Advanced Engineering Mathematics
Prof. D. N. Pandey
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture - 12
Fundamental Matrix for Non-Autonomous Systems

Hello friends. Welcome to this lecture. In this lecture, we will focus on the non-autonomous system of differential equation. In fact, if we recall, we have solve x dash=Ax and solve in the sense that how to find out a linearly independent solution and how to write down the general solution.

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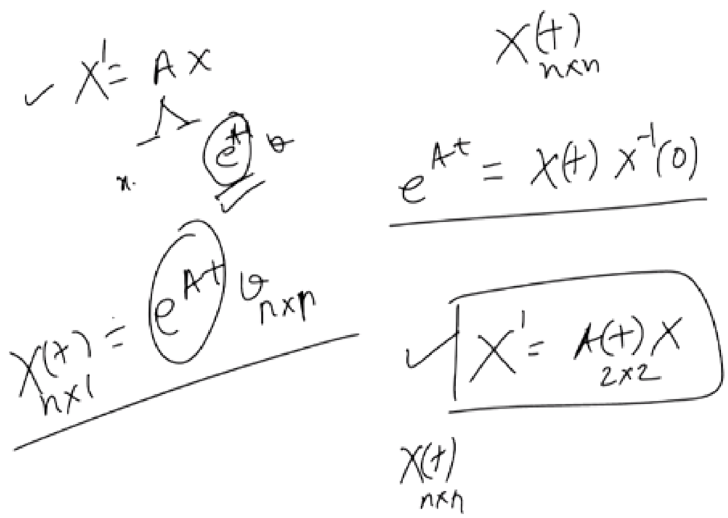

So here depending on the matrix A, we have calculated, we have 2 cases, case 1 and case 2. In one case, when we have n linearly independent Eigen vectors, then we found the n linearly independent solution and in the second case, when we do not have n linearly independent Eigen vectors, then we focus on finding e to power At*v. So e to power At is a matrix which needs to be found and here we have discussed how to find out e to power At.

And we have shown that once e to power At is known to us, then any solution x(t) can be written as e to power At*some vector nx1 here. So once we know e to power At, then any solution can be written as e to power At*v and we have discussed how to calculate e to power At with the help of other fundamental matrix solution and if we x(t), here this x(t) is nxn. If x(t) is a fundamental matrix solution, then e to power At can be written as x(t)*X inverse 0.

And once e to power At is known, then any solution of this systems x dash=Ax that is nx1 is given as e to power At*v nx1. Now we come to x dash=At*x. so this is a time dependent system of linear equation and here we have seen one example where we try to find out the fundamental matrix solution x(t) that is nxn. Here we have taken the example of 2x2 and it means that we have found fundamental matrix solution for 2x2.

But that we have found using conversion of this system into a single second order Cauchy-Euler equation that is a particular example. Now what happen when we are dealing this general matrix x dash=At x, then how we can look at, we want to know whether we have similar kind of features available in case of non-autonomous system or not.

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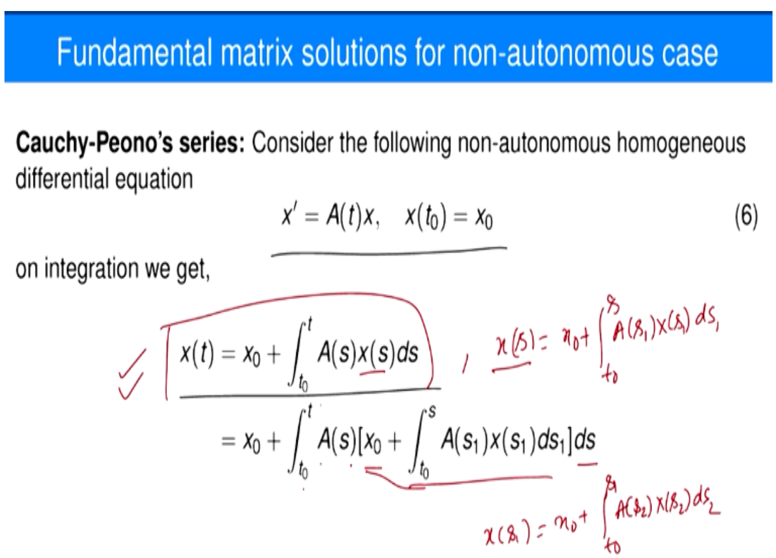

So to discuss this, let us start this lecture. So first we used to define the solution is the solution given in terms of Cauchy-Peano series, so what is this Cauchy-Peano series, let us try to understand. So here we have following non-autonomous homogenous differential equation x dash=e(t)x and x(t0) = x0. So we can integrate and we can find out the solution like x(t)=x0+t0 to t A(s) x(S) and here the only problem is that if we can integrate it, then we are done.

But if we cannot integrate, then we are stuck. So to avoid, to start this procedure that can we proceed further, here you realize that this is the expression of x(t)=x0+t0 to t A(s) s(s) ds. So I can use this again to calculate the value of x(s). So when you calculate X(S) using the same thing, then it is x0+t0 to S A of some S1 and X(S1) and of dS1. So using this expression for X(S), I can write the value of X(S)+t0 to S A(S1) X(S1) dS1.

So I can write down the next step as X0+t0 to t A(S), so I am writing the expression of X(S) that is X0+t0 to S A(S1) X(S1) dS1(dS). So this is the expression of X(S). Then, again we can use the same thing for X(S1), so we can write X(S1)=X0+t0 to S1 A(S2) now X(S2) dS2 and again put the value of X(S) in here and we can write here as we can simplify and we can write this as X0+t0 to t A(S) X0 dS+t0 to t t0 to S A(S) A(S1) X0 and so on.

So every time we are getting the expression like X(S1) here we are getting expression of X(S) X(S1) and then X(S2) and so on and every time we are using this expression and find out the value of X(S1) X(S2) and put it back.

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$$= x_0 + \int_{t_0}^{t} A(s)x_0 ds + \int_{t_0}^{t}\int_{t_0}^{s} A(s)A(s_1)[x_0 \cdots + ds_1 ds$$

$$= [I + \int_{t_0}^{t} A(s)ds + \int_{t_0}^{t}\int_{t_0}^{s} A(s)A(s_1)ds_1 ds + \ldots]x_0$$

Therefore, a solution of (6) is given by

$$x(t) = \phi(t, t_0)x_0, \quad = \phi(t, t_0)\, x(t_0) \tag{7}$$

where

$$\phi(t, t_0) = I + \int_{t_0}^{t} A(s)ds + \int_{t_0}^{t}\int_{t_0}^{s} A(s)A(s_1)ds_1 ds + \ldots \tag{8}$$

is called the state of transition matrix or principle fundamental matrix.

So in this way, we can get a kind of infinite series because this process is unending and we are getting an infinite series *x0. So we can write down the solution x(t) as phi (t, t0)x0 where phi (t, t0) is given as I+t0 to t A(s) ds+t0 to t, t0 to s A(s) A(s1) ds1 ds and so on. So this expression, we call this as the state of transition matrix or principle fundamental matrix. The reason why we call this as state of transition matrix, then if you look at the matrix expression given in equation #7.

If you look at, it is what x(t)=phi (t, t0)*x0. So it means basically it changes the initial condition x0 that is let me write it here this completely, this is phi(t, t0) and x at (t0). So basically it changes the state from x(t0) to xt. So this x(t0) to x(t) that is state at t. So that is why we call this as state of transition matrix. It changes the state from t0 to t, right and it is given by this.

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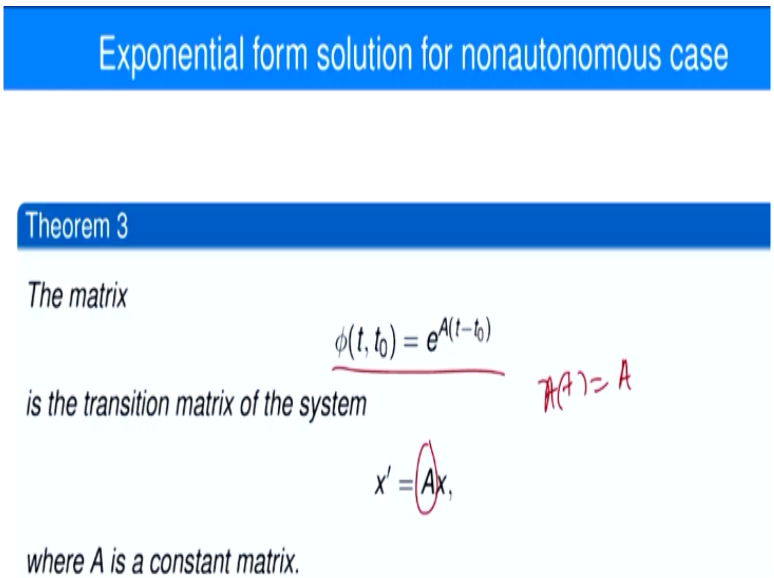

And if you see that in particular case, when the state transition matrix is reduced to e to power A(t-t0) when the At is ideally equal to A. So that is the contain of this theorem 3. So the matrix phi (t, t0), the state transition matrix is equal to e to power A(t-t0) is a transition matrix of the system x dash=A(x) where A is a constant matrix. So in case of constant matrix, this expression given in 8 is reduced to e to power A(t-t0) that is the contain of this theorem 3.

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Proof In the case of autonomous case $A(t) \equiv A$, the transition matrix (8) is

$$\begin{aligned}\phi(t,t_0) &= I + \int_{t_0}^{t} A ds + \int_{t_0}^{t}\int_{t_0}^{s} A^2 ds_1 ds + \ldots \\ &= I + A(t-t_0) + A^2\frac{(t-t_0)^2}{2!} + \ldots \end{aligned} \tag{9}$$

And hence the matrix

$$\phi(t,t_0) = e^{A(t-t_0)}$$

is the transition matrix of the linear autonomous homogeneous system

$$x' = Ax,$$

where A is a constant matrix.

So now let us look at the proof here. So in the case of autonomous case A(t) is and equal to A. So now we look at the expression of phi (t, t0), it is I+t0 to t, Ads+t0 to t t0 to s A square ds1 ds and so on. So if you look at this I can write as A*(t-t0), I can take it out A, similarly here I can take it out A square out and if you simplify, then it is what? It is t0 to t and t0 to s, then it is s-t0 ds and if you further do this thing, then it is s-t0 whole square upon factorial 2 t0 to t.

Then it is t-t0 square upon factorial 2 that is written here and so on and if you simplify this, then this is nothing but the exponential of e to power A, exponential series in terms of A(t-t0). So in the case of autonomous case, your state of transition matrix phi(t, t0) is given by eA to power t-t0. This is straight transition matrix of the linear autonomous homogeneous system x dash=A(x) where A is a constant matrix. This we know that how we can obtain this, okay.

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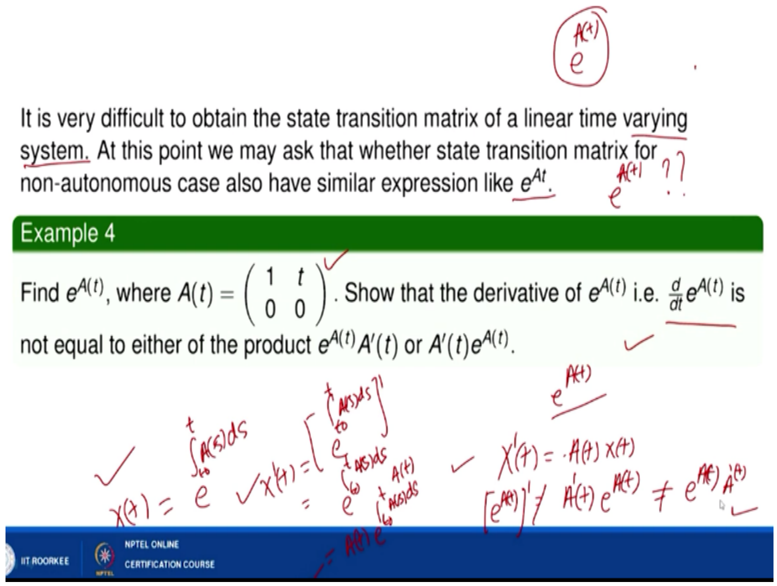

Now with this, we know that At is a constant matrix, then we know that the state of transition matrix is reduced to e to power A(t-t0) kind of thing. So we may have this question that what happen, can we define this similar kind of expression in case of non-autonomous case. It means that is it possible to write down the expression like e to power At kind of thing, right. So the answer is that in general, it may not be true.

But in some cases it may be your state transition matrix is given like e to power At kind of thing. So that is the content of the next few slides. So let us read this that it is very difficult to obtain the state transition matrix of a linear time varying system. Time varying system means that coefficient is depending on time. So at this point, we may ask that where the state transition matrix for non-autonomous case also have similar special like e to power At.

It means that we are expecting that whether this kind of thing is possible or not. So here first thing that we want to show that it may not be true in case of non-autonomous. So this example gives you that it may not be true. So find e to power At where et is given as 1 t 0 0 and with this, we want to show that derivative of e to power At that is d/dt of e to power At is not equal to either of the product e to the power At*A-t or A-t*e to power At.

It means that we may define e to power At, no problem, we can do that, but this will not solve x dash=At x(t) because if this candidate solve, it means that e to power At, derivative of this must be equal to At*e to power At and this example basically shows that that here the all matrix available for which the derivative of e to power At is not equal to this At*e to power At or A2*e to power At. So this is not equal, not equal to this also e to power At*At. This may not be equal.

Sorry, here it is A dash t. So here we want to show that even if we can define e to power At, then also e to power At derivative is not equal to A dash t*e to power At. In general, it is not true and it is also not equal to e to power At*A dash t and if it is not true, then it is quite difficult to define this kind of expression A(s)ds t0 to say t as a solution candidate x(t) for differential equation x dash t=Atx(t).

Because if this is a solution of this, then I need to find out the derivative here, which is nothing but e to power this t0 to tA(s)ds derivative of this is equal to the e to the power t0 to t dsA(s)ds*At or it may be like this At*e to power t0 to t A(s)ds. If x(t)=e to power t0 to t A(s)ds is a candidate for solution, then this must be equal, this and this must be equal, but this may not be true as we are trying to show with the help of this example.

So let us find out e to power At for this matrix At. So let us move here. So At is basically what?

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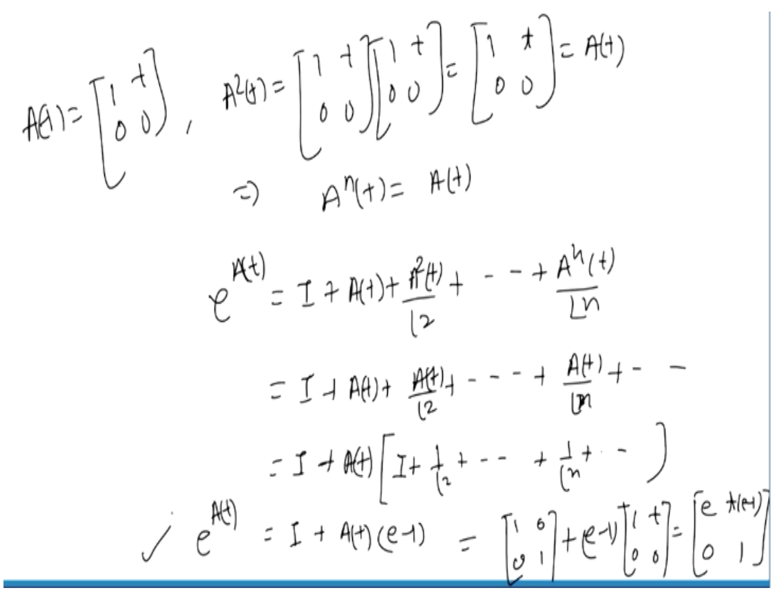

At is basically 1 t 0 0. Is it okay? So now you can calculate A square t, so A square t is basically 1 t 0 0 operating on 1 t 0 0 and you can check that it is nothing but this is 1 here and this is t here 0

0. So it is coming out to be At again. So it means that here you can check that A and t is nothing but your At, right. That you can check, no problem. So now you can calculate e to power At as I+At+A square t upon factorial 2 and so on An(t) upon factorial n.

So here it is I+At+At upon factorial 2 and so on. It is At upon factorial n and so on. So I can write this I, I can take it out, At out, so it is what? It is 1, basically it is I+ or you can say 1 and it is 1 upon factorial 2 and so on 1 upon factorial n and so on. So if you look at this is what I+At*this is nothing but the expression for e-1. So here e to power At is given as I+At*e-1 and if you simplify it is what? It is 1 0 0 1+e-*this At is 1 t 0 0.

So I can simplify and I can write it here as e and t*e-1 0 and 1. So that is the expression for e to power At. So, okay, now we want to find out the derivative of e to power At.

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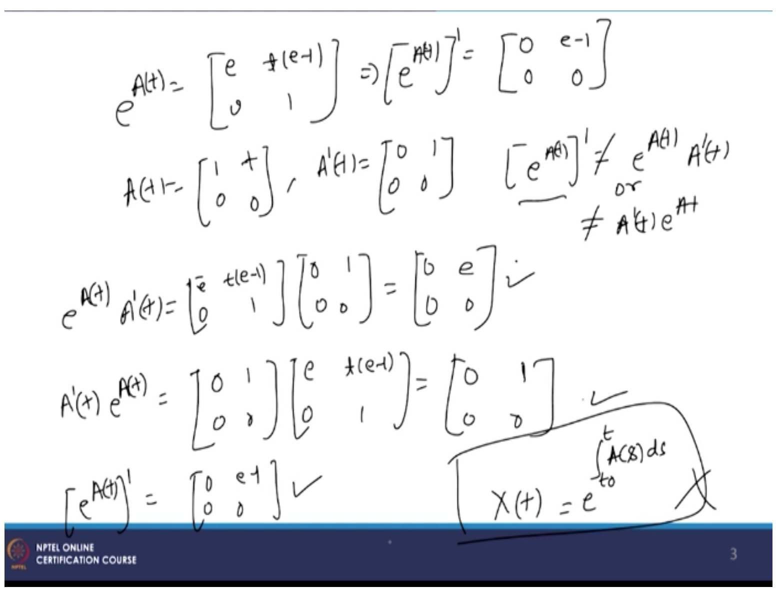

So now look at e to power At, let me write it again, e to power At is basically Ate-1 0 and 1. Now calculate e to power At derivative of this. So if you find out the derivative, derivative is nothing but 0, e-1, 0 and 0, yeah. So this is the derivative of e to power At dash. Now your At is what, At is given as 1 t 0 0, right. So you can calculate A dash t as what A dash t is 0 1 0 0, right and you want to show that this is nothing but that our expectation is this.

That if we calculate e to power At derivative, then it should be e to power At*A dash t or e to power A dash t*e to power At. So let us say that this may not be. So calculate e to power At*A dash t. So e to power At*A dash t. So what is this, e to power At is you have 0 e to power -1 0 0 and A dash t is basically what 0 1 0 0 and this is equal to what. This is 0, this is 0, this is 0, e to power At*A dash t, sorry e to power At is, sorry.

This is something long I did e to power At is not this, e to power At is this expression e te-1 0 and 1. So when you simplify it is what, this will give you 0 and this will give you 0 and this will give you e. So in place of 0 z is e, okay. Now let us calculate A dash t e to power At. It is what 0 1 0 0 and e to power At is ete-1 0 and 1. This is what, this is 0 here, this is 1 and this is 0 0, right. So A dash t e to power At is 0 1 0 0, e to power At A dash t is 0 e 0 0.

But e to power At derivative of this is equal to 0 e-1 0 0. So this is not equal to this, not equal to this. So what we have shown here, the derivative of e to power At may not be equal to e to power At*A dash t or A dash t*e to power At. So in this case, defining the expression, solution like e to power A(s)ds from t0 to t as x(t) is not correct, right. So for non-autonomous case, I cannot define this as a candidate for solution. So it is quite difficult basically.

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Here

$$e^{A(t)} = I + (e-1)A(t) = \begin{pmatrix} e & t(e-1) \\ 0 & 1 \end{pmatrix},$$

$$\frac{d}{dt}e^{A(t)} = \begin{pmatrix} 0 & e-1 \\ 0 & 0 \end{pmatrix}$$

and

$$e^{A(t)}A'(t) = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \text{ and } A'(t)e^{A(t)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So now let us check that what we have done here, we have calculated e to power At and that is coming out et-1 0 1, that I think we have shown here that e to power At is nothing but et-1 0 1, that is clear from this, d/dt of e to power At is 0 e-1 0 0 that also we have shown here 0 e-1 0 0 and e to power At A dash t, we have shown here e to power At A dash t is given as 0 e 0 0 that we have calculated and A dash t*e to power At.

A dash t*e to power At we have calculated as 0 1 0 0, so that we have also done and we have shown that it is this candidate is not equal to this or this, okay.

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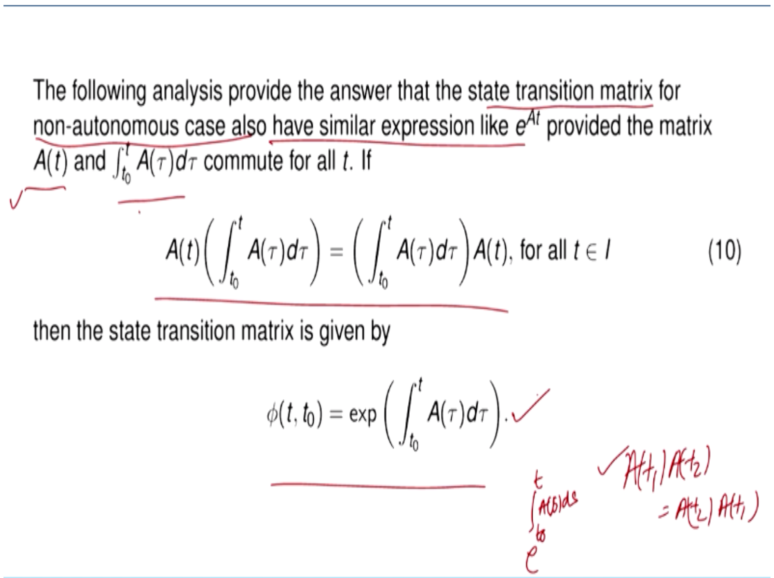

Now further it means that we say that in general it may not be true, but we can show that in some cases, we can consider e to power t0 to t A tau d tau as a solution candidate. So the following analysis will help us to identify the cases that the following analysis provide the answer that the state transition matrix for non-autonomous case also have similar expression like e to power At provided that the matrix At and t0 to t A tau d tau commute for all t. This is 1 case.

In fact, there are several cases available, 1 case is given by this another case is at if A(t1)*A(t2)=A(t2) A(t1) for all t1, t2, then also we can define solution like this, t0 to t A(s) d(S). So but in very special cases like this, this and this 1 more, you can define your state of transition matrix like this. So first let us focus here the condition that this At and this integral t0 to t A tau d tau commute, then we want to show that state transition matrix is given by exponential of t0 to t A tau d tau.

So the condition, which we are assuming is this that At*t0 to t A tau d tau=t0 to t A tau d tau*At for all t in the interval where we are looking at the solution, okay. So idea is that in this case your state transition matrix is given by exponential of t0 to t A tau d tau, okay. So let us prove it.

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To prove the claim, let us assume that the solution is

$$x(t) = \phi(t, t_0)x(t_0) = \exp\left(\int_{t_0}^{t} A(\tau)d\tau\right)x(t_0).$$

Using the series expansion of $\exp\left(\int_{t_0}^{t} A(\tau)d\tau\right)$; which is

$$\exp\left(\int_{t_0}^{t} A(\tau)d\tau\right) = I + \int_{t_0}^{t} A(\tau)d\tau + \ldots + \frac{(\int_{t_0}^{t} A(\tau)d\tau)^n}{n!} + \ldots$$

and $\left(e^{\int_{t_0}^{t} A(\tau)d\tau}\right)' = 0 + A(t) + \cdots + n\frac{\left(\int_{t_0}^{t} A(\tau)d\tau\right)^{n-1}}{\lfloor n} \cdot A(t) + \cdots$

So to show this, let us assume that the solution is x(t)=phi t0x(t0) that is exponential of t0 to t A tau d tau X(t0). So it means that what is the expression for this, exponential of t0 to t A tau d tau is this that I+t0 to t A tau d tau+t0 to t A tau d tau to power n upon factorial n and so on. So this is the expression for exponential of this.

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the series form of $\frac{d}{dt}\exp\left(\int_{t_0}^{t} A(\tau)d\tau\right)$; which is

$$\frac{d}{dt}\exp\left(\int_{t_0}^{t} A(\tau)d\tau\right) = A(t) + \left(\int_{t_0}^{t} A(\tau)d\tau\right)A(t) + \frac{\left(\int_{t_0}^{t} A(\tau)d\tau\right)^2}{2}A(t) + \ldots.$$

We get $= A(t) + A(t)\int_{t_0}^{t} A(\tau)ds + A(t)\frac{\left(\int_{t_0}^{t} A(\tau)d\tau\right)^2}{\lfloor 2} + \cdots$

$$X'(t) = A(t)X(t)$$

$$= A(t)\left[I + \cdots\right] = A(t)e^{\int_{t_0}^{t} A(\tau)d\tau}$$

$$\Rightarrow X' = A(t)X$$

So now since it is a candidate of a solution, so x(t) we need to find out the derivative of x(t), so find out the derivative of exponential of t0 to t A tau d tau and that we are writing as d/dt exponential of t0 to t A tau d tau is basically At+t0 to t A tau d tau*At and +t0 to t A tau d tau whole square upon factorial 2 At and so on. So this we have basically we are just differentiating term by term of this series. So here if you differentiate this, what you will get?

Exponential of t0 to t A tau d tau and we are finding the derivative here. So this will give you 0+ this will give you At and general term will give you n and this is what t0 to t A tau d tau power n-1 upon factorial n* derivative of this term that is At and so on. So this is the derivative we are assuming. I am assuming like this that we are differentiating this as this, okay. We have written it like this. Now so we can write down expression like this.

If you simplify, we can write it like this and here now we are equating both the things that X dash t=At and x(t). So when writing At*x(t), we are writing it like this and this is expression by differentiating d/dt of this x(t) and if we equate these 2, we will get equating the like powers of t on both the sides of the equation, we have t0 to t A tau d tau*At+1 upon 2 t0 to t A tau d tau whole square*At=At*t0 to t A tau d tau + so on.

So if we equate it, the corresponding power what you will get, that At, this At* this=this thing. So At*t0 to t A tau d tau=t0 to t A tau d tau=At. So when these 2 commute, then we can differentiate this thing and we can write here that if these 2 commute, then I can write At + this I can take it here At and t0 to t A tau d tau + here At you can take it out here and we can write t0 to t A tau d tau whole square upon factorial 2 and so on.

And now we can take it At out and this is nothing but I + so on, which is nothing but At*exponential of t0 to t A tau d tau. So it means that it will satisfy the differential equation X dash=At x, right. So in the case when the integral t0 to t A tau d tau commute with the matrix At, then we candidate this e to power t0 to t A tau d tau is a solution of this and we can call this as a state transition matrix, which transfer the state from x(t0) to x(t).

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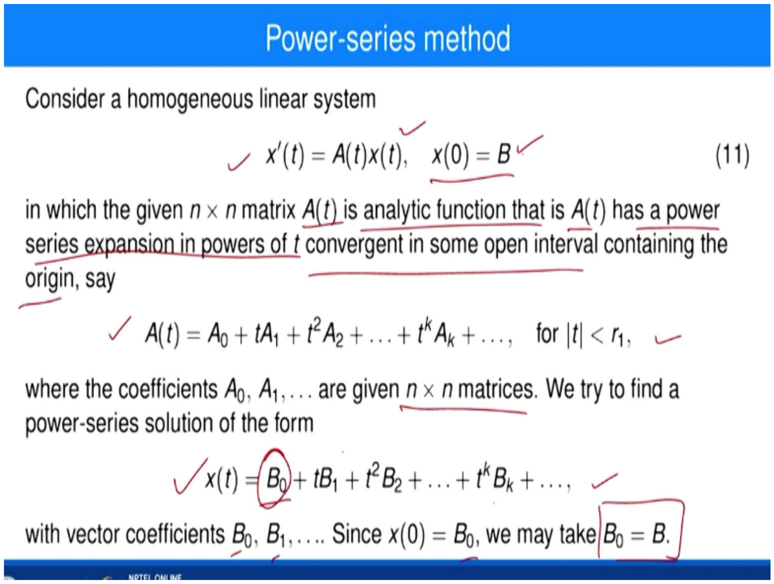

So it is in certain cases, when this At this integral commute, then we can write down your phi(t, t0)=exponential of t0 to t A of tau d tau, that is all. So here we have this state transition matrix. Now let us find out 1 more case when we can find out the solution of this non-autonomous system of linear equation and this case is the case when At is an analytic function of t. So for existence of solution, we have assumed that At is a continuous function on a closed interval t.

But finding the power series solution kind of thing, we assume that that At is an analytic function of t and it is similar to your finding the power series solution method of your second order linear differential equation. So it is similar to that, so let us consider this power series method. So considering homogeneous linear system X dash t=A(t)x(t) x0=B here, in which this nxn matrix At is analytic function. That means what that At has a power series, expansion in powers of t.

And it has some radius of convergence or we can say convergence in some open interval containing the origin. It means that I can have the expression like At=A0+t*A1+t square A2+t to power k Ak and this expression is valid for mod t<r1 and the coefficient A0 A1 are given nxn matrices and with if At has this kind of expression, then we want to find out the solution of the system x dash t=A(t)x(t) in the following form that x(t) can be written as B0+tB1+t square B2 and tkBk and so on.

Here these coefficient matrices B0, B1, and Bk are unknown and that we need to find out and here since x(0)=B is a given thing, then we can fix B0 as your B. So 1 thing is already obvious that B0 is nothing but the initial condition given as x(0)=B. The remaining B1, B2, Bk you need to find out. So x(t) is the solution of this differential equation 11.

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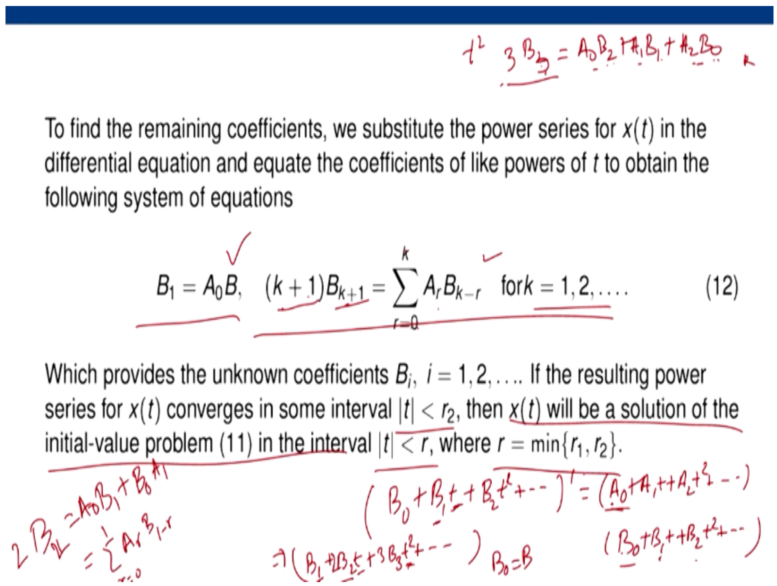

So to find the remaining coefficient, we substitute the power series for X(t) in the differential equation and equate the coefficient of like powers of t to obtain the following system of equation. So here we have B0+B1t+B2t square and so on=At is basically A0+A1t+A2t square and so on*B0+B1t+B2t square and so on. Now when you find out the derivative, derivative is basically B1+B2 2B2t+3B3t square and so on and that is equal to this thing.

So it means that if you equate the coefficient constant term, then we have B1 here and here the constant term is equal to A0 and B0, but B0 is already fixed as B. So B1 is given as A0B that is done. If you look at the coefficient of t, this is B1 here. So B1=now here the coefficient of t will be what, it is A0B1+A1B0, so it is 2. So here we can write it, it is A0B1+B0A1 or I can write this as summation r=0 to 1 ArB1-r, that is k-r.

And similarly we can find out the coefficient of B2, so B2t square will be coefficient of t square will be what it is B2 here. here it is coefficient of t square, sorry coefficient of t2 B2, sorry, we have to write it 2B2, sorry 2B2=this. So now coefficient of t square is 3B3, so

3B3=A0B2+A1B1+A2B0. So here 3B3 is basically what, this is k+1 Bk+1. If you look at here, it is, if you sum the indices, it is basically what, sum the indices is basically 2.

But here it is 3 and 3. So we can write it if sum of the indices of k, then it is k+1 and Bk+1. So I can write this as k+1, Bk+1=r=0 to k, arBk-r for k=1, 2, and so on. So we can say that if you equate the like powers of t, then we have the following recurrence relation to obtain the coefficient Bi, so once you know B1 in terms of B, then we can use this recursive equation 12 to find out the remaining coefficients Bi.

And if the resulting power series for Xe converges in some kind of interval mod of t <r2, the next it will be solution of the initial value problem 11 in the interval mod of t<r where r is the minimum of r1 and r2. So minimum of r1 and r2 means if this series exist for mod of t<r2, and this series for At exist for mod of t<r1, then the solution will exist in mod of t<r where r is the minimum of these r1 and r2.

So that will give you the solution given in terms of power series that is x(t)=B0+tB1+t square B2 and so on. So that is the idea of power series solution.

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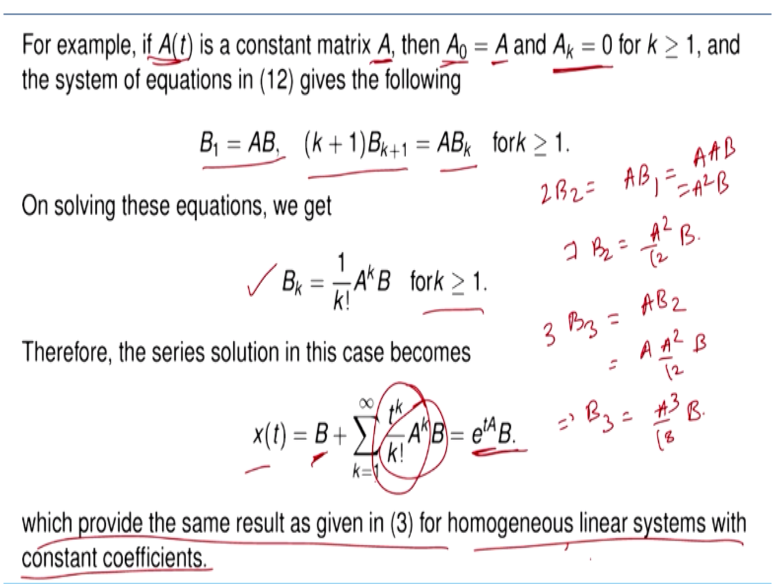

And here let us consider very simple example when this matrix At is again a constant matrix that is A, then we say that what method whatever we have discussed is reduced to the method we have already discussed. Now in case of At is a constant matrix, then A0=A and the remaining Ak is simply 0 and hence the system of equation that is B1=AB and k+1, Bk+1=ABk because all other terms simply vanish, because Ak=0 for k>/=1.

So here we can find out B1 as AB and remaining you can find out using this. Here we can simply say that B2 is what 2B2 is basically AB1 where B1 is given as AB, so we can write it here. This has A B1 is AB. So it is A square B. So B2 is basically what, B2 is A square upon factorial 2B. Similarly, we can calculate B3, B3 is basically what, B3 is basically 3B3=AB2. AB2 is A A square upon factorial 2 B and when you divide by 3, then B3 is basically A cube upon factorial 3*B.

So when you simplify, then what is your X, X(t) is basically B+k=1 to infinity and here coefficient tk and Bk. Now Bk is given by A to power k B upon factorial k. So when you put the value of Bk, that is given as 1 upon factorial k A to power k*B for k>/=1, then X(t) can be written as k=1 to infinity t to power k upon factorial k AkB. Now if you look at this expression is what, this expression is nothing, but e to power tA-I and I is kept here.

So we can summarize and we can write x(t) as e to power tA*B, so which provide that the same result as given in 3 for homogeneous linear system with constant coefficient. So it means that in case when At is your constant matrix, then this solution is reduced to e to power tA*B. So this gives some of the possible way to define the solution for non-autonomous cases. So far, we have discussed both the autonomous cases and non-autonomous cases.

Of course, the methods available are more in the case of autonomous case, but in case of non-autonomous methods are quite limited and it is very difficult to obtain the state of transition matrices. So with this we cover almost relevant thing for solution of finding the solution of linear homogeneous equation. So far we have not discussed any problem related to non-homogeneous system.

So in next class, we will discuss the solution of linear non-homogeneous system of differential equation with the help of the solution of linear homogeneous system of differential equation. So in next class, we will focus on non-homogeneous system of linear equation. So with this, I end this lecture. We will continue in next lecture. Thank you very much for listening. Thank you.