown that any solution of x dash = A x is also stable and so it means that if all the eigenvalues of A has negative real part, then zero solution is stable and hence every solution of x dah = Ax is a stable solution. So, with this I stop our discussion and in next lecture, we will discuss the; its second and third part.

And then we will also discuss some nonlinear system of differential equation and we will discuss the stability of solutions of nonlinear system, with this I stop, we will continue in next lecture, thank you very much for this.