Dynamical Systems and Control
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Lecture – 14
Stability of Systems: Equilibrium Points

Hello friends. Welcome to this lecture. In this lecture, we will start working with nonlinear system of differential equation. So consider the following nonlinear system of differential equation.

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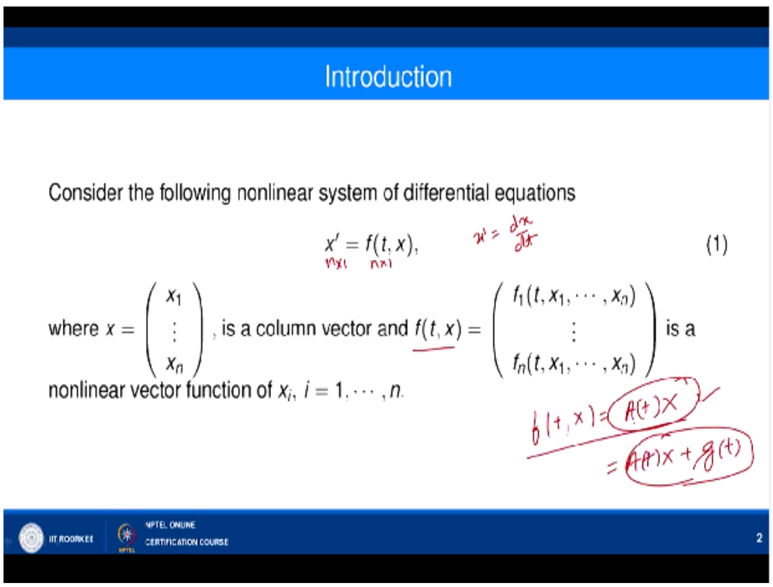

x dash=ftx. Here x dash means your dx/dt and x is a vector, column vector function x1t2 xnt and ftx is given as f1t,x1 to xn and fnt, x1 to xn. So basically it is n*1 here, it is n*1 here. So basically it represent the nonlinear system of differential equation. Here f is nonlinear in terms of its variable x1 to xn. And here we have seen a particular case of this nonlinear system that when ftx is only written as AtX, then we say that it is a nonlinear in terms of say in variables of x.

So this we have already seen. Not only this we have seen, we have also seen the following case ATX+your gt or ft. So we have discussed several methods to find out the solution of AtX and solution of AtX+gt in previous few lectures. And in this lecture, now we will focus when ftx is not reducible to any of these 2 forms, that is ftx is not a linear operator in terms of x but it is a nonlinear vector function of xi's, okay.

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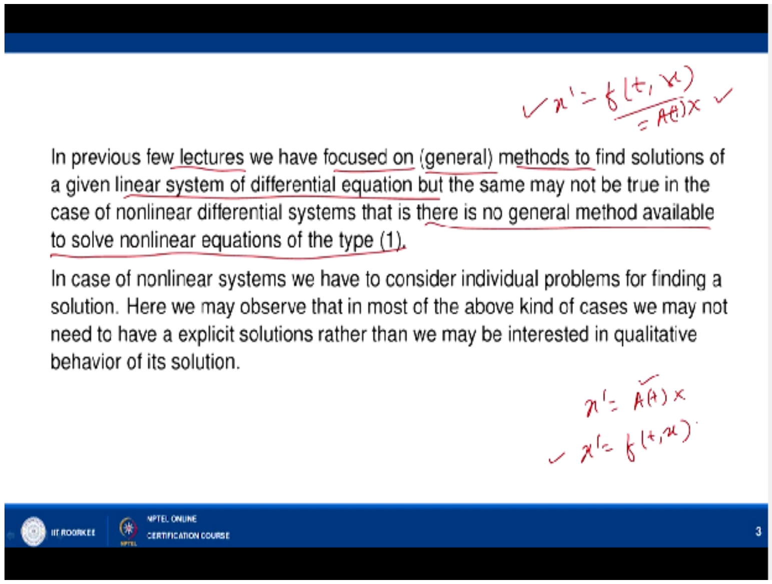

So as we have pointed out in previous few lectures, we have focused on methods, some methods, in fact, general methods to find out solution of a given linear system of differential equation but the similar thing is not applicable in case of nonlinear system that is x dash=f of t, x. The reason is that though when this ftx is your AtX, then we may have some method to find out the general solution of x dash=ATX.

But given this nonlinear system x dash=ft,x, there is no general method available to find out a solution of this nonlinear equation. In fact, here in case of nonlinear system, we have to say find out the solution depending on the given form of f. It means that there is no general method available to solve nonlinear equation of the type x dash=ft,x. So in case of nonlinear system, we have to consider individual problem rather than the general problem for finding a solution.

Here we may observe that in most of the above kind of cases, we may not need to have an explicit solution rather than we may be interested in qualitative behaviour of its solution. So it means that here when we take this problem x dash=AtX, then whatever be the form of At, whether it is a constant matrix or matrix of say depending on t, we have certain method to find out the solution of this problem x dash=AtX.

And here, we require only the condition on At that it is a continuous or it has a power series solution or it is a constant, we have a general method to find out the solution. But when we consider the nonlinear problem, then we do not have a unified method to find out the solution of this nonlinear problem. So it means that it is kind of a very say big problem to solve a nonlinear problem.

So in general, nonlinear problem may not be solvable in terms of say elementary functions. So what we can do now? So then we realize that most of the cases which are linked with these nonlinear systems, there we may not find out, we may not need the exact solution but we may require some properties of solution. For example, solution is bounded or not or solution is tending to some limit or not. For example, let us consider the population model.

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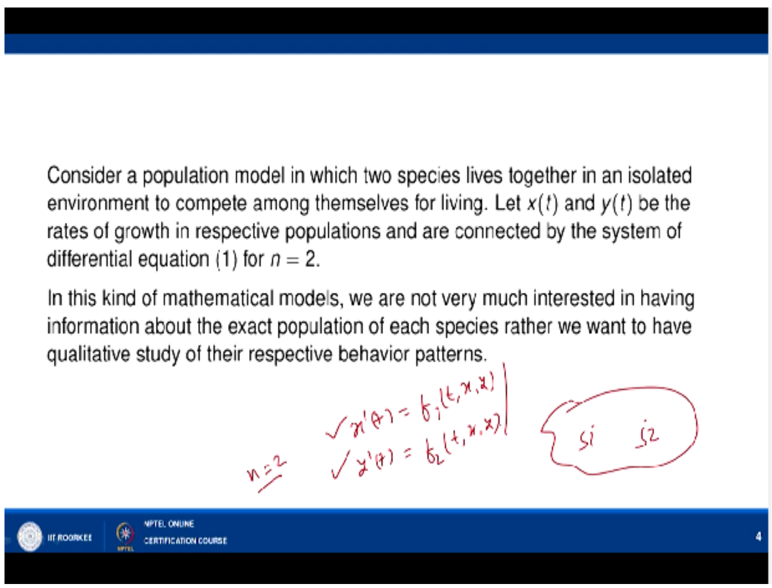

So consider a population model in which 2 species lives together in an isolated environment to compete among themselves for living. So we are taking a particular case of population model. So we have a some kind of organism and here we have 2 species, species 1 and species 2. We can write it S1 and S2. And they are kind of isolated from the outer surroundings and they are living together and we try to look at the population model based on this.

So here, let us assume that let xt and yt be the rates of growth in respective population and they are connected by the system of differential equation of the type 1 for n=2. It means that we have system like x dash t=f1tx,y and y dash t=your f2t,x,y. So it is a particular case of the previous system for n=2. Now here we may write down the particular form of f1 and f2 depending on this behaviour of S1 and S2.

For example, we may consider 1 species as prey and another species as predator. Then this may represent the prey-predator model and then depending on the assumptions we may have several model coming out of this. But we say that all these model can be categorized like this that x dash t=f1t,x,y and y dash t=f2t,x,y where x and y represent the growth in the population of x and y. Now in this kind of mathematical model which is a population model in this case.

We are not very much interested in finding the exact value of xt and yt. So here we may not say be interested that at say t=, say, 20 unit, what is the population of species 1 or species 2. Rather than we may have several other queries which are qualitative in terms of, qualitative rather than quantitative. So let us consider what are the common kind of properties we are interested in. For example, we may think about the following few possibilities.

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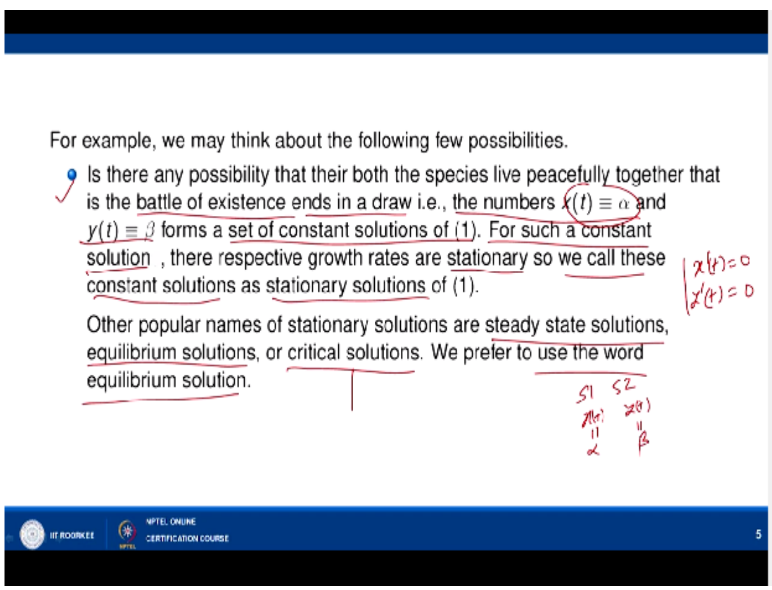

First possibility is that is there any possibility that both the species live peacefully together that is the battle of existence that they are living together and they are fighting among themselves for say space, for food and so it means that a continuous battle is started. Now we try to know that is there any possibility that both the species live together and say peacefully, in the sense that their population is independent of time t.

So it means that they are existing in kind of a equilibrium that there is no change of population of one species and in a similar way, there is no change in the population of the second species. So it means that we are interested to know that whether the battle of existence ends in a draw. It means that the number xt=alpha and yt=beta forms a set of constant solution of one. So it means that the population of species 1 and species 2 is not changing with respect to time t.

So we have a kind of an equilibrium and they are living together. So this fight is now ended in a kind of a draw. So it means that when there is no change, rate of change, then we can say that we have a constant solution of this particular problem x dash=f1 and y dash=f2. And for such a constant solution, their respective growth rates are stationary. Respective growth rate stationary means your x dash t=0 and y dash t=0.

And so rate is stationary. It means that in this situation, we simply say the constant solutions are stationary values, stationary solutions of the system x dash=ft,x. And other popular name for these stationary solutions are steady state solutions, equilibrium solutions, or critical solution. And there is another way to define this critical solution. But we are using the differential critical solution for this stationary solutions also.

And now onward, we prefer to use the word equilibrium solution, that they are living in an equilibrium position. So it means that both the species, say S1 and S2 have, say constant, say number constant population, say xt and yt. We will take some constant value that is alpha and beta here.

So it means that the first query is that whether, we are really not interested in what is the actual value of xt and yt, but rather we are interested in whether that they are living together and the rate is not changing with respect to time t. So this is our first query that whether we have an equilibrium solution or not. Now second query we may ask that suppose we have an equilibrium solution.

So it means that the 2 species living together and the population remain unchanged means population is not changing with respect to time t. And suddenly we disturb the equilibrium position. It means that suppose we add say few numbers of species 1 in that equilibrium position. So we have say environment and suppose we add few members of species 1 into that environment.

So it means that now your equilibrium position, equilibrium solution is now perturbed. Then we want to know that whether this equilibrium solution will be regained or it will disturb the equilibrium solution and ultimately, the say species 1 will be dominated or say species 2 will be dominated, that we need to consider.

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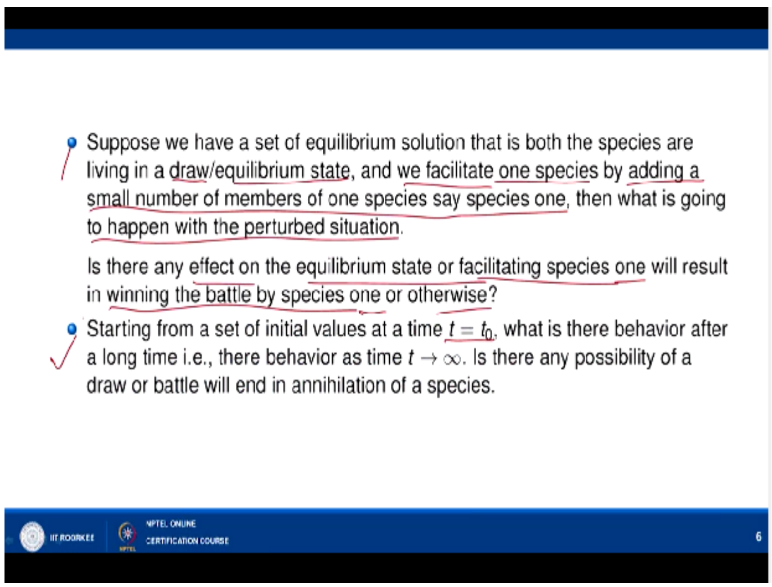

So second query we may consider like this that suppose we have a set of equilibrium solution, that is both the species are living in a draw or equilibrium state. And we facilitate one species by adding a small number of members of one species, say species 1. Then what is going to happen with the perturbed situation, right.

So we want to know that is there any effect on the equilibrium state or facilitating one species that is species 1 will result in winning the battle by species 1 or the otherwise. Why the condition may happen that the other species may also kind of win. We can consider like this that 1 is prey and other one is predator. So if we have more number of food, then there is a chance is that the other species will grow faster and it may happen that ultimately they will win the battle of the existence.

So this is our second query that if we perturb our equilibrium solutions by small amount, then whether the equilibrium solution will be regained or it will go to some other value. That whether xt will tend to 0 or yt tending to 0 or some other thing, okay. So that is our second query. And the third query is that starting from a set of initial value at time t=t0, what is the behaviour after a long time.

So it means that we have some starting point let us say t=t0 and we want to consider the same environment after say very longtime and we wanted to know the behaviour of the population of species 1 and 2. Whether this battle, long battle of existence will be say settle in one position or it is in an equilibrium or any other. So it means that it may happen that your xt will tend to 0 or both xt and yt end in a some kind of a draw.

Or it may happen that xt is very large and yt is also tending to very large value. So all these things we want to consider. In fact, the first query is related to the existence of equilibrium solution. We will see how we can consider this problem. This is very important problem. And the second condition, the perturbation problem. This problem is related to the stability of the solution of x dash=ftx.

And third one, which is related to long time behaviour is related to asymptotic behaviour of the system x dash=ftx. So 1 problem is the problem of finding the equilibrium solution. Another one is problem of stability. And the third problem is related to asymptotic behaviour of say system x dash=ftx, whta should be the behaviour at say very large time.

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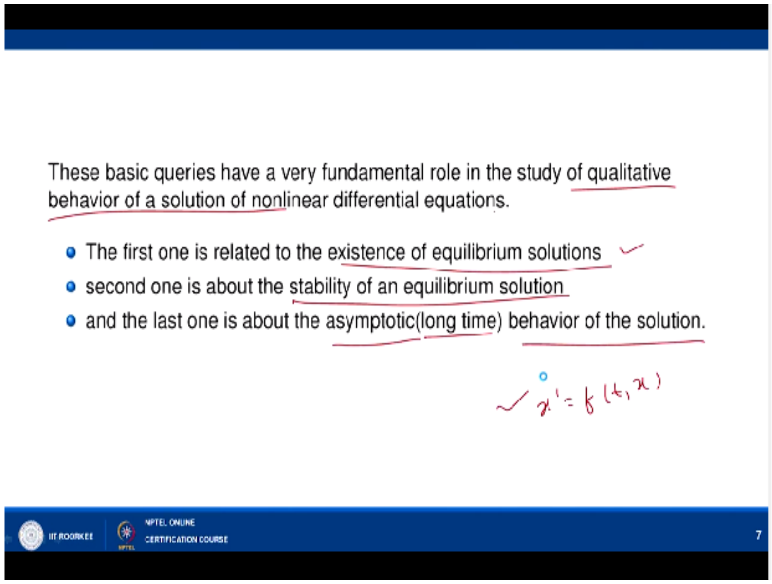

So these basic queries have a very fundamental rule in the study of qualitative behaviour of a solution of a nonlinear differential equation. So first one is related to existence of equilibrium solution. Second one is about the stability of an equilibrium solutions. And the last one is about the asymptotic or say long time behaviour of the solution.

So it means that if we summarize whatever we have discussed is the following that though we are not able to solve completely the system x dash=ftx that is defining the complete solution or finding a solution of a non-linear system may be very difficult but in that kind of situation, we are searching new qualitative behaviour of the solution of this non-linear system. So rather than going for quantitative behaviour of solution of nonlinear system, we are interested in finding the qualitative behaviour or qualitative properties of solutions of nonlinear system, okay.

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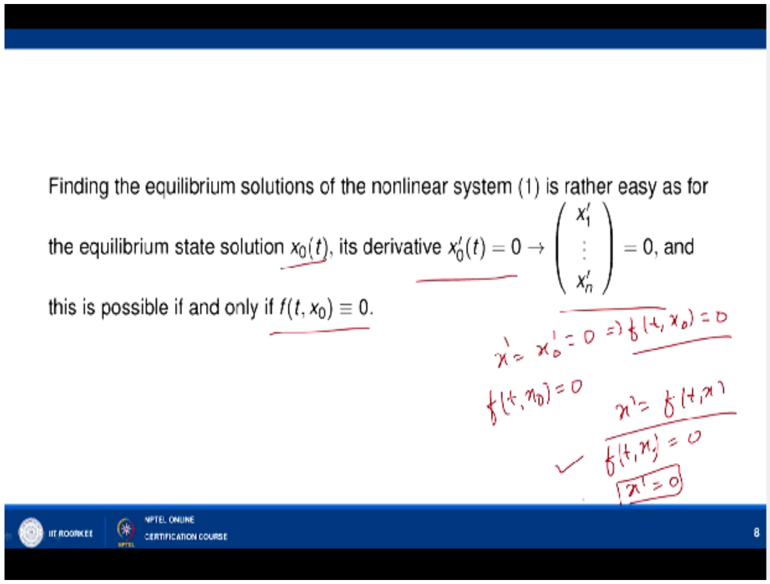

So let us say discuss this question one by one. So first problem is finding the existence of equilibrium solutions. So let us consider this problem. So finding the equilibrium solutions of the nonlinear system 1 is comparatively easy, easier than the other 2 problems. And here, this we can find out by observing this fact that at equilibrium state solution, say it is x0t, it is not depending on t.

The derivative is 0 at the constant solution x0. So derivative is 0 means x1 dash to xn dash=0. It means that you are at constant solution say x=x0, your f of t,x0 is basically 0, right. So it means that at x=x0, the derivative is basically what? It is 0 and this implies that f of t,x0=0. So we can easily find out the equilibrium solution by equating the, so if the system is given by x dash=ftx, we can find out the equilibrium solution by taking ftx=0.

And find out the solution of this nonlinear system and whatever be the solution, that will give you the solution for x dash=0. It means that they will give you the solution of stationary solution. They will give you the stationary solution or we say equilibrium solutions. So now let us consider few examples. So first one is quite a trivial example.

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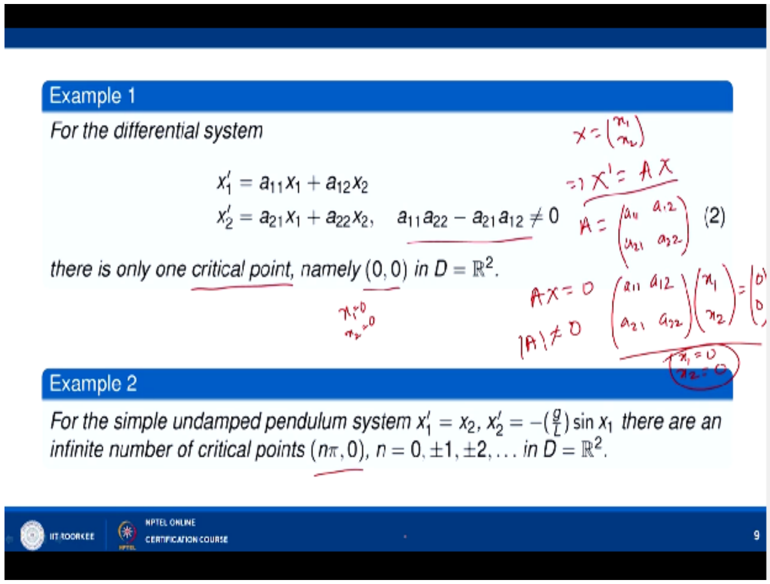

So for the differential system x1 dash=a11x1+a12x2 and x2 dash=a21x1+a22x2 and provided that the following quantity that is a11a22-a21a12 is not equal to 0 and you want to find out the equilibrium solution of this. If you look at, I can write this as say x=x1 and x2 and this can be written as X dash=AX, where A is given as say a11 a12 a21 a22 and we want to find out the equilibrium solution, means we want to find out the solution for AX=0.

And so it means that here we have to find out the solution of a11 a12 a21 a22 and x1 and x2=0. And we already know that this is a system of linear equation. And the system will have a solution provided this determinant of A is invertible or not invertible. So here it is given as a11a22-a21a12 is non-0. So it means that determinant of A is basically non-0.

So it means that in this case, A is invertible and hence, the system AX=0 will have only a trivial solution. So it means that here x1=0 and x2=0 is the only equilibrium solution. It means that these are the only points for which X dash=AX will have a stationary solution, that is X dash=0 for x1=0 and x2=0. So it means it has only 1 equilibrium solution, not solutions, solution may be more but here we have only 1 equilibrium solution, that is 0 0.

Now consider the second example that is for the simple undamped pendulum system that is x1 dash=x2 and x2 dash=-g/L sin of x1 where g is the gravitational force, L is the length of the pendulum, sin x1. There are an infinite number of critical points. So how we can find out the critical points.

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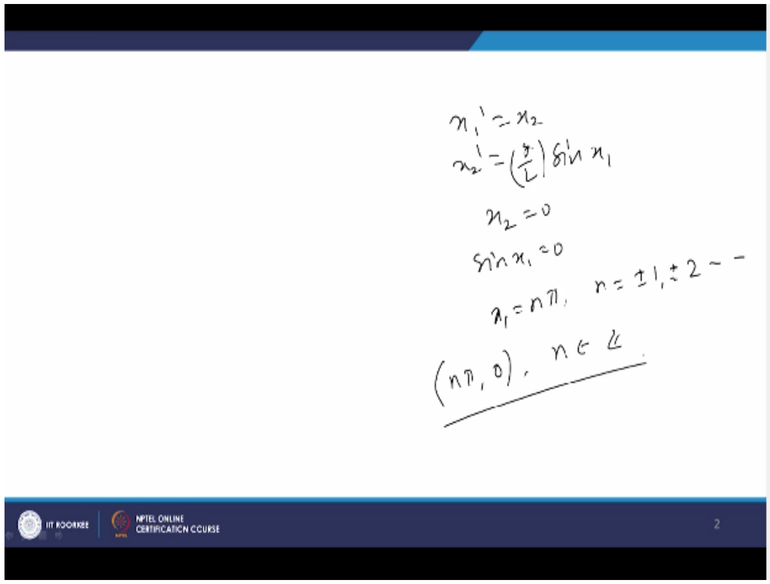

So if you look at here, here we have x1 dash=x2 and x2 dash=g/L sin of x1, right. So it means that here your x2 must be 0 and sin of x1 must be 0. So this gives you, this directly you will get x2=0. But this will be 0 provided that x1=n pi where n is your +-1, +-2 and so on. So it means that here your solution will be written as n pi and 0. So n is coming from z. So in this case, we have infinite many number of say equilibrium solutions. So in this case for the case of undamped pendulum system, we have n pi, 0 infinite number of critical solutions or equilibrium solutions.

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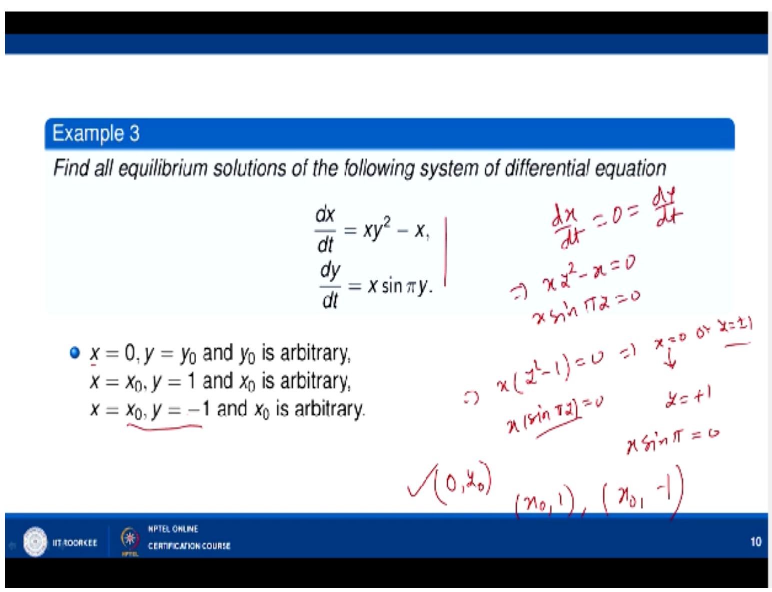

Now go on to next example that find all equilibrium solution of the following system of differential equation, that is dx/dt=xy square-x and dy/dt=x sin pi y. So to find out the equilibrium solution, we have to equate dx/dt=0=dy/dt. So when you equate this to, then we have xy square-x=0 and n sin pi y=0. If you consider the first equation, it is xy square-1=0 and second one is x sin pi y=0.

So if you look at the first problem, this will give you x=0 or y=+-1. And if you consider this x=0 as a possible case, then this second equation is automatically satisfied. Whatever be the value of y, it will be automatically satisfied. So one possible solution is 0 and y0 where y0 is any arbitrary value for this y. And other possibility is when we take y=+-1. If we take y=+1, then the first one, first equation will be satisfied that is dx/dt will be 0.

And if you look at the second equation, second equation x sin pi will also be 0. So it means that if y=1, then irrespective of the values of x, both the equation, both the dx/dt=0 and dy/dt=0 will be satisfied. So it means that the other solution will be x0,1 where x0 is any arbitrary value and similarly we can prove that if we take y as -1, then also I can take any value, any arbitrary value of x.

So it means that here for this particular problem, here we have 3 kind of equilibrium solutions. First is x=0. In that case, y can take any arbitrary value y0. So one way is that 0, y0 be one kind of a solution where y0 is an arbitrary. Another equilibrium solution is x0, 1 where x0 is an arbitrary value for this x. Similarly, x0, -1 is also be your equilibrium solution for the given system of differential equation, okay. So this is one way.

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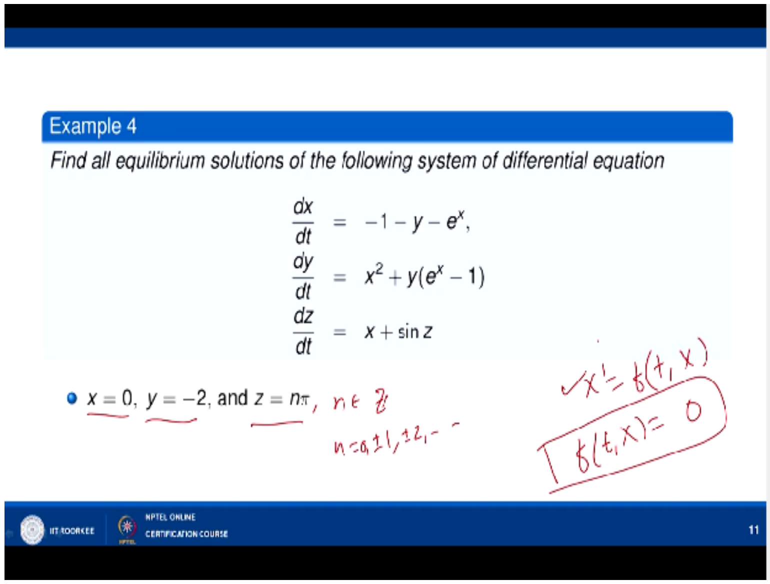

Another is find all equilibrium solutions of the following system of differential equation, dx/dt=-1-y-e to the power x. dy/dt=x square+y e to the power x-1, x+sin z. So if you look at, let me write it here.

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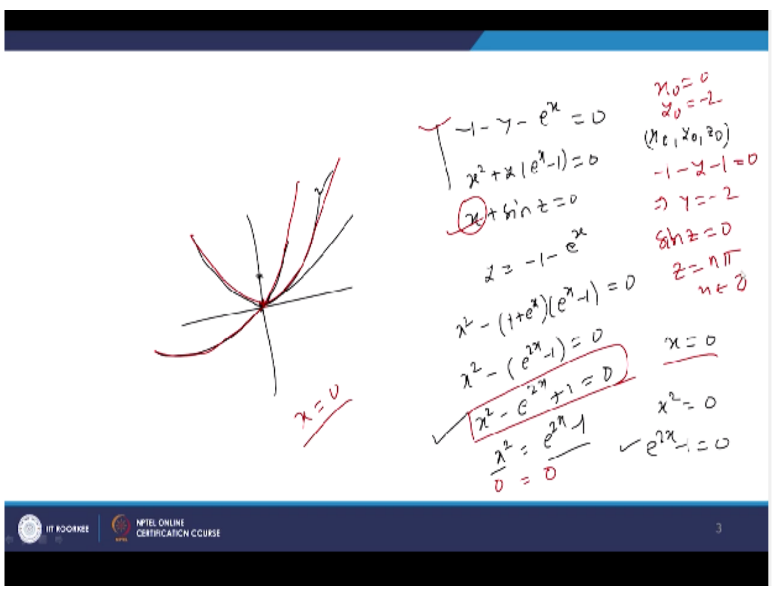

So first one is -1-y-e to the power x=0 if you want to find out the solution. Second is x square+y e to the power x-1=0. And third one is x+sin z=0. So to have a equilibrium solution, that equilibrium solution must satisfy these 3 equations. So if you look at, here it is quite difficult to find out the, say, solution x0, y0 and z0 which satisfy this. So here what we try to do?

Here if you look at the first 2 equations and from this equation, we can simplify for one say let us say x or say y. Let me write it here. I can write here the value y from the first equation. I can find out it is -1-e to the power x. So you can use the value y here and we can write x square-, you can write it 1+e to the power x*e to the power x-1=0. Or I can write it x square-, here I can write e to the power 2x-1=0.

Or we can write x square-e to the power 2x+1=0. So here if you look at, this is an equation in terms of x. But it is not an algebraic equation. It is a transcendental equation and solving this is a quite a difficult thing. But this we can solve by writing that it is x square=e to the power 2x-1. So basically the solution of this equation will be the, say, intersection of the curve y=x square and y=e to the power 2x-1, means let us say consider the 2 system, x square.

So x square is like this and e to the power 2x-1. If you consider the graph of e to the power x, it is something like this, say passing through the point 0,1. So but it is e to the power 2x-1, so it will pass through, say, here. So it is something like this that x tending to 0, it will give you what? As x tending to infinity, it will go to infinity. And x=0, this will give you value what? This will give you value 0 here.

So it is basically as x tending to -, it is tending to 0. So it is something like, this is quite large and we can consider something like this, okay. This may be this or this, we have to check, okay. So if you look at, the intersection point is basically the origin because both will pass through origin. So we can say that the intersection point is that x square=0 and e to the power 2x-1=0 or we can say that x=0 is one possibility.

And for this, this is also satisfied. So 0,0 is one intersection point of this, okay. So here as we have pointed out that these 2 curves, the x square that is this and let me write it here. So this is your e to the power 2x-1, so it will go like this. So it means that they will intersect only at one point that is the origin. So it means the curve, this x square and e to the power 2x-1 will cut at one point that is the origin, say x=0.

So for x=0, this will be 0 and this is also equal to 0. So here we have only one, say root available for this equation x square-e to the power 2x+1. So it means that x=0 will be one solution. So it means that if x=0, then from the first equation, we can find out the value for y that is -1-y-1=0. So this will give you y=-2. So x0=0, y0=-2. We have already taken. Now look at the third equation, it is x+sin z=0.

Now x=0, so sin z=0 and we know that this has infinite many solutions n pi where n is coming from the integers. So in this case, our set of equilibrium solutions are x=0, y=-2 and z=n pi where n is coming from say z, right. So it means that n is taking value +-1, +-2 and so on. So we are able to find out equilibrium solution of nonlinear system x dash=ftx by solving the system, nonlinear system ftx=0.

And the solution of this ftx=0 will give you the equilibrium solution or stationary solution of this nonlinear system. So this will solve the problem, though problem of the first query, that is related to finding the equilibrium solution of the nonlinear system. Here we will stop and now we will continue our study in next lecture. Thank you very much. Thank you.