Advanced Engineering Mathematics
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Lecture - 13
Solution of Non-Homogeneous Systems

Hello friends. Welcome to this lecture. In this lecture, we will continue our study of linear system and if you recall in previous 2 lectures, we have discussed the solution process of finding, finding solutions of the homogeneous linear differential equation. You can understand like this that in previous few classes, we have discussed x dash=A(t)x. How to find out a solution of this and x dash=A(x).

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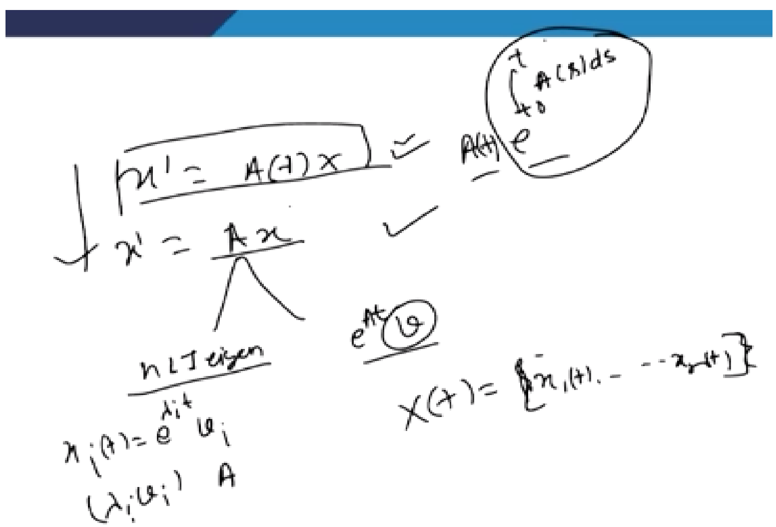

So first we have discussed the existence and uniqueness criteria and then we have found the solution method to find out the fundamental matrix solution of these 2 system. So in this, this is corresponding to autonomous system, where this matrix A is not depending on time t, then we call the system as autonomous system, while this is system is called non-autonomous system and in the case when A is having n linearly independent Eigenvectors.

Then we know how to find out a solution and we have found the solution xi(t)=e to power lambda i(t)vi where lambda vi are Eigen pair of the matrix A and when we are not able to find out n linearly independent Eigenvectors, then we have seen the process of finding the solution of the form e to power At*v and based on the type of V, we have found the solution using this concept and we have seen that how to calculate fundamental matrix solution x(t).

Where x(t) is nothing but x1(t) to xn(t) where each Xi is a solution of x dash=Ax and also for non-autonomous system also, we have discussed certain methods to find out the solution of this and we have also seen that in general this e to power t0 to t Asds will not work as a fundamental matrix of solution here. So here we have seen that under certain conditions as At and this matrix if they commute, then we can consider this as a say possible form of a solution of x dash=A(t)x.

And we have seen certain other type of solution of x dash=A(t)x. Now we will proceed further and now we will focus on non-homogeneous system of linear equation. Now consider the following linear non-homogeneous differential equation x dash=A(t)x+f(t).

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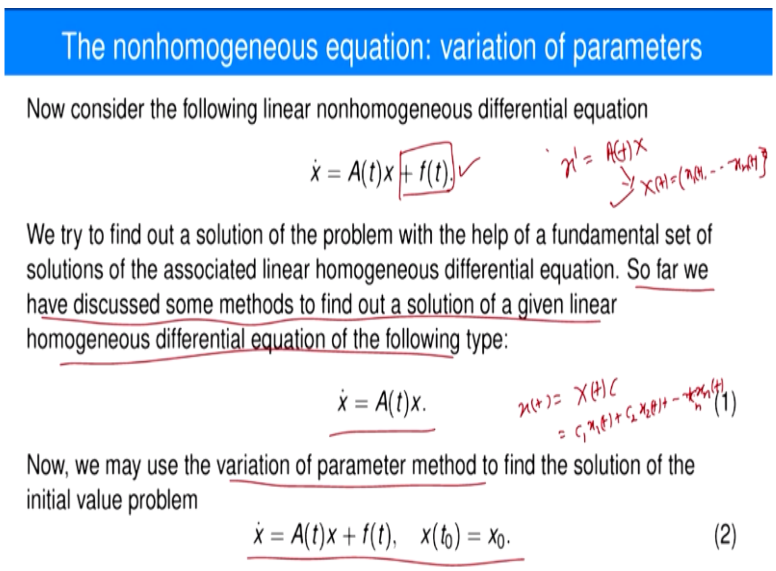

So the only is different from this 2 previous thing that here now we have one more term extra that is f(t), which is known as forward term, external forward term, we can consider that. So if we have this system x dash =A(t)x+f(t), then we call this system as linear non-homogeneous differential equation and we try to find out a solution of the problem with the help of a fundamental set of solution of the associated linear homogeneous differential equation.

So to find out the solution of this, we consider the associated homogeneous linear system, that is x dash=A(t)x and then we know the fundamental matrix solution is already available that is x1(t) to xn(t). So once we have the fundamental matrix solution for homogeneous part, then we try to find out the solution for this linear non-homogeneous differential equation, okay. Now, so far we have discussed some method to find out a solution of a given linear homogeneous differential equation of the following type.

That is x dash=A(t)x. Now we may use the variation of parameter method to find the solution of the solution of the initial value problem x dash=A(t)x+f(t) and x(t0)=x0. So what is variation of parameter method. If you recall, the general solution of x dash (t)=Ax is written as x(t)=x(t)*matrix C where C is or you can simply write it like C1x1(t)+C2x2(t)+Cn xn(t). Now here if you look at C1, C2 and Cn are all constant.

Now if you consider a very, or we can say that, you can take any arbitrary value of these constant and it will still serve as a solution. Now I can say that if C1, C2, and Cn are variable constant or you can say the parameters, then this will act as a solution of, under certain condition this will act as a solution of the non-homogeneous problem. So it means that we are varying our parameters such that this will serve as a solution of non-homogeneous problem. We will try to understand what is, I mean through this.

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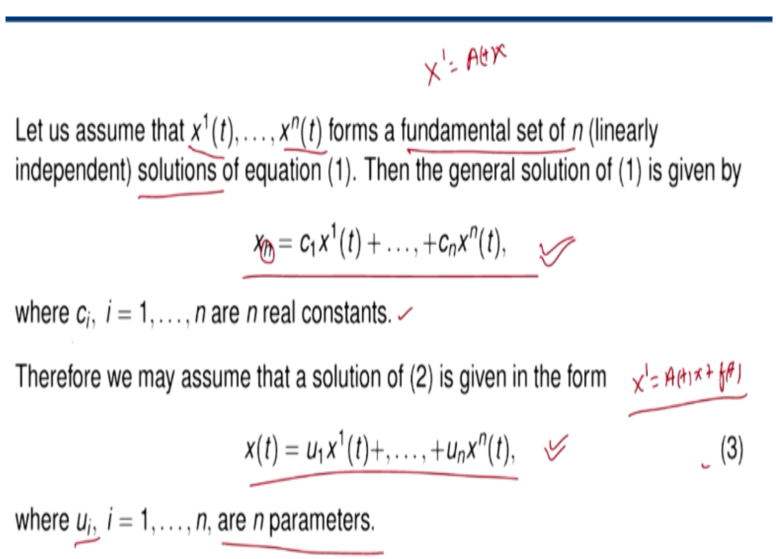

So here let us assume that x1(t) to xn(t) forms a fundamental set of n solution, fundamental set of n solution means that all these are linearly independent and they are solving the system x dash=A(t)x, right. They are solution as well as they are linearly independent. Then the general solution of 1 is given as XH, XH here means homogeneous part, so the solution of homogeneous linear equation is given as C1x1(t)+C2x2+Cnxn(t) here.

Where Ci i from 1 to n are n real constants. Now therefore, we may assume that a solution of 2 that is x dash=A(t)x+f(t), we may assume that it will the form x(t)=U1x1(t)+U2x2(t)+Unxn(t) here, where Ui are n parameters. So if you look at what is the difference between here and here, if you look at here, these C1 to Cn are constant, but here your U1 to Un are parameters. So we are basically varying these parameters so that this term 3 will act as a solution of the non-homogeneous problem x dash=A(t)x+f(t). So just look at how this will work.

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We may rewrite the above equation in the following matrix form

$$x_{NH}(t) = X_H(t)u(t),$$

where $X_H(t)$ is a fundamental matrix solution given by $X_H(t) = [x^1(t), \ldots, x^n(t)]$ and

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}.$$

$x' = A(t)x$

So we may rewrite the above equation in the following matrix on that is XNH=XH*U(t) where NH is the solution of non-homogeneous problem x dash t is the solution matrix or fundamental matrix solution given by this matrix x1(t) to xn(t). We are calling it fundamental because x1 to xn are given n linearly independent solution of x dash=A(t)x. So it means that we are writing our solution of non-homogeneous problems as XH(t)*U(t).

Now we need to find out the matrix Ut, where Ut is given as U1t2, U1t, this is a column matrix where a component of column matrix U1t to Untr, some unknown function, which we need to determine, so that this will act as a solution of non-homogeneous problem.

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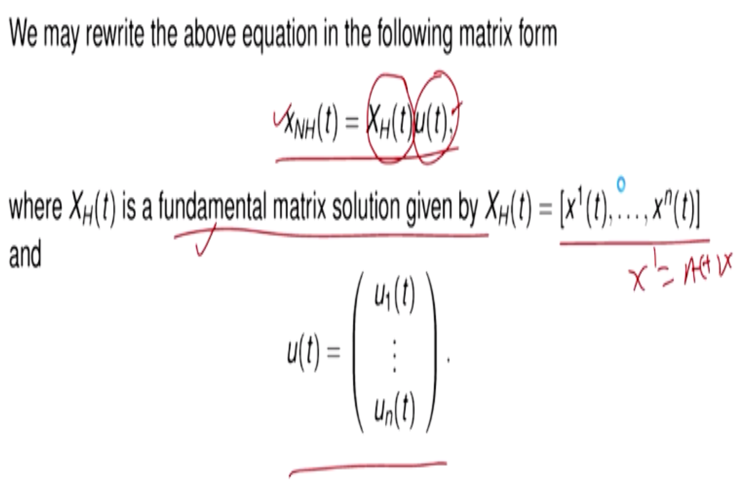

So now using the expression of x(t) in the differential equation x dash=A(t)x+f(t) and integrating between t0 to t, we have the following formula. Let us try to understand that how we got this formula 4 here. So please look at here, the solution is this part, let me look at here. So here we have x dash=A(t)x.

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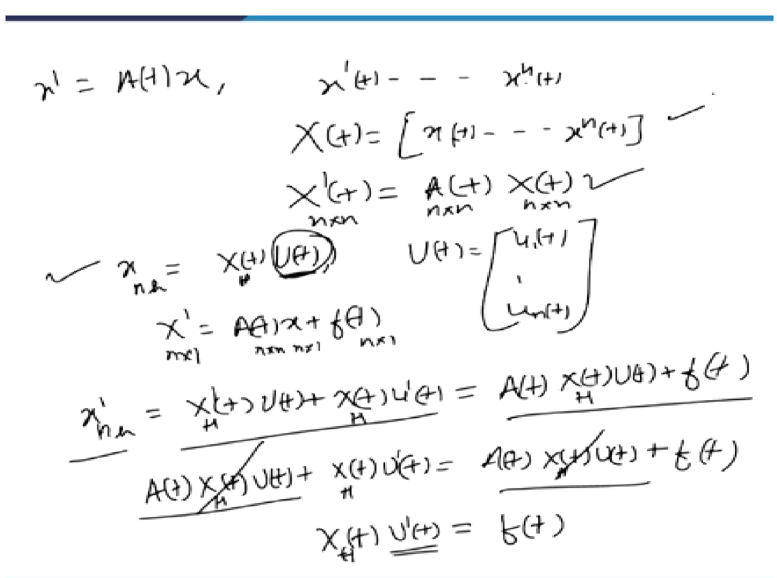

And we have solutions X1t to say Xnt, Xnt here. So here we have X1t to Xnt. So with the help of these n linearly independent solution, we form matrix like X1t2, Xnt, right. So it means that this will satisfy the following matrix differential equation x dash(t)=A(t)x(t) here. So here it is nxn, here it is nxn, here it is nxn. Now we claim that this XNH=Xt and Ut where Ut=U1t to Unt. These are some n function, which we need to find out and this is known X(h).

Now let us try to find out this Ut such that this will act as a solution of x dash(t)+f(t) here. Please note down this is nx1, this is nxn, this is nx1, so let us verify here. So if it is a solution of this, it will satisfy this, so it means that let us find out Xnh dash=x dash t and Ut+x(t) U dash(t) and we want that it should be equal to your At, X is here as x(t) and U(t)+f(t). So it means that we need to find out this Ut in a way such that x dash nh.

Which is coming out to be this must be equal to this. Now x dash(t), here it is XH. So here XH is a matrix solution of this matrix equation that is x dash(t)=A(t)x(t). So using this, we can write this as A(t)XH(t)Ut+XH(t)U dash(t)=A(t)x(t)U(t)+f(t). Now if you look at this, these 2 are equal, so we can cancel out here and what is left here is x(t) x dash t U dash t=f(t), but we are interested in finding this Ut, so we can take.

Since XH(t) is basically a fundamental matrix solution, so inverse exists, so we multiply by the inverse of this and we can write here as X inverse x(t)U dash t= sorry x(t) is here. Let me write it in a clear manner.

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$$U'(t) = X_H^{-1}(t) f(t) \qquad t_0$$

$$U(t) - U(t_0) = \int_{t_0}^{t} X_H^{-1}(s) f(s)\, ds$$

$$U(t) = U(t_0) + \int_{t_0}^{t} X_H^{-1}(s) f(s)\, ds$$

$$x_{nh} = X_H(t)\, U(t)$$

$$x_{nh} = X_H(t)\, U(t_0) + \int_{t_0}^{t} X_H(t)\, X_H^{-1}(s) f(s)\, ds$$

$$x_{nh}(t_0) = X_H(t_0)\, U(t_0)$$

$$\Rightarrow U(t_0) = X_H^{-1}(t_0)\, x_{nh}(t_0)$$

So we can multiply by the inverse of this and we can have this XHt inverse of f(t), but since it is U dash t, so let us integrate to find out Ut, so let us integrate between t0 to t where t0 is initial point and t is some point lying in some interval. So this will, when we integrate we have Ut=Ut-Ut0=t0 to t X inverse(s)f(s)d(s), right. So we need to find out Ut, so Ut is basically Ut0+t0 to t XH inverse(s)f(s)d(s).

Now Ut is known to us, so with the help of Ut, now we can find out the solution of non-homogeneous problem that x(t)*Ut. So this I can write it x(t)Ut0+t0 to t x(t) X inverse, this is XH, so XHt*X inverse(s)f(s)d(s). So this is the solution of non-homogeneous problem that is XHtUt0+t0 to t XHt X inverseh(s)f(s)d(s). The only problem is that what is the value of Ut0. So if you look I can use this equation to find out the value of Ut0.

You can find out nh at t0=XH(t0)Ut0. So we can multiply by the inverse Xt0 since X is a fundamental matrix solution, so inverse exist for all t, so in particular for t0 also, so I can find out the value of Xt0=XH inverse t0*Xnht0 and we have, okay. So we can write this and we can have our solution like this.

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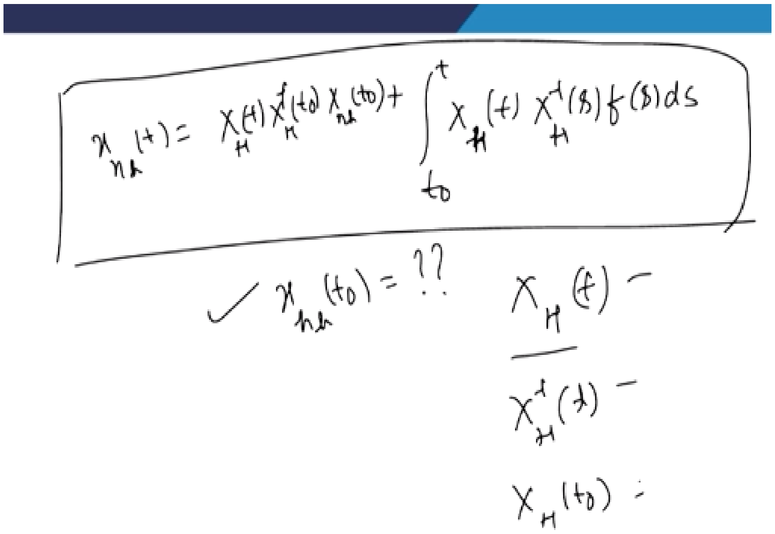

So xNH(t)=XH(t), now X inverse t0 and Xnht0+t0 to t XH inverse t. This is not inverse, it is XH(t) and XH inverse as f(s)d(s). Now it is written completely and so this will act as a, this is already known to us that XNH(t0) is known to us. It is already given with the initial value problem. So this is known to us and if you are able to find out XH(t), then we can calculate XH inverse of t also and we can calculate XH(t0) also. So all these are calculable when XH(t) is known to us.

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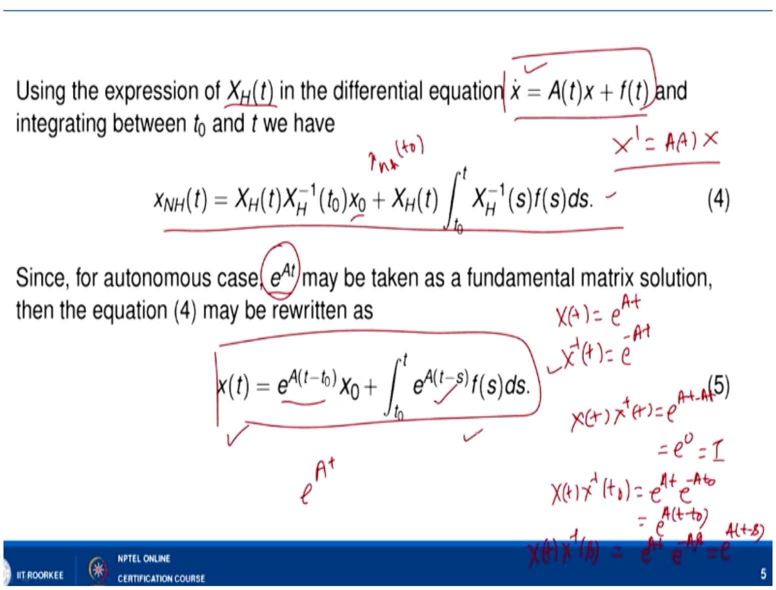

I can say that using the expression XH(t) in the differential equation x dash=A(t)x+f(t) and integrate between t0 to t, we have the following formula xNH(t)=XH(t) X inverse H(t0)x0 where is x0 is basically XNH at t0. If you look at here, it is already given here that x(t0)=x0. So using this, I can write this XNH(t0) as x0+XH(t) since it is this integration is respect to X, so I can take it out XH(t) and we can write t0 to t XH inverse as f(s)d(s).

So is when X fundamental matrix solution of this system XH=A(t)X is known to us, then we can write down the one particular solution of x dash=A(t)x+f(t) and it is given by equation number 4. Now since for autonomous case, we know one fundamental matrix solution given by e to power At. So e to power At may be taken as a fundamental matrix solution, then this equation number 4 may be written in this following manner x(t)=e to power At-t0.

This expression is I am using for XH(t) XH inverse(t0). If you look at x(t)=e to power At, then X inverse of t, you can verify that it is nothing but e to power –At that you can verify that Xt*X inverse t=e to power At and e to power –At. Now since these 2 matrix commute with each other, so we can write this as x(t) X inverse(t)=e to power At-At that is e to power 0, this is nothing but identity. So this will act as an inverse matrix of this x(t)=e to power At.

So once we know the inverse, then we can calculate x(t)*X inverse(t0) that is nothing but e to power At*e to power –At0 and we can simply write e to power At-t0. Similarly, we can calculate x(t) X inverse(s), this is nothing but e to power At e to power –A(s) that is e to power At-s, that is what is written here. So I can say that a solution of non-homogeneous differential equation is given as x(t)=e to power At-t0x0+t0 to t e to power At-sf(s)d(s).

So it means that we need to find out e to power At for the homogeneous part, then we can write down the solution of non-homogeneous part using the equation number 5. Now once we have the formula with us, let us see some example and see how we can calculate this.

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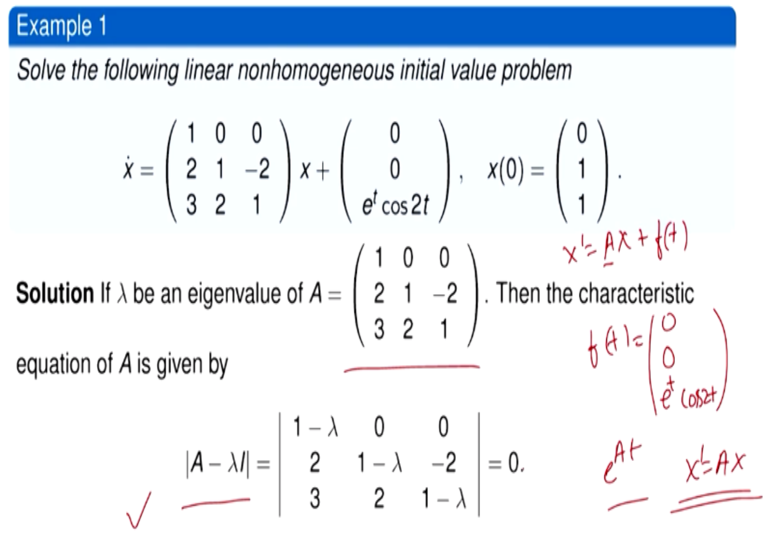

So let us consider the first example, solve the following linear non-homogeneous initial value problem x dash=1 0 0 2 1 -2 3 2 1 x+0 0 e to power cos(2t) with initial condition that x(0)=0 1 1. So I can write this as system x dash=A(x)+f(t) where A is the matrix given like this 1 0 0 2 1 -2 3 2 1 and f(t) I am writing this as 0 0 e to power t cos(2t), okay. Now we need to calculate the fundamental matrix e to power At.

So for that since it is a constant matrix, let us find out e to power At with the help of fundamental matrix of solution. So for that we need to find out 3 linearly independent solution of x dash=Ax. So for that let us first find out the Eigenvalues of the coefficient matrix A. So for that consider the characteristic equation of A that is determined of A- lambda I=0 that is 1-lambda 0 0 2*1-lambda-2 3 2 1-lambda=0. When you simplify.

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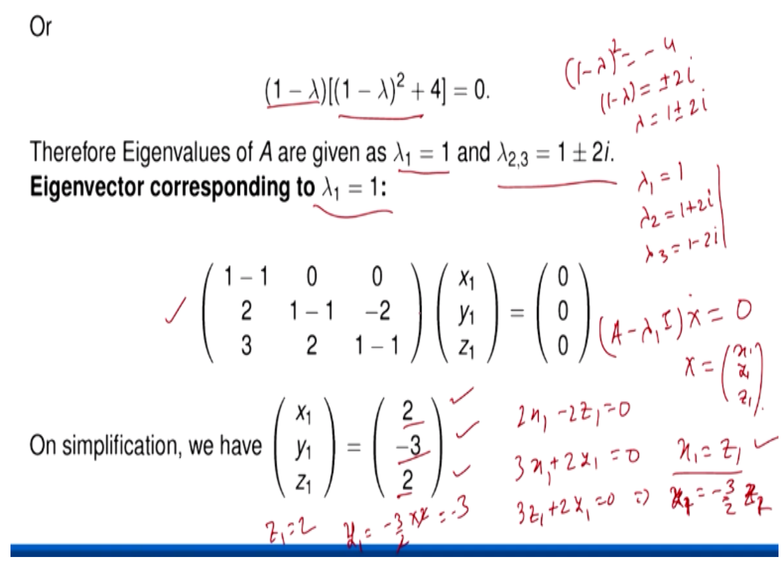

You will have the following expression that is 1-lambda*(1-lambda) square+4=0. So this will give you a simple rule that is lambda1=1 and if you simplify this, this is what 1-lambda whole square-4, but this is possible when lambda take the complex value. So 1-lambda=+/-2I. So lambda=1+/-2I. So we call this as lambda 2, 3 that is 1+/-2I here. So if you look at lambda1=1, lambda2=1+2I and lambda 3=1-2I. So if you look at all 3 are distinct.

So it means that if all 3 are distinct, it means that we have guarantee of having 3 linearly independent Eigenvectors in this case and we can find out 3 linearly independent solutions and hence we can find out a fundamental matrix solution in a quite easy manner and so first find out the Eigenvector corresponding to lambda1=1. So for that, we need to solve this A-lambda1 I*this x=0 here.

So here let us put lambda1=1 and try to solve here, where x is given as x1, y1 and z1. So it is the following equation 1 -1 0 0 2 1 -1 -2 3 2 1 -1 x1 y1 z1=0 0 0. So if you look at the first equation, first equation is nothing but 0 0 0, so it will not give any equation in terms of x1 y1 z1. If you look at the second equation, it is 2x1-2z1=0 and 3x1+2x2=0. So if you look at here we can easily find out your x1 as z1 and with the help of x1, you can put it here.

And you can write it here 3z1+2x2=0. So it means that x2 is given as -3x2x(2), sorry z1 here, sorry it is some problem here, it is y1 it is basically. So it is 3x1+2y1 and it is y1. So we can write it y1=-3/2x1, sorry 3/2z1. So now x1 can be written as z1 and y1 can be written as -3/2z1. So we can take some suitable value for z1. So let us take value z1=2, then I can write y1 as -3/2*2 that is -3 here and your x1 is nothing but z1.

So Eigenvector is coming out to be x1 that is 2, y1 that is -3 and z1 is 2. So 2 -3 2will be the corresponding Eigenvector and so we can find out linearly independent solution corresponding lambda 1=1 as e to power lambda 1 tv1.

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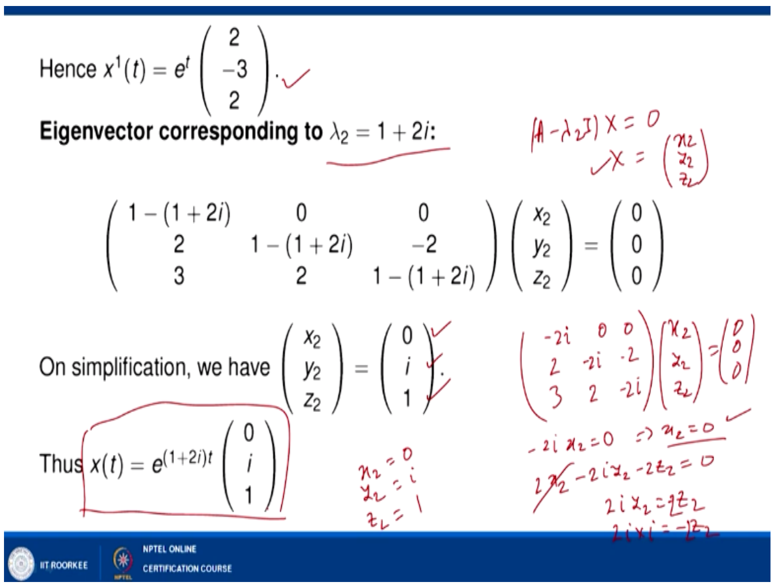

So e to power lambda 1, lambda 1 is 1, so e to power t 2 -3 2 is 1 linearly independent solution. So x1(t)=e to power t 2 -3 2 is basically linearly independent solution of this. Now second thing is find out the Eigenvector corresponding to lambda 2=1+/-2I and we have already discussed this case that when we have complex Eigenvector case, then we have Eigenvalues comes in pairs and Eigenvector also comes in pairs.

So here we know that how to find out 2 solution out of this. So let us first find out Eigenvector corresponding to lambda 2=1+2I. So we simply solve A-lambda 2I, your x=0 here. Here x let us assume that it is x2, y2 and z2. We need to find out this x. So let us put the value of lambda 2 that is 1+2I, so we can have 1-1+2I 0 0 x2 y2 z2 and so we can have this. Now when you simplify, then we have the following equation -2I 0 0 and 2 here -2I -2 3 2 -2I x2 y2 z2 and that is 0 0 0.

So one thing is very much clear here that -2I x2=0 means that your x2 has to be 0. So if you look at the second equation is 2x2 -2I y2 -2z2=0. My x2 is 0, so this part is gone, so we can write down 2I y2=z2, yeah –z2. So now I can put the value of y2 or z2 and we can find out the corresponding value of, so x2 is 0. Now put y2 as I, so if you put x2=0, and y2=I, then z2 you can find out as, it is 2I*I=-z2. So this is 2I square that is -2 and that minus will cancel out.

So z2 will come out to be, I think there is something wrong here. It is 2z2 here, okay. So z2 is coming out to be 1. So your x2 y2 z2 is coming out to be 0 I and 1. So solution, the complex solution will be given as x(t)=e to power 1+2It 0 I 1 and when you solve, this is a complex valued solution and if you solve the other way around that is 1-2I, then also you can find out the Eigenvector, just complex conjugate to this Eigenvector.

And will get a similar kind of solution for other one that is 1-2I. So first find out the real and imaginary part of this complex valued solution.

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Now

$$
\begin{aligned}
e^{(1+2i)t}\begin{pmatrix}0\\ i\\ 1\end{pmatrix} &= e^{t}(\cos 2t + i\sin 2t)\left[\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + i\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right] \\
&= e^{t}\left[\cos 2t\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} - \sin 2t\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right. \\
&\qquad \left. + i\sin 2t\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + i\cos 2t\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right] \\
&= e^{t}\begin{pmatrix}0\\ -\sin 2t\\ \cos 2t\end{pmatrix} + ie^{t}\begin{pmatrix}0\\ \cos 2t\\ \sin 2t\end{pmatrix}.
\end{aligned}
$$

So we can write e to power (1+2i)t 0 i 1 as e to power t cos2t+ i sin2t 001 +i *010 that we are writing as 0 I 1. So here we are writing the real part different and imaginary part different. So now we multiply here. So first write down the real part that is e to power 2 cos2t 001 that this thing and when you multiply these 2 term will have negative part that is sin2t 010. Similarly, we can find out the complex part that is e to power t cos2t with 010 that is we are writing here and then this with this.

So sin2t 001, so sin2t 001 e to power t we are taking out. So we can write down the imaginary and real part in this way. If you add this, what you will get e to power t and 0 and it is –sin 2t cos 2t and here imaginary part is ie to power t and then it is cos 2t sin 2t. So here this is the real part and this is the imaginary part and we have already seen that real and imaginary also forms linearly independent set of solutions. This we have not proved.

We have just proved that if xt=yt+izt is a complex solution, then yt and zt also is a solution of x dash=A(t)x(t). This we have proved, but we have not proved that yt and zt are linearly independent. In fact, we can prove in an easy manner. You simply just show that yt and zt are solutions corresponding to say different, distinct Eigenvectors corresponding to distinct Eigenvalues.

So this you try and we can prove that yt in general that if xt=yt+izt is a complex valued solution, then yt and zt are 2 real valued linearly independent solution of x dash=A(t)x(t) that you can verify. In fact, you can verify like this that xt=yt+izt, then you can find out the X bar t as yt-izt. So we can write down yt and zt in terms of xt and x bar t and then you can show that this xt and x bar t are linearly independent. So yt and zt are also linearly independent.

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Therefore

$$x^2(t) = e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix}$$

and

$$x^3(t) = e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}.$$

You try, the characteristic of the matrix is that now we are able to find out 2 new linearly independent solution x2t=e to power t0cos2t sin2t and real part as xt=0sin2t-cos2t that is this thing. So here we may observe this thing that x3t we have written as et0sin2t-cos2t, but here it is just minus different, so it is 0 –sin2t cos2t, but anyway if x3t is a solution, then –x3t is also a solution of x dash=A(t)x, okay.

With this, we can say that x2 and x3 are 2 linearly independent solution of x dash=A(t)x.

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Now $x^1(0) = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$, $x^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $x^3(0) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Therefore the solution $x(t)$ of our initial value problem must have the form

$$x(t) = c_1 e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix}.$$

So now we can write, we can check that x10 is 2 -3 2, x20 is 0 1 0 and x30 is 0 0 -1 and we can check easily that these 3 are linearly independent Eigenvectors of r3. So it means if these 3 are linearly independent Eigenvectors of r3 and x1, x2, and x3 are 3 solutions of x dash=A(t)x, then it will form 3 linearly independent solutions of x dash=A(t)x and we can write down the general solution of our initial value problem as this xt=c1e to power t 2 -3 2+c2e to power t 0 cos2t sin2t +c3e to power t 0 sin2t –cos2t.

So general solution of homogeneous problem is written like this. Now with the help of this, we can form the fundamental matrix of solution.

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A fundamental matrix solution of $\dot{x} = Ax$ may be given as

$$X(t) = \begin{pmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t \cos 2t & e^t \sin 2t \\ 2e^t & e^t \sin 2t & -e^t \cos 2t \end{pmatrix}.$$

That is this is your x1t, this x2, it is x3. So we can write down the fundamental matrix of x dash=Ax may be given as x(t)=2et-3et 2 et 0 to e to power t cos2t sin2t 0 e to power t sin2t –e to power t cos2t and once your x(t) is given to us and we have already checked that it is, these 3 solutions are linearly independent solution. So x(t) is an invertible matrix, that is non-singular matrix.

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Observe that $X(0) = \begin{pmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$ and $det(X(0)) = -2 \neq 0$, therefore X is an invertible matrix and its inverse matrix X^{-1} is given by

$$X^{-1}(0) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Hence, we can calculate the fundamental matrix solution e^{At} as follows:

$$\begin{aligned} e^{At} &= X(t)X(0)^{-1} \\ &= e^t \begin{pmatrix} 1 & 0 & 0 \\ \frac{-3}{2} + \frac{3}{2}\cos 2t + \sin 2t & \cos 2t & -\sin 2t \\ 1 + \frac{3}{2}\sin 2t - \cos 2t & \sin 2t & \cos 2t \end{pmatrix}. \end{aligned}$$

So we can observe the value of X(0), which is coming as 2 -3 2 0 1 0 0 0 -1 and we can check that determinant of X(0) is coming out to be non-zero and hence we can find out the inverse of X(0) that is 1/2 0 0 3/2 1 0 1 0 -1. You can find out the inverse using your properties of matrix and you can easily find out the inverse of X(0) and it is given as X inverse(0). So idea is to calculate e to power A(t), the fundamental matrix solution e to power A(t) is now calculated as x(t)x0 inverse.

We are writing e to power t x(t) is already given to us, X inverse(0) we have just calculated. So we multiply x(t) and X(0) inverse and we have the following solution, following matrix as a fundamental matrix solution e to power A(t).

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Consequently

$x(t)=e^{A(t-t_0)}x(t_0)+\int_{t_0}^{t} e^{A(t-s)} f(s)\,ds.$

$$x(t) = e^{At}\begin{pmatrix}0\\1\\1\end{pmatrix} + e^{At}\int_0^t e^{-s} \tag{6}$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ \frac{-3}{2}+\frac{3}{2}\cos 2s+\sin 2s & \cos 2s & -\sin 2s \\ 1+\frac{3}{2}\sin 2s-\cos 2s & \sin 2s & \cos 2s \end{pmatrix} \times \begin{pmatrix}0\\0\\e^t\cos 2s\end{pmatrix} ds \tag{7}$$

Hence we get

$$x(t) = e^t\begin{pmatrix}0\\ \cos 2t-(1+\frac{1}{2})\sin 2t \\ (1+\frac{1}{2})\cos 2t+\frac{5}{4}\sin 2t\end{pmatrix}.$$

So now e to power A(t) is given to us, then we can use our formula to find out the solution of non-homogeneous problem that is x(t)=e to power A(t-t0)+t0 to t e to power A(t-s)f(s)d(s). So here e to power At is already known to us. T0 is coming out to be 0, so we can write e to power A(t-t0) as e to power A(t)*this is x(t0). So x(t0) is 0 1 0 +e to power At0 to t. Now we are writing e to power –A(s) as e to power At is given here, then e to power –A(s) is simply replace t by –s and we can write down e to power –A(s) and we can write it like this.

So here you can check that it is actually written as e to power A-s and we can have this. And multiply by f(s) that is 0 0 e to power t cos2sf(s) we are writing here. We can multiply this expression into e to power –s, you can integrate with respect to 0t and then multiply e to power At and we simplify and we can have this as a solution x(t)=e to power t 0 cos2t -1+1/2 sin2t and 1+1/2 cos2t+5/4 sin2t and this process is quite lengthy and tiring, so we can see that even though we are able to find out e to power At, but this calculation is quite lengthy, I will say.

So finding the solution of non-homogeneous problem using variation of parameter method is quite difficult. So another alternative method available to find out the solution of non-homogeneous linear system is the use of nice guess. So if you look at f(t) is 0 0 e to power t cos2t, so it means that here you can say that this we can find out the possible way of solution is something like alpha*e to power t cos2t+beta*e to power t sin2t.

And then we try in to find out alpha beta in a suitable manner such that it will satisfy this non-homogeneous problem. This is something, which you do for the scalar valued function. So you can see that both are giving you the same result, that is given here as this. Now let us consider one more example, so that this can be simplified in a better way.

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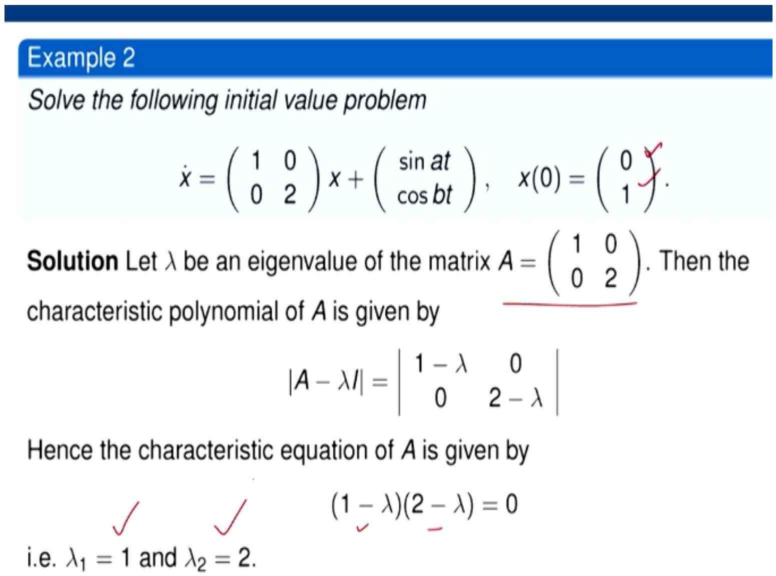

So let us consider 2x2 matrix case, where this can be further simplified. So solve the following initial value problem x dash=1 0 0 2 x+sin at cos bt, initial condition is given as x(0)=0 and 1 here. So forcing to find out e to power At at fundamental matrix solution of x dash=Ax where A is given as 1 0 0 2. So we have taken very simple case. If you look at this, nothing but diagonizable case. So we can find out the Eigenvalues in a very easy manner.

That is nothing but the diagonal entry of this matrix. So lambda 1=1 and lambda 2=2 Eigenvalue of matrix A.

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Eigenvector corresponding to $\lambda_1 = 1$:

$$\begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

On simplification, we have an eigenvector $v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Eigenvector corresponding to $\lambda_2 = 2$:

$$\begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

On simplification we have $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So we can find out the Eigenvector corresponding to lambda 1=1 that is coming out to be 1 0, so we can simply write it here that y1 is coming out to be 0. So x1 you can take arbitrary value. In particular, I am taking value as 1. So Eigenvector corresponding to lambda 1=1 is 1 0. So we can write down the first solution x1(t)=e to power t 1 0. Similarly, we can find out the Eigenvector corresponding to lambda 2=2 and you can see that your x2 is coming out to be 0 here.

So y2 is, you can take any arbitrary value, we have taken the particular value that is 1. So Eigenvector corresponding to lambda 2=2 is given as 0 1. So your x2(t) is given as e to power 2t and 0 and 1. So x1t and x2t are coming out to be very easily.

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Then a fundamental matrix solution of $\dot{x} = Ax$ may be given as

$$X(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}.$$

Since $X(t)$ is a nonsingular matrix, we can calculate its inverse as:

$$X^{-1}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the fundamental matrix solution e^{At} is given by

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}$$

And we can write down the fundamental matrix solution as x1t, x1t is et 0 and x2t is 0 e2t, right. So that we have written here. So this we can write e to power t0 and this we can write it 0 e to power 2t. So with the help of x1t x2t we have written xt as et 0 0 e to power 2t and we can write down x(0) is nothing but 1 0 0 1. So it is an identity matrix, so it is invertible. So xt is a fundamental matrix solution.

So we can find out the inverse of this and since it is identity matrix inverse is coming out to be identity equal. So e to power At is nothing but the xt*x inverse 0, so that is coming out to be xt itself. So e to power At is coming out to be e to power t 0 0 e to power 2t. So now we can find out the inverse of this.

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Consequently

$$
\begin{aligned}
x(t) &= e^{At}\begin{pmatrix}0\\1\end{pmatrix} + e^{At}\int_0^t \begin{pmatrix} e^{-s} & 0\\ 0 & e^{-2s}\end{pmatrix}\begin{pmatrix}\sin as\\ \cos bs\end{pmatrix} ds\\
&= e^{At}\begin{pmatrix}0\\1\end{pmatrix} + e^{At}\int_0^t \begin{pmatrix} e^{-s}\sin as\\ e^{-2s}\cos bs\end{pmatrix} ds\\
&= e^{At}\begin{pmatrix}0\\1\end{pmatrix} + e^{At}\begin{pmatrix} \frac{e^{-s}}{(1+a^2)}[-\sin as - a\cos as]\\ \frac{e^{-2s}}{(4+b^2)}[-2\cos bs + b\sin bs]\end{pmatrix}_0^t
\end{aligned}
$$

And inverse of this is nothing but so e to power –As you can write simply as e to power –s 0 0 e to power -2s, right. Now we can write down the formula here. Formula is this, xt e to power At, initial condition x(0) is 0 1+e to power At 0 1 0 t e power –s 0 e0 e to power -2s sin as sin bs ds. This is as it is. Now e to power At we have taken out 0 t, if you multiply what you will get, it is nothing but e to power –s sin as and this is nothing but e to power -2s cos bs ds.

So we can integrate this is nothing but e to power At 0 1 +e to power At and 0 t and now rather than we can take integration of this matrix is nothing but integration of each component. Now you can find out the integration of e to power –s sin as and e to power -2s cos bs and it is coming out to be this that e to power –s/1+a square-sin as-a cos as. Similarly, we can find out 0 to t e to power -2s cos bs as e to power -2s 4+b square-2 cos bs+b sin 2s 0t. So now we just plug in the limits here and we can plug in the limits.

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$$
\begin{aligned}
&= \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} + \begin{pmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{pmatrix} \\
&\quad \begin{pmatrix} \frac{e^{-t}}{(1+a^2)}[-\sin at - a\cos at] + \frac{a}{1+a^2} \\ \frac{e^{-2t}}{(4+b^2)}[-2\cos bt + b\sin bt] + \frac{2}{4+b^2} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} + \\
&\quad \begin{pmatrix} \frac{1}{(1+a^2)}[-\sin at - a\cos at] + \frac{ae^{t}}{1+a^2} \\ \frac{1}{(4+b^2)}[-2\cos bt + b\sin bt] + \frac{2e^{2t}}{4+b^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{(1+a^2)}[ae^{t} - (\sin at + a\cos at)] \\ \frac{1}{(4+b^2)}[(6+b^2)e^{2t} + (-2\cos bt + b\sin bt)] \end{pmatrix}
\end{aligned}
$$

And we can have like 0 e to t that we have just calculated e to power At*0 1 that is coming out to be 0A2t+this is e to power At*this, this we have plug in the value. So here when we multiply and we have the following value e to power –t 1+a square –sin at –a cos 2at+this thing and in this we have plug in the limits of 0t and when you simplify you have this solution. This is quite lengthy, I will say, though it is 2x2 case, but still it is quite lengthy.

We say that though it is a very good method and you can find out the solution of almost all the problems, but it is quite lengthy. So I suggest that you can also consider the solution system which involves the guessing, say finding the undetermined coefficient method of undetermined coefficient you can try finding the solution of non-homogeneous linear system. So with this I end this lecture.

We will continue our study in next lecture. So in this lecture, we have discussed the solution of non-homogeneous linear differential equation with the help of solution of homogeneous linear differential equation and we have used the method of variation of parameter method in this lecture and also observed that though this process is quite lengthy, but it is quite reliable and you can find out the solution of almost all the linear non-homogeneous problems with the help of the present method, which we have just presented.

Other alternative methods available are method of undetermined coefficients and other method. So with this we end our lecture. Thank you very much for listening this. Thank you.