Dynamical Systems and Control
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Lecture – 15
Stability of Linear Autonomous Systems - I

Hello friends. Welcome to this lecture. In this lecture, we will start, basically continue our discussion of queries which we have post in the beginning of study of say system of nonlinear equation. So second very important problem or question is that if we have an equilibrium solution of a system of nonlinear equation that is x dash=ftx.

Then we want to know that if we perturb our equilibrium solution, then what will happen? It means that whether the equilibrium solution will be regained or we have some completely different solution possible. Whether the solution will tend to 0 solution or solution will be unbounded solution. So that decision based on the second query we want to discuss in this particular lecture here.

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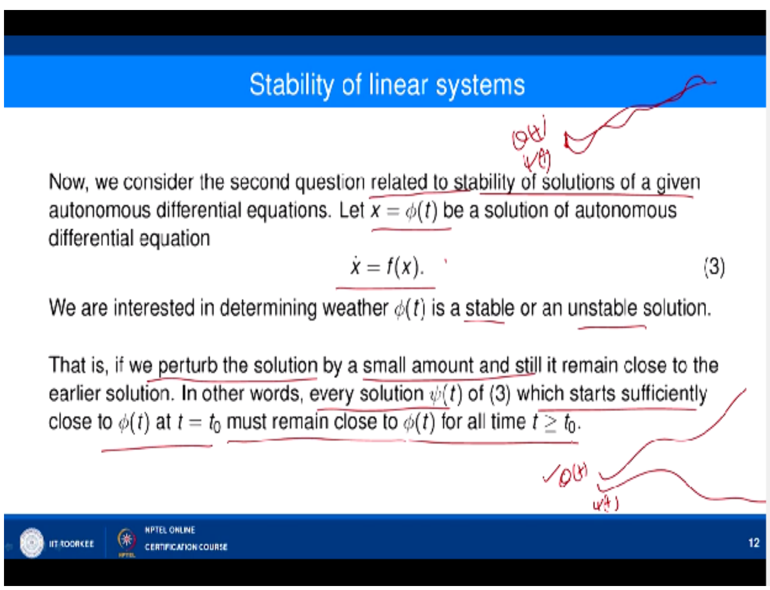

So here we consider the second question which is related to stability of solutions of a given autonomous differential equations such as say x dash=f of x. And let x=phi t be a solution of autonomous differential equation x dash=fx. And we are interested in determining whether phi t is stable or an unstable solution. So this is the problem which I wanted to discuss in this lecture.

But before that we need to understand what do you mean by stable or unstable solution.

So first let us try to consider the stable solution. We want to, suppose we perturb the solution by small amount and the solution, the new solution that is the, when we have a solution, we perturb the solution. And now we look at the future behaviour of that perturbed solution. And if the perturbed solution will remain close to the original solution, the unperturbed solution, then we say that our solution, unperturbed solution is a stable solution.

Otherwise, we call that solution as unstable solution. Or in other words, every solution psi t of the system x dash=f of x which start sufficiently close to phi t at t=t0 must remain close to phi t for all time t>t0. So it means that suppose we have a solution, say this solution. So we call this as phi t, right. Then consider a solution, say some other solution let us say psi t which start very near to this, that is only this difference.

And if it will remain almost like this, then we say that solution will be stable solution. But in place of this, if we have phi t like this and if we have a psi t like this and it will go, say away from this phi t, then we say that our phi t solution is an unstable solution rather than stable solution. So this is main intuitive idea. Let us write down the definition of the stable solution or unstable solution in a precise manner that is in terms of mathematical language.

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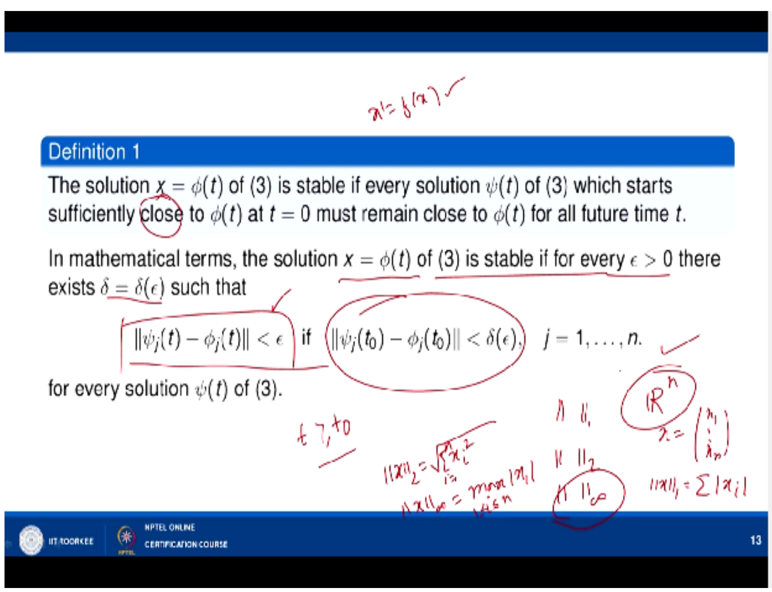

So the definition goes like this. The solution x=phi t of the system x dash=f of x is stable if every solution psi t of 3 which starts sufficiently close to phi t at t=0 must remain close to phi t for all future time t. Now here we need to define what do you mean by close, right. So to define closeness in terms of mathematical language, we need to define the distance function which we have already discussed that the distance function in case of vector valued functions is given by the norm of that function.

So here in mathematical terms, the solution x-phi t of 3 stable if for every epsilon>0, there exist a delta, of course this delta will depend on the value epsilon, such that the psi jt-phi jt norm of this<epsilon if norm of psi jt0-phi jt0<delta. So it means that if at the initial point if they are close by, then after all future time that is t greater than or equal t0, they remain close in this sense that norm of psi jt-phi jt<epsilon for all t greater than or equal t0 and for all solution, psi t of your x dash=f of x.

Then we call phi t as a stable solution. And if it is not happening for at least 1 solution of x dash=f of x, then we call our solution as unstable solution rather than stable solution. So it means that, if you want to check for say stability of 1 solution that is phi t, consider all the solutions of x dash=f of x. And if they satisfy this, then we say that our solution is stable solution, x=phi t is a stable solution.

Otherwise, if we can find out at least 1 solution psi t such that initially they are close in the sense here that norm of psi jt0-phi jt0<delta. But this second inequality may not hold. That is norm of psi jt-phi jt may not be less than epsilon. Then we call our solution phi t as an unstable solution, okay. So this is the definition in terms of norm here. Now here this norm is any suitable norm on R of n.

So here as we have discussed, we have different type of norm, norm of 1, norm of 2 and infinity norm. These are the commonly used norm we are using here. So let me write it here that if x=, let me write it x1 to xn, then norm of x1, norm of x is nothing but summation mod of xi, right. And 2 norm of x is basically it is under root of say summation xi square, that is this. And infinity norm of x is basically say maximum of mod of xi.

So depending on the situation, we will use any of these norm. But here in particular, the coming few slides, I will prefer to use this infinity norm. You can use any of the norm and so we already know a result from function as is that infinite dimension spaces like Rn, all the norms are equivalent. It means that the qualitative properties will remain unchanged. Quantitative properties may differ but qualitative properties like stability will remain unchanged, okay.

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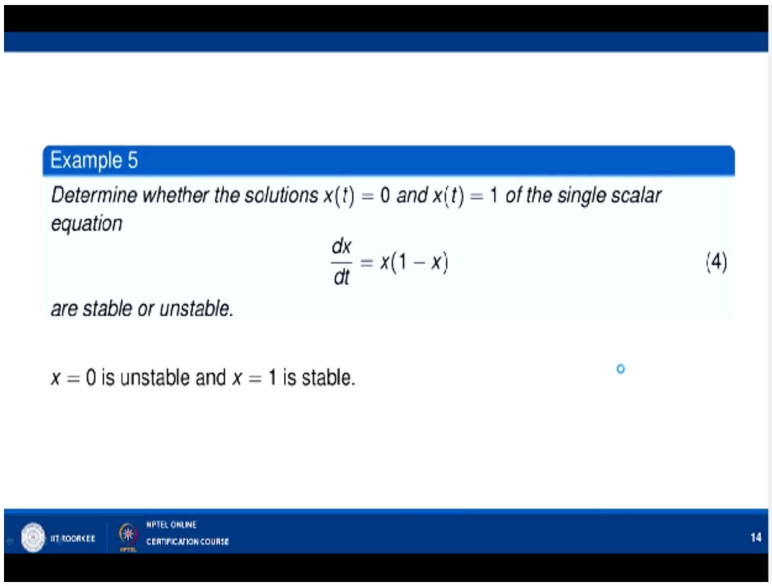

Now let us consider one problem based on this that in this problem, determine whether the solution xt=0 and xt=1 of the single scalar equation dx/dt=x*1-x are stable or unstable. So here we have say, let me write it here.

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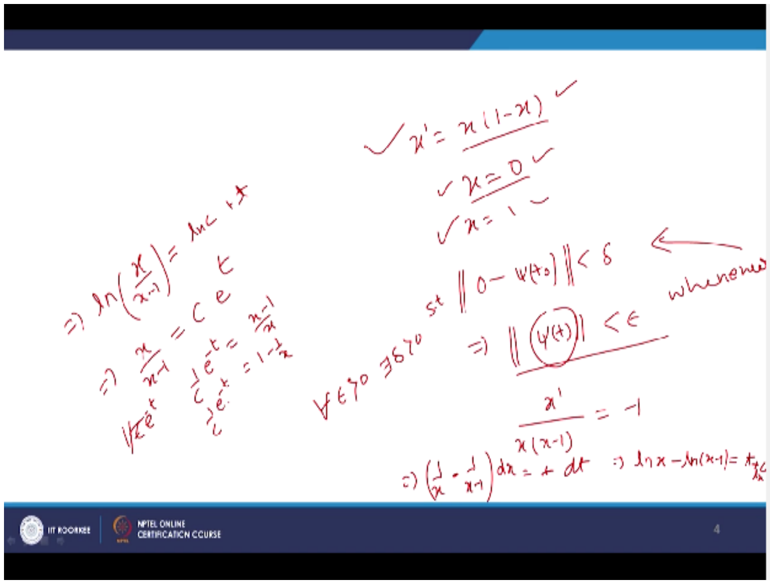

We have x dash=x*1-x and we already know that equilibrium solutions are given by the solution of x*1-x=0 or we can say that x=0 will satisfy this and x=1 will also satisfy this. So these are 2 solutions of this. And they are of course, they are more than these 2 solutions. And we want to check the stability of these 2 equilibrium solutions.

So let us first find out the solutions of x dash=x*1-x because if you want to check for the stability of x=0 solution, it means that we have to check 0-psi t0<delta implies that psi t norm, basically here it is norm, is less than epsilon. So if for every epsilon>0, there exist delta>0 such that this is happening whenever this is happening, right. So it means that we need to find out, this should be true for every other solution of the system x dash=x*1-x.

So let us first find out solutions of x dash=x*1-x so that we can work with the definition of the stability solution, stability of 0 solution. First let us talk about the stability of 0 solution, then we can talk about the stability of solution x=1. So let us first find out the solution here. So here we have x dash/x*x-1=-1 and that for simplicity, I am writing. So it is, let me write it here x and it is x-1 dx=-dt.

Now here I can write it x-1-. So it is x-1-x, it will give you -1 and it is -1. Is it okay? So it is the solution here. Let me write it here +. Is it okay? So this is the differential equation. Now let us solve this and if you solve this, we have ln x-ln of x-1=t, right. So we can simplify it further, t+some constant, let us say, you can write it say c here or let us say ln c, you can write it. So we can write down this as ln of x/x-1=say, you can write it here, c* here or you can write it here ln c*t.

So we can write down the solution like x*x-1=ce to the power t here. Is it okay? So solution can be simplified like this. Now we want to find out the solution xt, so let us simplify further. So here we have, I can write it here. So this is ce to the power -t1/c-, let me write it here. So it is 1/ce to the power -t=x-1/x, that is 1-1/x=1/ce to the power -t. Let us simplify 1/c as c. So I can write it here.

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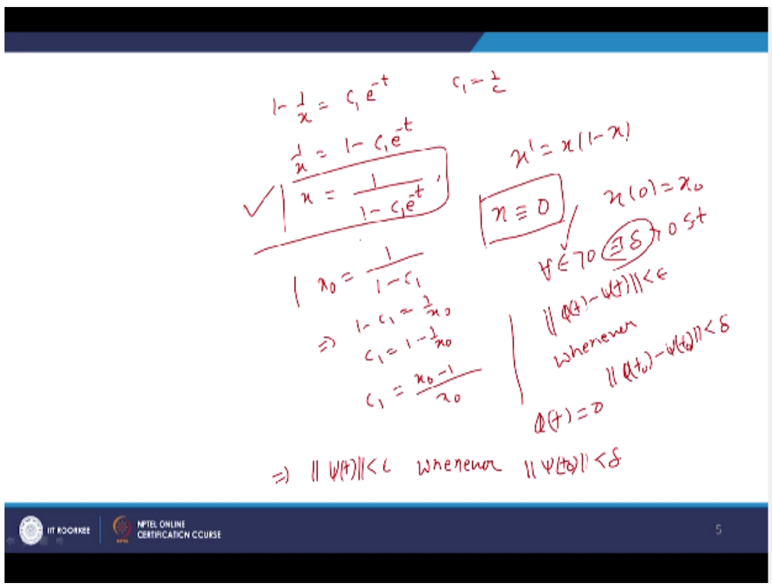

1-1/x=some c1e to the power -t. So where c1 I am writing as 1/c. So I can write this as 1/x=1-c1e to the power -t. Or I can write it x as 1/1-c1t to the power -t, right. And where c1 is some constant which can be determined using the initial condition. So this is your arbitrary solution of x dash=x*1-x here, okay. So now let us find out the behaviour of say x ideally equal to 0 solution where we want to check the stability of this solution.

So let us say that this c1 you can identify, let us say that what is the value of c1 here. Let us say that x of 0=some x0, okay. So I can find out this as, this is x0=1/1-c1. So you can find the value of c1 here from solving this equation, okay. So c1 will be what? So 1-c1=1/x0. So c1=1-1/x0. So we can write it, this is as x0*x0-1. So c1 is, you can use this relation. So it means that we are assuming that x0 is non-0.

Otherwise, we have to use some other equation, okay. So initial condition is like this, okay. So now we want to check the stability behaviour of x=0. So here we want to apply this, the definition that for every epsilon>0, there exist a delta>0 such that the, here phi t-psi t norm<epsilon whenever phi t0-psi t0<delta. Basically it is an epsilon delta game.

You give us an epsilon and we need to find out a delta such that this happen. Now here phi t is your 0 and psi t, we have already obtained like this. So it means that this reduces to what? This reduces to this that the norm of psi t<epsilon whenever norm of psi t0<delta, okay. So now look at the solution here and look at the psi t here. So your psi t is 1/1-c1e to the power -t.

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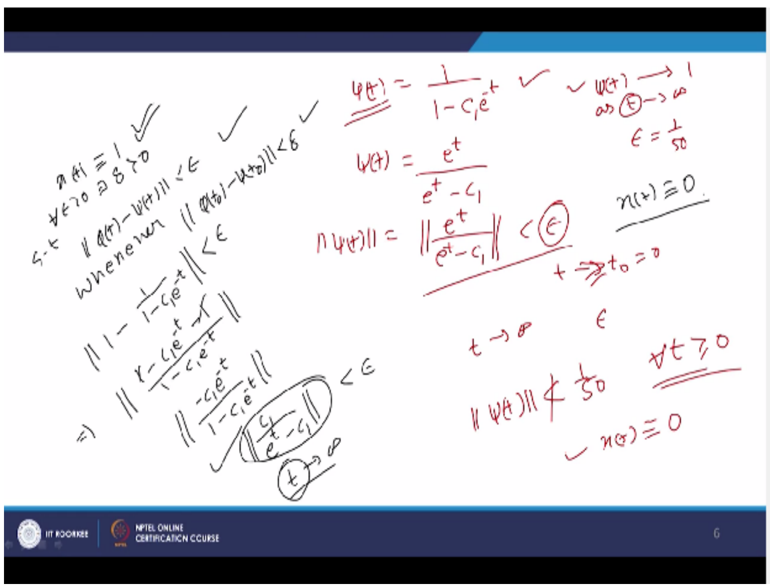

So your psi t is 1/1-c1e to the power -t. Or I can simplify it further. I can write it e to the power t t to the power t-c1 and that is your psi t. I am writing it like this. Look at the norm of psi t. So norm of psi t will be what? Norm of e to the power t/e to the power t-c1. And we want to make this quantity small, say less than epsilon and we want to see.

But if you look at here, as t tending to, say going bigger than t0 or say t is bigger than or equal to some t0 where t0 is the initial condition. Here I have taken t0 as 0. That if t is bigger than 0 and for any given epsilon, I cannot make this quantity, this inequality true. Why? Because at t tending to infinity, then e to the power t is tending to infinity. Is that okay?

So this can be, if we take, it means that I can consider like this that if epsilon, so let me write it here, that as t tending to infinity, this value psi t which is 1/1-c1e to the power -t, if t tending to infinity, then this value is tending to 0. So it means that psi t is tending to value 0, sorry, psi t is tending to 1 as t tending to infinity, right. So it means that if I choose epsilon very small compared to this one, say 1/, say, 50, then I can find out some suitable time t such that the norm of psi t is not less than this 1/50, right.

Why? Because as t tending to infinity, your psi t is tending to what? Is that okay? So it means that for given epsilon which I am choosing as 1/50, I can find out a time t such that norm of psi t is bigger than 1/50 because this should be true for all t>t0, that is 0 here. So since it is true for all t, in particular I can take very large value of t and we can choose very large value of t such that norm of psi t is bigger than 1/50, right.

So it means that here your solution, say xt ideally equal to 0 is not a stable solution because whatever be the initial condition, your solution psi t will not be bounded by arbitrary epsilon which is less than 1. So it means that xt ideally equal to 0 is not a stable solution. Is that okay? So now let us look at one more problem that is now let us consider the problem of finding the stability of xt ideally equal to 1 solution.

So in this case, we consider this for every epsilon>0, there exist a delta>0 such that phi t-psi t<epsilon whenever, you please remember this, whenever phi t0-psi t0<delta, okay. So let us first look at here. Now phi t is 1 here, psi t is given as 1/1-c1 e to the power -t and we want to make this small, that is less than epsilon. So first look at this and then try to see. Then if you look at, it is what?

Let me, 1-c1e to the power -t-1/1-c1e to the power -t, this we want to show that it is less than epsilon if it is possible. When you simplify, it is what? It is -c1e to the power -t/1-c1e to the power -t, right. Now if you divide by e to the power t, this is what? c1/e to the power t-c1. Is that okay? So now here we can simply say that as t tending to infinity, right, this e to the power t-c1 is tending to infinity. So it means that this quantity can be made arbitrary small for choosing very large or choosing sufficiently large value of t.

So it means that for any epsilon, I can choose the t such that this quantity can be made less than epsilon. Is that okay? So it means that for every epsilon, I can choose any delta in this case because here we are not getting any condition on this delta. In fact, I can choose any delta. So in this case, I can find out an epsilon such that norm of phi t-psi t<epsilon for all t>0. So in this case, this is say coming out true.

So it means that xt ideally equal to 1 is a stable solution but the xt ideally equal to 0 is not a stable solution.

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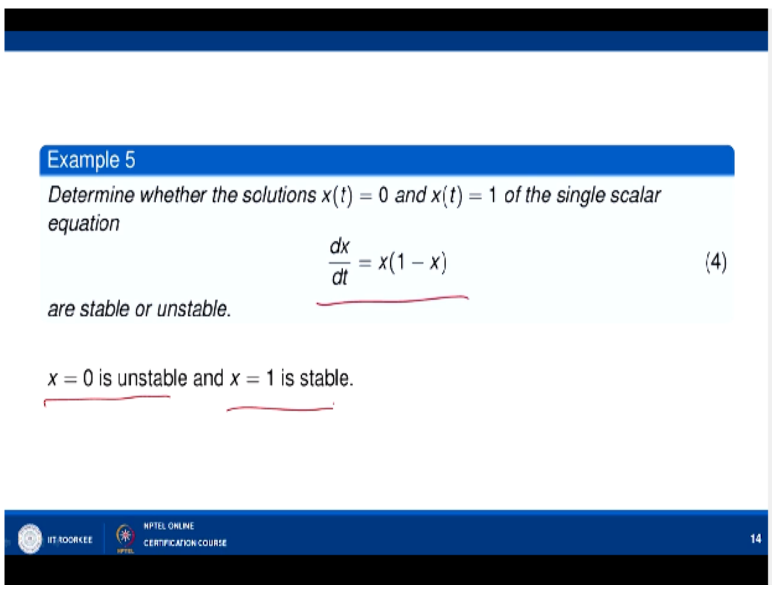

So what we have shown here that for dx/dt=x*1-x, your x=0 is an unstable solution while x=1 is a stable, okay.

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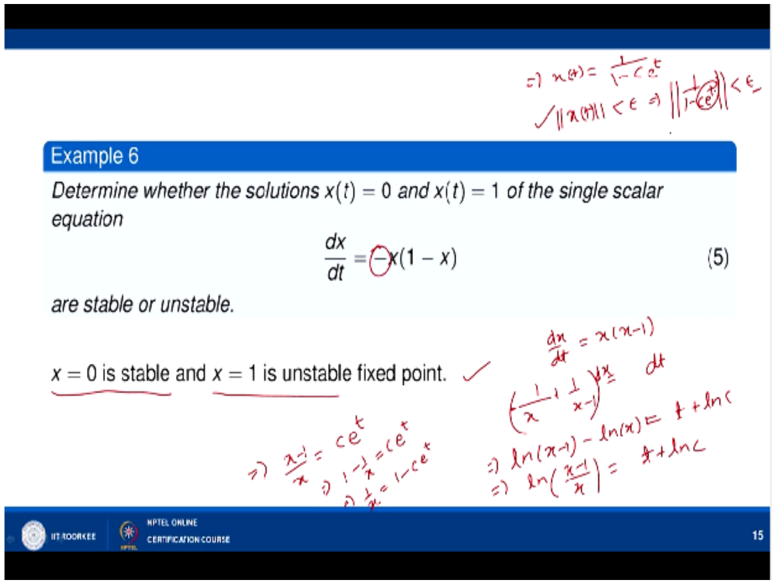

So now look at the next example. So here next example is determine whether the solution xt=0 and xt=1 of the single scalar equation dx/dt=- of x*1-x are stable or unstable. So here we have just changed this - sign. If you look at the previous problem, here it is +x*1-x and here we have -x*1-x. So if you look at, here your answer is just reverse. Here we have, look at x=0 is unstable and x=1 is stable.

But here if you look at x=0 is now stable and x=1 is say unstable which is, earlier it was stable. So just by putting - sign here, your behaviour is just reversed and how we can prove it? And this can be proven like this. I can write dx/dt=x*x-1, right. So I can write these as 1/x x-1=dt, let me write it here, it is dx. So I can write it here x-1 - here and + here. Is that okay? So I can write this as ln of x-1-ln of x, I can put =t+, say ln c.

So I can write it here as ln of x-1/x=this t+ln c and you can write here x-1/x=ce to the power t, right. And we can find out this is what? 1-1/x=ce to the power t or I can write it here 1/x=1-ce to the power t or I can simply say that this implies that xt=1/1-ce to the power t. If you look at in previous, we have what? Here we have xt=1/1-ce to the power -t. And here we have 1-ce to the power t, right.

And now I am not doing. You please carry out that here just by changing this - sign, everything will be unchanged. Let me do it for x=0 now. So for 0, I need to look at only the norm of this xt, right. And we can find out this, the constant c depending on the initial condition, I can say. So we want to show that can it be made arbitrary small. So if you look at here, the norm of this is 1/1-ce to the power t.

This we want to make arbitrary small. So as we already pointed out that as t tending to infinity, this quantity is very large now, right. So ce to the power t is going to be very large as t tending to infinity. And that makes 1/1-ce to the power t kind of a small. So as t tending to infinity, this quantity 1/1-ce to the power t is tending to 0. So it means that I can find out, for arbitrary small, I can find out the t such that this quantity can be made less than epsilon.

So it means that in this case, this is corresponding to the case of xt ideally equal to 0. So it means that in this case, 0 solution is stable solution and I leave it to you that prove that xt ideally equal to 1 is an unstable solution. So this I am leaving it to you.

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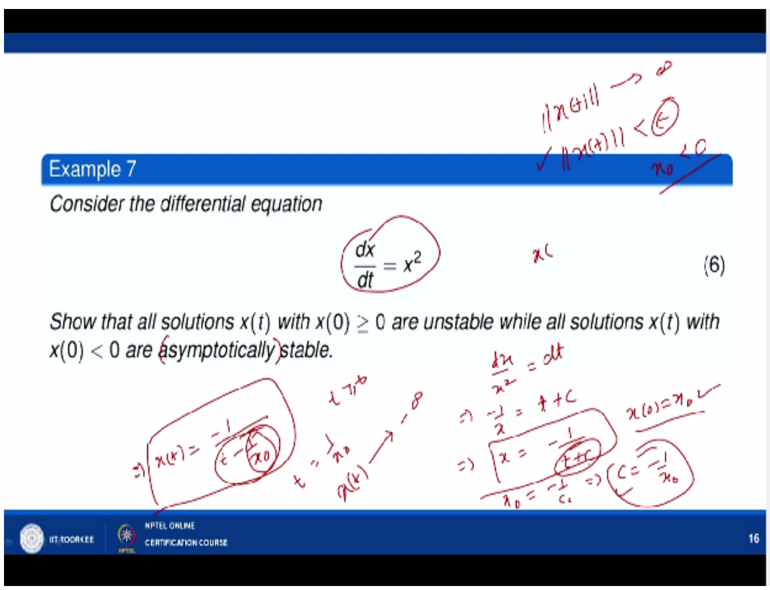

Now look at the next problem. So consider the differential equation dx/dt=x square. And in this case, we show that all solution xt with x0 greater than or equal 0 are unstable while all solutions xt with x0<0 are asymptotically stable. We need to understand what do you mean by asymptotically stable. So here right now I can understand that it is stable right now, okay. So let us simplify dx/dt=x square.

So here your stability is depending on the initial condition that if initial condition is non-negative, then it is stable. If initial condition is negative, then it is unstable solution. So this kind of a stability we can say that it is half stable in the sense, okay. So now how we can simplify this. So dx/dt=x square I can simplify find out dx/x square=dt and this is what? -1/x=t+c. If we integrate this, then I can get x=-1/t+c, right.

So this is the solution we can easily find out, just it is a separation of variable case. Now we must find out the value of c. So the x of 0 is, let us say call it x0 here. So x0, let me write it here, x0=-1/c. So this I can find out c=-1/x0. So I can simplify, I can write down the arbitrary solution of this as -1/t, +c I can write it -1/x0. Is that okay? So this is your arbitrary solution. Every solution of say equation number 6 is written like this.

Now x0 is the initial condition that is x0=0. Now we want to prove the rest. So here if you look at x of 0 greater than or equal to 0. If x0=0, then what you will get? Then the c is not defined, right. So I can say that in this case your this quantity is -infinity or we can say that xt is infinity. So it means that if x0=0, then we have only solution which I can obtain from this is an infinity solution.

But we can say that we have a, so it means that here your 0 solution is an unstable solution. Because here what I am doing? I am just finding the norm of xt, right. So it means that I am talking about the 0 solution that 0 solution is unstable solution. So we can easily check that we have a 0 solution of this. So it means that 0 solution is unstable. Now look at the case when x0 is positive.

So if x0 is positive, then c is a negative value or I can say that when x0 is positive, then this quantity is positive. So it means that since t is starting from 0, t is greater than or equal to 0, then we can simply say that in particular if t is starting from 0 and if we take the value 1/x0, since x0 is positive, so t will assume the value 1/x0 when it is starting from 0.

So it means that as t is tending to the value 1/x0, then the xt is tending to -infinity. So it means that solution is going to be unstable solution, right. So it means that when x0 is non-negative, then your solution xt is going to be very large whether it is positive infinity or minus infinity, norm of xt is tending to infinity, right. So it means that if x0 is positive or non-negative, then 0 solution is unstable solution.

Now look at the solution x0<0. When x0<0, then this c is basically positive solution. Or I can say that here this 1/x0 is negative and hence t-1/x0, it means that the whole thing is positive now. It means that for t>0, we do not have any value of t for which this can be made 0, right. So it means that or I can look at like this that here c is negative, so it means c is positive now. So it means that t+c will remain positive for all value of t and hence, x will be taking -1/t+c.

And as t tending to infinity, x will tend to the value 0, right. So it means that I can make this quantity, norm of xt can be made epsilon for arbitrary epsilon for choosing large values of t. So it means that when x0<0, for given epsilon, I can choose large value of t such that norm of xt<epsilon, right. So it means that the solution is tending to 0 for large values of t, means as t tending to infinity, solution will tend to 0.

So it means that norm of xt<epsilon, can be achievable, right. So it means that here in this case, if initial condition is non-negative, then 0 solution is unstable. And if initial condition is say negative, then solution is asymptotically stable means it is stable and as t tending to infinity, solution is tending to 0. So this is the analysis of this example 7.

And in next lecture onward, we will discuss certain result by which we can easily find out the stability of a given solution. So first we will discuss the case of linear system and then we will consider the general nonlinear system of differential equation. So that will continue in next lecture onward. With this, I will finish this lecture here itself. Thank you for listening. Thank you.