Advanced Engineering Mathematics

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Lecture - 11
Fundamental Matrix - II

Hello friends. Welcome to this lecture. In this lecture, we will continue our study from the previous lecture. In previous lecture, we have discussed the method to find out fundamental matrix solution. In fact, we have defined what is fundamental matrix solution and we have seen certain properties of fundamental solution. For example, if you have any nearly independent solution of x dash=Ax, then you can form a matrix.

And we can call that matrix as a fundamental matrix solution. And also we have shown that any matrix is said to be a fundamental matrix solution, provided that x dash=, it satisfies this differential equation, x dash=Ax. And determinant of that matrix is non-zero for all t in that particular interval. Then we call that matrix as a fundamental matrix solution and based on that we have proved that e to power At is also a fundamental matrix solution.

And also that we have shown that, if we have two fundamental matrix solution, X(t) and Y(t), then they are related by a non-singular matrix and we can write one fundamental matrix as a, like Y(t)=X(t)*constant matrix C, where C is a non-singular matrix. Now we start after that,

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Theorem 1

Let $X(t)$ be a fundamental matrix solution of the differential equation $\dot{x} = Ax$. Then,

$$e^{At} = X(t)X^{-1}(0).$$

Proof Let $X(t)$ be any fundamental matrix solution of $x' = Ax$, then using the previous lemma and the fact that e^{At} is also a fundamental solution we have

$$e^{At} = X(t)C. \qquad (1)$$

Let $t = 0$ in (1), we have $I = X(0)C$ which implies that $C = X^{-1}(0)$. Hence

$$e^{At} = X(t)X^{-1}(0).$$

Now we proved that let X(t) be a fundamental matrix solution of the differential equation x dash=Ax, then we can calculate e to the power At=X(t)*X inverse 0. In fact, if you recall while in the beginning when we started solving this problem, x dash=Ax, then we have proved that that e to the power At is a solution of this and corresponding to any vector e to power Atv is a solution of this.

So there problem of finding e to power At is quite difficult and we have obtained some alternative method to find out the solution, linearly independent solution of x dash=Ax, but now with the help of those linearly independent solution, now we want to calculate this matrix e to power At and so that e to power At into any vector of size nx1 will serve as a solution of x dash=Ax. So basically we are trying to find out e to power At with the help of a linearly independent solution of x dash=Ax.

So now we say that once we have a linearly independent solution x dash=Ax, then we can form a fundamental matrix solution, call this as X(t), then we say that that e to power At, we can calculate as X(t)*x inverse 0. So let us prove this theorem, which is proved like this that X(t) be any fundamental solution of x dash=Ax, then using the previous lemma and the fact that e to power At is also a fundamental solution that we have proved in previous lecture.

So we have shown that your X(t) is fundamental matrix and e to power At is also a fundamental matrix. So it means that I can write X(t) as e to power At*C or I can write e to power At=X(t)*C, where C is a non-singular matrix and we want to find out the value of this non-singular matrix. Now for this, let us take t=0 here and when t=0, then e to power At is reduced to i and X(t) is reduced to X(0) i*C.

Now since X is a fundamental matrix, so determinant of X0 is non-zero, it means that we can say that X0 is an invertible matrix. So multiplying by X0 inverse throughout the equation, we can say that your C is coming out to be X inverse 0. So we can say that once we have the value of X inverse 0, then we can write down the fundamental matrix e to power At in terms of X(t) as e to power At=X(t)*X inverse 0. So this is, this looks very tiny result.

But it is very powerful in the sense that from any fundamental matrix solution, you can calculate e to power At by just writing X(t)*X inverse 0, okay and once e to power At is given to you, then you can take any constant vector of size nx1 and you can write down the solution of x dash = Ax as e to power At*that vector v.

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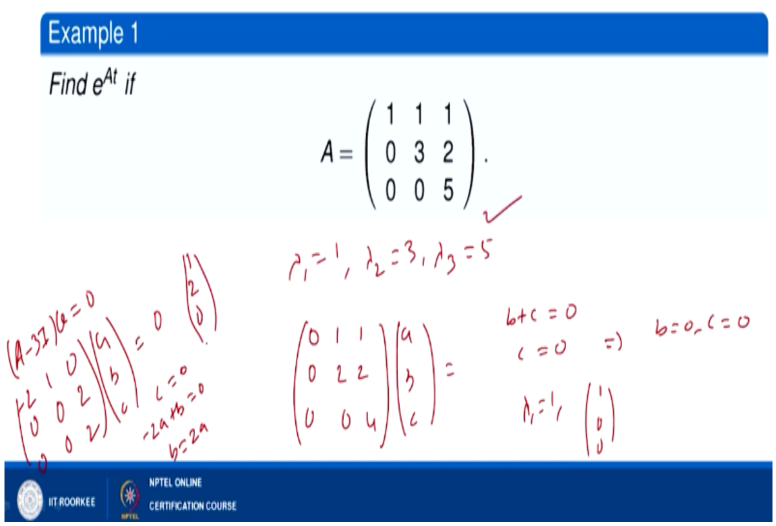

Now let us take one example, so that we can give the illustration of this theorem 1. So here in example 1, we need to find out e to power At where A is given as 1,1,1 0,3,2 0,0,5. Now here, we have taken a very simple example to discuss because it is an upper triangular matrix and

Eigenvalues, you can easily calculate as 1,3, and 5. So here your lambda 1=1, lambda 2=3 and lambda 3=5 and since it is, we have distinct Eigenvalues, so Eigenvectors we can easily calculate.

We can calculate corresponding to one as 0,1,1 0,2,2 0,0,4 and you can write it A, B, C and you can calculate that B+C=0 and C=0, so this implies that your B=0 and C=0. So we can find out the Eigenvector corresponding to lambda 1=1 say 1,0,0. You can calculate it like this.

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Solution To find e^{tA} we first find a fundamental matrix solution of $X' = AX$ and then we use the result of the Theorem 1.

Since A is an upper triangular matrix and hence eigenvalues of A are nothing but the diagonal entries of A i.e. $\lambda_1 = 1, \lambda_2 = 3$ and $\lambda_3 = 5$.

Since we have three distinct eigenvalues so corresponding eigenvectors are also going to be linearly independent eigenvectors and hence we will get 3 linearly independent solutions of the form $e^{\lambda t}v$.

(i) For $\lambda_1 = 1$, we may obtain $U = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ as an eigenvector. Hence, one solution corresponding to $\lambda_1 = 1$ is given by $x_1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Here we can say that to find out e to power and in this way we can proceed further. So let us start with this solution procedure. So to find e to power tA, you first find out a fundamental matrix solution of x dash=Ax and then we can use the result of previous theorem and as we have pointed out that A is an upper triangular matrix, and hence Eigenvalues of A are nothing but the diagonal entries that is lambda 1=1, lambda 2=3, and lambda 3=5.

That we have just pointed out. Now since we have 3 distinct Eigenvalues, so corresponding Eigenvectors are also going to be linearly independent Eigenvectors and hence we have 3 linearly independent solution of the form e to the power lambda t*v. So for lambda 1=1, we have already obtained that 1 0 0 is an Eigenvector. So solution is given as X1t=e to power t*1 0 0.

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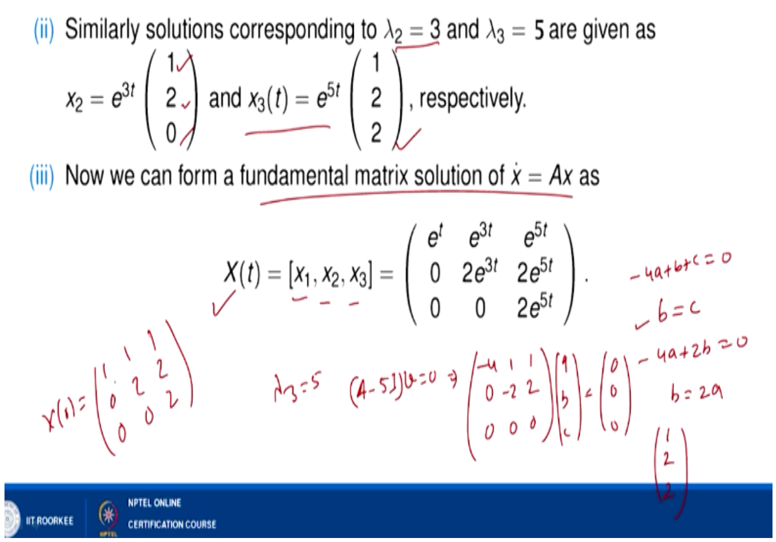

Similarly, we can find out solution corresponding to lambda 2=3 and lambda 3=5. So we can find out that it is nothing but 1 2 0. Let us calculate the Eigenvector corresponding to 3. So for that we simply say that A-3i will be what, A-3iv=0. So it is what -2 1 0, 0 0 2, and 0 0 2 and A, B, C=0. So it means that here C is 0 and we can write -2A+B=0. So B=2A you can say. So we can write down solution as, if A is 1, then B is 2, so 1 2 0 is 1 Eigenvector.

So it is written as 1 2 0. So in solution corresponding to lambda 2=3 is given as e to power 3t*1 2 0. Similarly, for lambda 3=5, A-say 5i*v=0 implies that it is, so it is, let me write it here. What is the matrix, 1 1 1. So here it is -4 1 1 0 and 3-5, it is -2, 2 0 0. Here it is and it is some A, B, C=0 0 0 and here we can simply say that it is what, it is B=C and -4A+B+C=0 and in second, you can get -2B+2C=0, so B=0. So -4A+2B=0. So B you can write it 2A.

So we can simply say that if we take A as 1, then it is 1 and then B is 2 and then C is again 2. So here we can say that corresponding to lambda 3=5, Eigenvector is given as 1 2 2. So X3(t) is given as e to power 5t. So now we can find out a fundamental matrix solution of x dash=Ax as X(t) as X1 as first column, X2 as second column and X3 as third column. So X1t is e to power t 0 0 and X2(t) is e to power 3t 2 e to power 3t 0 and X3(t) is e to power 5t 2e to power 5t and 2e to power 5t.

So we have obtained one fundamental matrix solution of x dash=Ax.

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To find e^{At}, we use the result of Theorem 1 which says that $e^{At} = X(t)X^{-1}(0)$. So evaluate X^{-1} at the point 0 as follows:

$$X^{-1}(0) = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

And hence,

$$e^{At} = \begin{pmatrix} e^{t} & \frac{-1}{2}e^{t} + \frac{1}{2}e^{3t} & \frac{-1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ 0 & e^{3t} & -e^{3t} + e^{5t} \\ 0 & 0 & e^{5t} \end{pmatrix}.$$

$$= I + At + \frac{A^2t^2}{\lfloor 2} + \cdots$$

Now to find out e to power At, we need to use the formula e to power At as X(t)*X inverse 0. So we need to calculate X inverse 0. So X inverse 0, so let us find out what is X(0). So X(0) is nothing but, it is 1 0 0, say it is 1 2 0, it is 1 2 2 and we can find out the inverse using any of the method whether it is usual method or Gaussian elimination method, any method you can apply and you can find out X inverse 0 as this. So X1 inverse 0 is 1 -1/2 0, 0 1/2 -1/2, 0 0 1/2.

And you can simply say that here X zero has determinant non-zero. So X zero is invertible, so X inverse 0 exists and now you can calculate e to power At as X(t)*X inverse 0 and which we can calculate just like this. So here we can calculate e to power At in a very concise manner once we know a fundamental matrix solution, otherwise calculating e to power At for matrix A is quite difficult, because it is an infinite series and sum of infinite series is quite difficult.

So this infinite series is now converging to this matrix, which is given here. So in this way, this calculating e to power At using a given fundamental matrix solution is quite useful, okay.

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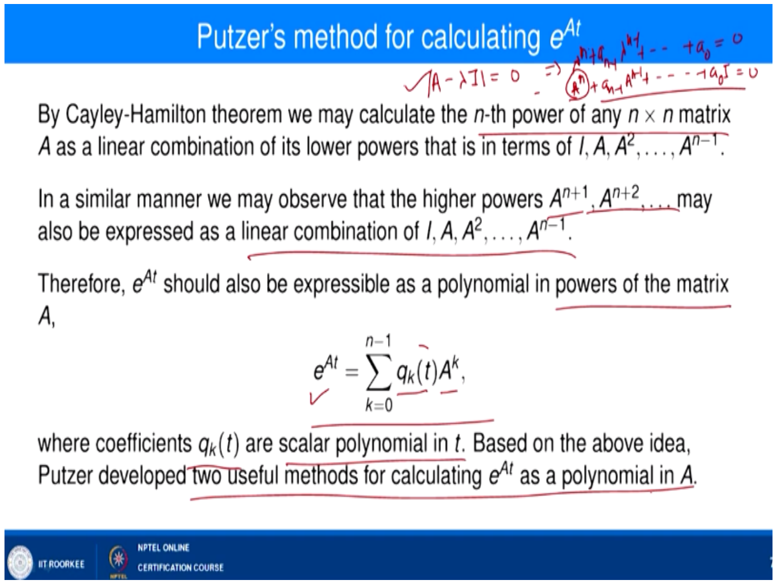

Now there are several algorithms available to calculate e to power At. In fact, more than 3 are available, but here I am just listing only one and this method is known as Putzer's method of calculating e to power At and the idea behind calculating e to power At is the Cayley–Hamilton theorem. Cayley–Hamilton theorem says that it is basically Hamilton theorem says that every square matrix satisfies 8 characteristic polynomials.

So it means that characteristic polynomial is characteristic equation is A-lambda i=0 and it says that when you explain, you will get a polynomial in terms of lambda and by solving this only we are getting the characteristic roots. So here Cayley–Hamilton theorem says that matrix A satisfy this characteristic equation and with this we can say that the n-th power of any nxn matrix can be written as linear combination of its lower power.

That is in terms of iA A square and so on. In fact, basically it is what, you can simply write this as simply lambda 2 power n+An-1 lambda n-1+1+this A0 you can write it as 0 and when A satisfies, then A to power n+An-1 A to power n-1 and so on, right and A0i=0. Now here we can write down the An as linear combination of iA, A square up to An-1. So that is the idea we can get from Cayley–Hamilton theorem.

That A to the power n can be written as linear combination of its lower powers. So if e to power n can be written as linear say lower powers of iA, A square and An-1, so An+1 can also be written as linear combination of iA, A square and An-1 and hence all the further powers of A may also be expressed as a linear combination of iA, A square and An-1 and in this way, we can simply say that e to power At can be written as powers of the matrix*some polynomial*powers of matrix A.

So I can write e to power At as summation K=0 to n-1 qk(t)*A to power k. Now the coefficient qkt are scalar polynomial t and based on this method, we suppose that there will be 2 useful method to calculate e to power At as a polynomial in A, but here we will take only one, sample of 1, if you want to discuss this, then I may refer a book by Tom Robusta. There you can get both, there we can have more than 2 method to calculate this e to power At.

So I am just giving you one idea of this algorithm. In this method, we are not calculating e to power At in direct powers of A to power k, but some function of matrix A. So here this theorem 2 gives the Putzer method to calculate e to power At.

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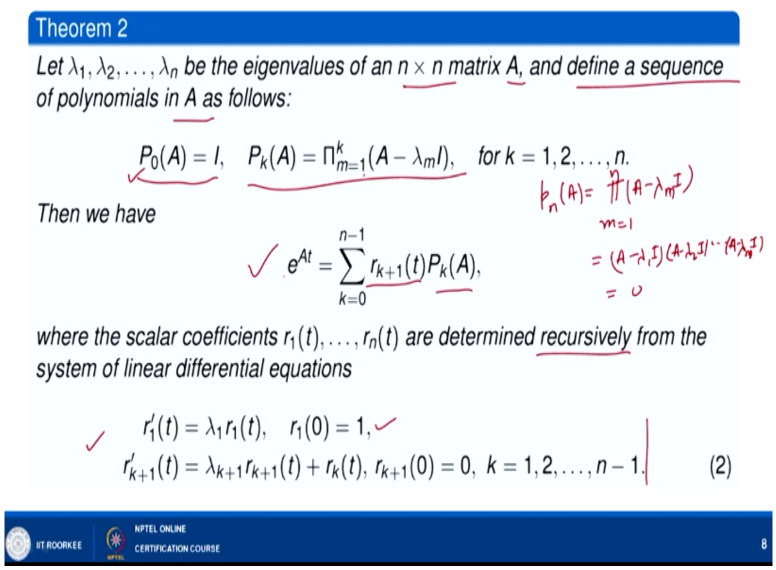

So theorem goes like this that let lambda 1, lambda 2, and lambda n be the Eigenvalues of nxn matrix A, then we define a sequence of polynomials in A as follows: P0(A)=I Pk(A) is basically product of A-lambda I, m is from 1 to k. So it is not given directly in terms of A, but it is lower powers of A is given. So Pk is basically product of m=1 to k, A-lambda mI for k=1 to n. please observe here that P and A is basically what.

Product of m=1 to n A-lambda mI and if you look at, this is basically product of what, product of A-lambda 1I, A-lambda 2I, and so on it is A-lambda nI, and this is nothing but the value 0. So Pn(A) is having value 0 and all other Pk(A) is defined like this. Then, we can write e to power At in terms of linear combination of these PI(A) as follows: e to power At is written as k=0 to n-1, rk+1t*Pk(A).

Where the coefficient rk+1t can be obtained using recursive relation of system of linear differential equation as follows, where r1 dash(t)=lambda 1 r1(t) r1(0)=1 and r dash (k+1t)=lambda k+1rk+1t + rkt where rk+1 (0)=0 for k=1 to n-1. So if you look at these some system of linear differential equation of first order and we can easily solve these scalar values, linear differential equation of first order.

After solving these coefficients in a recursive manner and having Pk(A), we can calculate e to power At. So first let us prove this and then we will certain example based on this theorem. So let us prove this.

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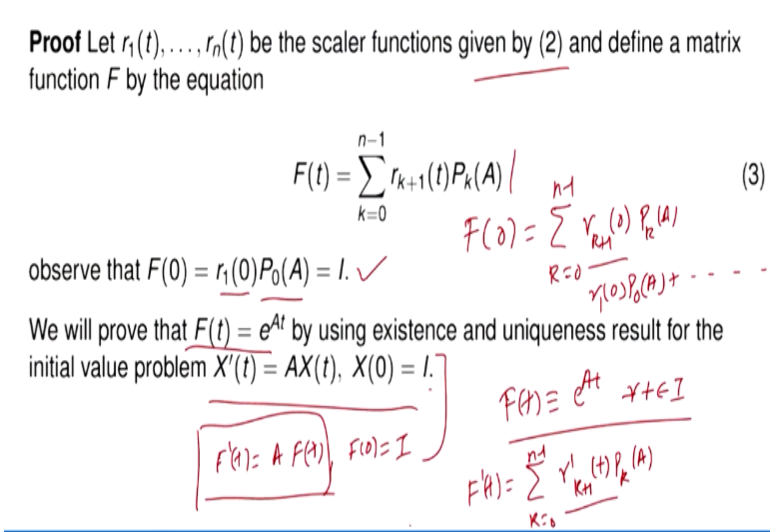

So let r1 to rn(t) be the scalar function given by 2 and define a matrix function F by the equation F(t)=k=0 to n-1 rk+1t Pk(A) that is what is here. This we assume as some F(t) and we want to show that e to power At is your F(t) and here we use the existent and uniqueness theorem to show the equality and here, please observe that the value of F(0) is r1(0) P0(A)=I. If you look at here it is what F(0) is basically summation k=0 to n-1 rk+1(0) Pk(A).

Now if you look at what is the value of rk+1(0), then go back here rk=1(0) is 0 for k=1 onward. So it means that that r2(0) is 0, r3(0) is 0 and so on, so only non-zero value is attained at r1. So, it means that when you put k=0, then we have r1(0) and P0(A) for all other values we have value 0. The scalar coefficient r2(0), r3(0) these are all 0. So r1(0) P0(A) is basically, r1(0) is basically 1 and P0(A) is defined as identity as we define here.

So this means that F(0) is identity and F(t) is given as this. So here we want to prove that F(t)=e to power At by using existent and uniqueness result of the initial value problem X dash (t)=AXt, X(0)=I. If you look at here e to power At satisfies this initial value problem because e to power At if you differentiate, then we will get A e to power At and e to power calculated at 0 is given as I. So if we can show that F(t) also satisfies this differential equation.

That is F dash (t)=AF(t) and F(0) this we have already obtained that it is F(0)=I, then since this initial value problem must have only unique solution, so F(t) has to be ideally equal to e to power At for all t in that interval I. So that proves our result. So only thing we have to show now is that F dash (t)=AF(t). For that let us differentiate this expression given in 3 and see what we are getting.

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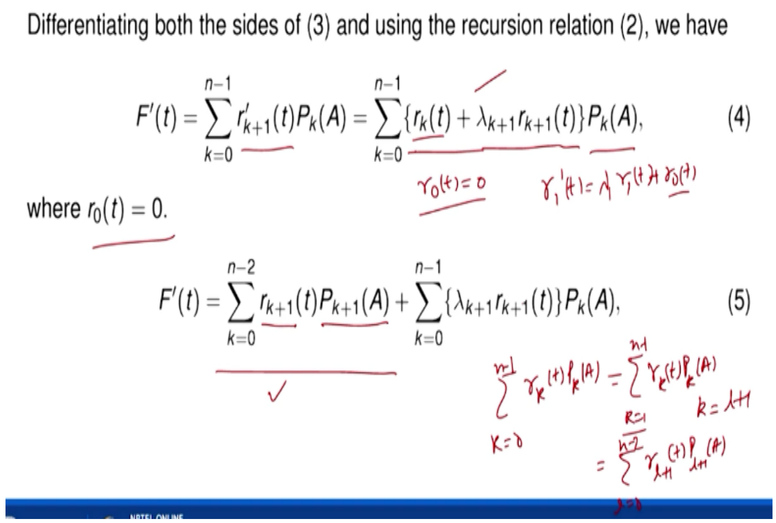

So differentiating both the sides of equation number 3, using the recursive relation 2, we have the following thing. So F(t) is this and if you differentiate this, what you will get, let me write here, F dash (t)=summation k=0 to n-1 r dash (k+1t) Pk(A). So this is independent of t, so only thing you have to write down r dash (k+1). So k=0 means r1 dash (t), so let me use here r dash (k+1t)= the following thing. So let me write it here and we can say that it is.

That lambda (k+1) rk+1t+rk(t) that is the expression for r dash k+1t and the Pk that we are just putting. So now since here your summation starts from k=0 to n-1, So when you put k=0 we have r0(t) and r0(t) we are defining it 0, why because for k=0, the expression is r dash t. So r dash t is basically what r1 dash (t)=lambda 1, let me write it here, r1 dash(t) is basically r1 dash is lambda 1 r1(t), so it is lambda 1 r1(t).

So it means that here your r0 is 0, we have assumed. So r0(t)=0 we are assuming and we can write this expression. So now we simplify it little bit further and we write k=0 to n-2, first term the rk(t)*Pk and this summation is starting from k=0 to n-1, but when you put k=0, then the r0(t0), so actual summation will start from 1. So 1 means r1(t) onward. So rather than writing k from 0 to n-1, we are writing as k from 0 to n-2 and we write rk+1(t)*Pk+1(A).

So here what we have done is this rk(t) and Pk(A) and that is k=0 to n-1. If you put, this is nothing but k=1 to n-1 rk(t)Pk(A), because corresponding to k=0, your r0(t) is 0. So now you simply replace in place of k, you can write it l+1, so it means that I can write here that this implies that l value starts from 0 to n-2 and this is rl+1(t)Pl+1(A), that is what we have written here that k=0 to n-2 rk+1(t)*Pk+1(A) and second term is as it is.

That is k=0 to n-1 lambda k+1rk+1(t)Pk(A). Now we want to show that f dash (t)-AF(t)=0. So for that we can look at here.

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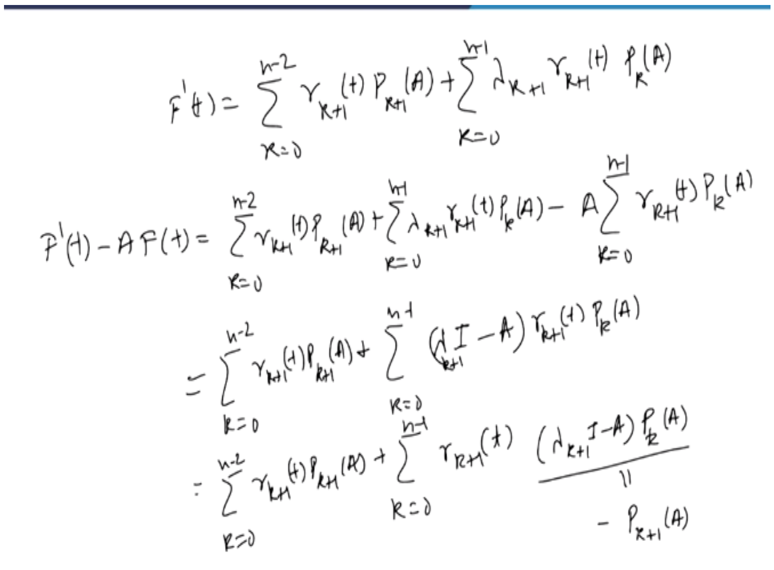

We have F dash (t), F dash (t)=summation k=0 to n-2. This rk+1(t) your Pk+1A+summation k=0 to n-1 and what is here. It is lambda k+1 rk+1(t)*Pk. So lambda k+1 rk+1(t)*Pk(A). Now look at F dash (t)-AF(t). So let me write it here k=0 to n-2 rk+1(t)Pk+1(A)+summation k=0 to n-1 lambda k+1 rk+1(t) Pk(A)-A*F(t). F(t) is what k=0 to n-1, let me write it. This expression it is rk+1(t) Pk(A). So that we are writing here rk+1(t) and Pk, right.

Now let us calculate this value. So here I will write it in summation k=0 to n-2 rk+1(t)Pk+1(A) +k=0 to n-1 and we can write it here lambda I-A, lambda k+1I-A rk+1t*Pk, is it okay. So we can write it like this. This we can write it k=n-2 rk+1(t) Pk+1(A). Now if you observe here that this rk+1(t) is just a scalar quantity. I can take it out. So rk+1(t) I am writing here, k=0 to n-1 and this is what lambda k+1I-A*Pk(A).

Now my claim is that this is nothing, but –Pk+1(A). So how we can look at. If you look at the expression for this Pk, now what is the value of Pk+1, so Pk+1(A) is basically A-lambda k+1I*Pk(A), right, but here we have lambda k+1I-A*Pk(A), so that I am writing as –Pk+1, so that I am writing as –Pk+1(A). So you are writing this as –Pk+1(A), then I can write this as summation.

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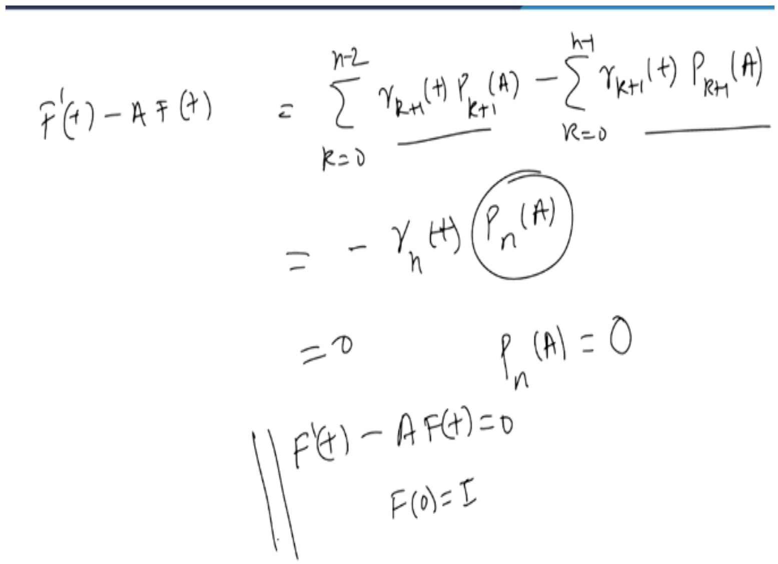

So it is basically F dash (t)-AF(t)=summation k=0 to n-2 and what is here rk+1tPk+1(A). So here rk+1t rk+1tPk+1(A)-summation k=0 to n-1 and what is here it is rk+1t, so rk+1t and Pk+1(A), right. So if you look at everything will be canceled out, only this last term will be left that is corresponding to k=n-1, because both are same up to n-2 here. So what is left here is –rn(t) and Pn(A), is it right here. It is t. Now what is Pn(A).

So Pn(A), this is Pn(A) and we have already calculated Pn(A) as 0. So it means that this is nothing but 0. So it means that F dash(t)-AF(t)=0 and we already know that F(0)=identity. So what we have shown here that F(t) satisfy this initial value problem, but e to power A(t) is already satisfying this equation and we have done. So that is we wanted to prove here. So here from this onward we have.

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Now, we subtract $\lambda_n F(t) = \sum_{k=0}^{n-1} \lambda_n r_{k+1}(t) P_k(A)$ to obtain the relation

$$F'(t) - \lambda_n F(t) = \sum_{k=0}^{n-2} r_{k+1}(t)\{P_{k+1}(A) + (\lambda_{n+1} - \lambda_n)P_k(A)\}.$$

But from (6), we observe that $P_{k+1}(A) = (A - \lambda_{n+1} I)P_k(A)$, so

$$\begin{aligned} P_{n+1}(A) + (\lambda_{n+1} - \lambda_n)P_n(A) &= (A - \lambda_{n+1} I)P_n(A) \\ &\quad +(\lambda_{n+1} - \lambda_n)P_n(A) \\ &= (A - \lambda_n I)P_n(A) \end{aligned}$$

And there is an alternative way to handle this situation. The alternative way is that you might separate lambda and F(t) and then we simplify this, that I am not going to do it.

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Therefore Equation (5) becomes

$$\begin{aligned} F'(t) - \lambda_n F(t) &= (A - \lambda_n I)\sum_{k=0}^{n-2} r_{k+1}(t)P_k(A) \\ &= (A - \lambda_n I)\{F(t) - r_n(t)P_{n-1}(A)\} \\ &= (A - \lambda_n I)F(t) - r_n(t)P_{n-1}(A). \end{aligned}$$

By the Cayley-Hamilton theorem we have $P_n(A) = 0$, so the last equation becomes

$$F'(t) - \lambda_n F(t) = (A - \lambda_n I)F(t) = AF(t) - \lambda_n F(t),$$

from which we find that $F(t)$ satisfies the differential equation $F'(t) = AF(t)$. Since $F(0) = I$, the uniqueness theorem shows that $F(t) = e^{tA}$.

$F(t) = e^{At}$ $\forall t$

But we just simply reaching up to this point that F(t) satisfy the differential equation F dash (t)=AF(t) and F(0)=I and we already know that e to power A(t) is satisfying this and by existent and uniqueness theorem, we simply say that F(t)=e to power A(t) for all time t. So that is the proof. So e to power A(t) can be written as the expression, which we have already wanted to prove is this, that e to power A(t)=summation k=0 to n-1 rk+1t*Pk(A).

Where rk(t) is defined in a recursive manner using the relation given in 2. So this proves the theorem 2 and we can use this result to find e to power A(t) for a given matrix.

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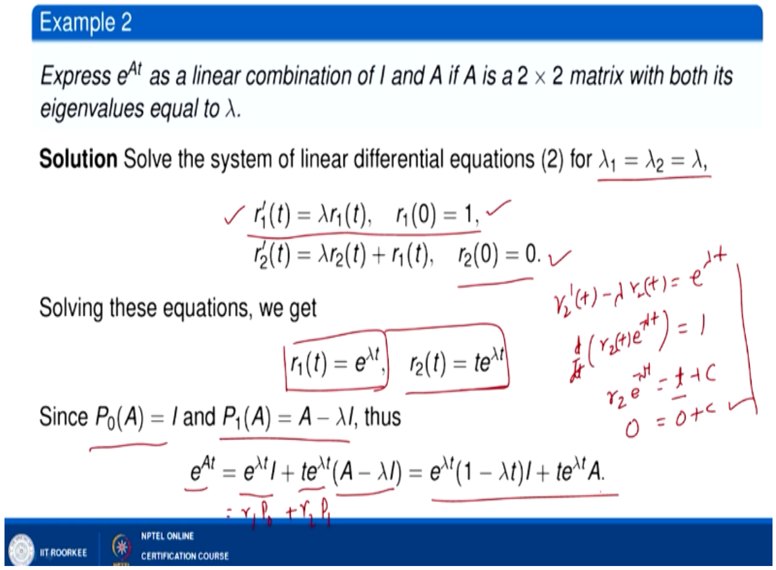

You may wonder that why we have discussed several methods to find out e to power A(t), then why this method is given to us. In fact, because of its nature, you can find out a computer program to calculate e to power A(t) in a very easy manner. So this algorithm, this may be treated as algorithm to calculate e to power A(t) using your computer also. So it may be a little bit difficult when you use this algorithm to solve your e to power, to find out e to power A(t) using your pen and pencil.

But it is quite useful when you write this as an algorithm. Anyway so now example 2 is that express e to power A(t) as a linear combination of I and A when A is a 2x2 matrix with both of its Eigenvalue is equal to lambda. So here the condition is that both of the Eigenvalues is equal to lambda, then how we can find out e to power A(t). So first things since it means what is given here that lambda 1=lambda 2=lambda.

And the scalar equations are given as r1 dash=lambda r1(t) and r1(0)=0 and r2 dash(t)=lambda r2(t)+r1(t) r2(0)=0. We can easily solve this equation r1 dash(t) is nothing but r1(t)=lambda (t). So basically this is some constant time e to power lambda (t) and that constant will be fixed by this initial condition that is r1(0)=1. So r1(t)=e to power lambda (t). Now to find out r2 dash (t), we have r2 dash(t)-lambda r2(t)=e to power lambda (t).

We can simply say that if you multiply by e to power –lambda t throughout, then I can write this as d/dt of r2(t) e to power –lambda t=1 and then if you integrate, then r2 e to power –lambda t=t + some constant thing and you can fix your constant by condition r2(0)=0, so it is 0=0+C. So C is coming out to be 0. So r2 e to power –lambda t=0 or we can say that r2(t) is given as te to power lambda t. I am using one method. You can use any of the method which you know.

So here r1(t) and r2(t) is known to us. P0(A) is given as I and P1(A) is nothing but A-lambda I. So now e to power A(t) is written as this r1P0(A) that is e to power lambda t*P0. So let me write it here. It is basically r1P0+r2P1 and so on. So r1 is e to power lambda t P0 is I+r2 is te to power lambda t and P1 is A-lambda I and we can write down the expression for e to power A(t) as e to power lambda t. This 1-lambda t*I+te to power lambda t*A.

So this is true for any matrix whose size is 2x2 and all the Eigenvalue is equal to lambda. Is it okay. So now, so we have calculated e to power A(t) in this case when A is 2x2 matrix and both the Eigenvalues are same and it is equal to some value, let us say lambda. Now let us consider 1 more example. Here also we need to find out e to power A(t). Again the matrix is of size 2x2, but the Eigenvalues are distinct, that is lambda is not equal to.

So in this case, let us first solve the system of linear differential equation r1 dash t=lambda r1(t) and r1(0)=1 and r2 dash (t)=mu r2(t)+r1(t) where r2(0)=0. So as we have pointed out the r1(t) is nothing but e to power lambda t and if you want to solve the second equation, then it is what r2 dash(t)-mu r2(t)=e to power lambda t. So here if you multiply by e to power –mu (t), then what you will get. Here we can get d/dt(r2t)*e to power –mu(t)=e to power lambda-mu and t.

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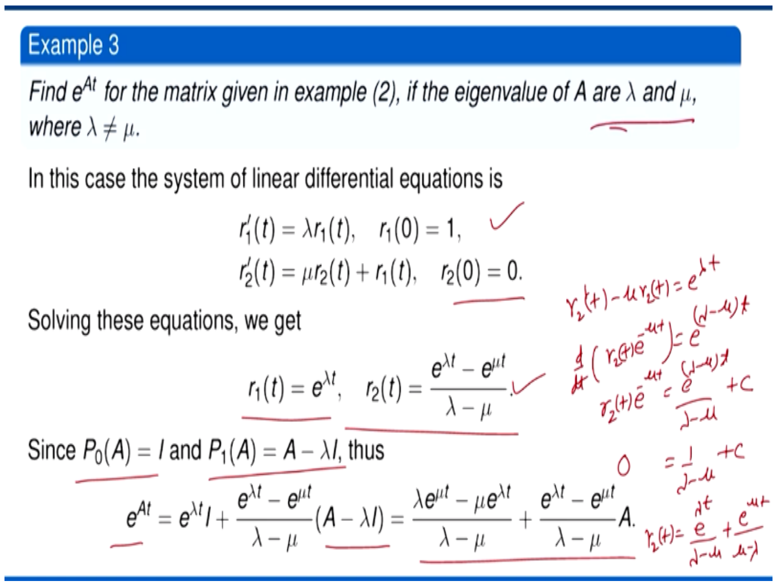

So this we can write it r2(t) e to power –mu(t)=e to power lambda-mu t/lambda-mu+some constant. Now r2(0) is 0, so this is 0 and this is 1/lambda-mu+C, so constant is given as the minus of lambda-mu. So constant is this lambda-mu and you can put it here, then you will get this as. So let me write it here r2(t) is basically multiply e to power mu(t). So here we have e to power lambda t/lambda-mu+e to power mu(t)/mu-lambda.

So when you simplify, you will get this, that e to power lambda t-e to power mu(t)/lambda-mu. So here we can get r2(t). So now P0(A) is as it is, it is I and P1(A) is A-lambda I, then you can write e to power A(t) as e to power lambda tI + this r2(t) that is e to power lambda t-e to power mu(t) upon lambda –P*P1(A) that is A-lambda I and when you simplify, you will get this expression.

So this expression will give you the expression for e to power A(t) for 2x2 matrix where both the Eigenvalues are unequal or not equal. So with this, I stop here. Discussion of finding the fundamental matrix e to power A(t) for the autonomous case and next lecture we will try to focus on non-autonomous case and try to see whether this similar kind of expression is available in the case of non-autonomous system or not.

So with this that is all and will continue in next lecture. Thank you very much for listening. Thank you.