Dynamical Systems and Control
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Lecture – 01
Formulation of Dynamical System - I

Hello friends. Welcome to this lecture. In this lecture, we start with the decision of dynamical system. So let us discuss what is we are going to discuss. So first let us define what is known as system.

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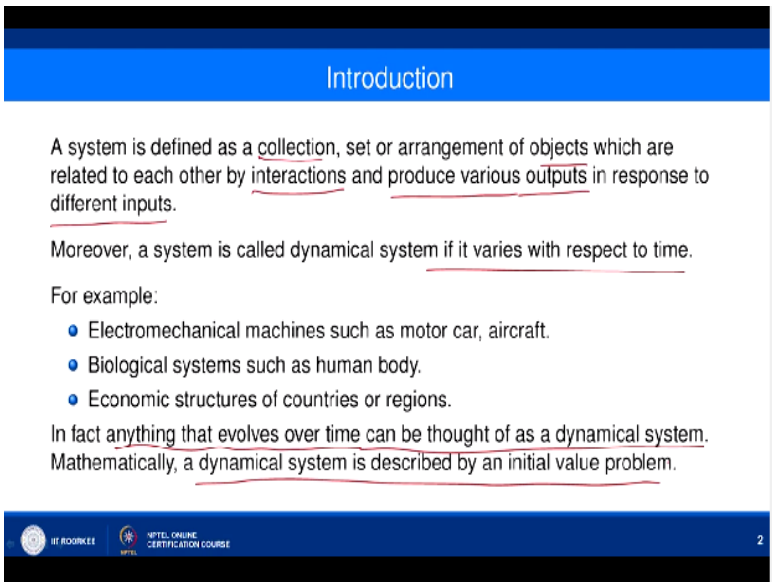

So a system is defined as a collection, set, or arrangement of objects. Basically, it is either collection of objects, set of objects or arrangement of objects in a way that they are related to each other by interactions and produce various outputs in response to different inputs. So if we have say some objects, they are collected or they are arranged in a way that they are interrelated and they produce different outputs corresponding to different inputs.

We call that kind of arrangement as a system. And a system is called dynamical system if it varies with respect to time t. So for example you take electromechanical machines such that motor, aircraft, washing machine, all these kind of electromechanical machines are coming under this dynamical system. And biological systems such as human body and the body of different species, they are all coming in the dynamical system.

And economic structures of countries or regions. So if you take, consider any country and consider the economic structure of that country, then it becomes a system which varies with respect to time t. And we call the structure as a dynamical system structure. And in fact, anything that evolves over time, can be thought of as a dynamical system. So any system which evolves over time, we call that as a dynamical system.

And if we model that dynamical system in terms of mathematical model, then generally a dynamical system is described by an initial value problem of differential equation. So in this particular lecture, we focus on concept related to initial value problem, what is initial value problem, what is differential equation and try to see certain examples of dynamical system at the end. So let us start with the history of differential equation.

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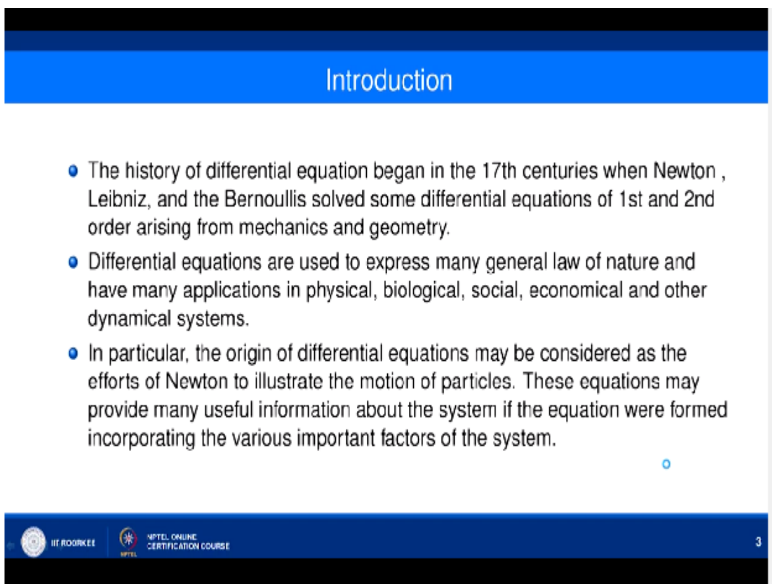

So the history of differential equation began in the 17th century when Newton, Leibniz and Bernoulli's brothers solved some differential equation of first and second order arising from mechanics and geometry. So in the beginning, all these problems started with a problem of mechanics problem of geometry and they tried to solve these problems using geometrical tools and all that.

And in the process, they tried to solve some first and second order differential equation. Now differential equation are used to express many general law of nature and have many applications in physical, biological, social, economical and other dynamical systems. And in particular, we can consider the origin of differential equation as the effort of Newton to illustrate the motion of particles.

And these equations may provide many useful information about the system if the equation were formed incorporating the various important factor of the system. So if you consider a real bird problem and then you consider the change in some kind of dependent variable and that dependent variable if it is a very important factor of the dynamical system, then by looking at their dynamical system, by solving their dynamical system, we are able to predict the behaviour of the dependent variable which plays a very vital role in that particular dynamical system.

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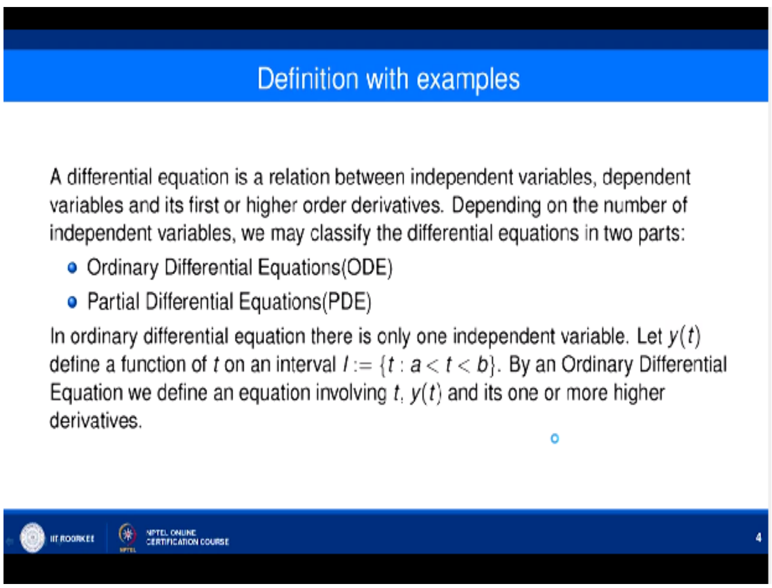

So let us say that a differential equation is a relation between independent variables. So most of the time when you deal with dynamical system and when you rewrite in terms of mathematical terms, it turns out to be a differential equation. So most often it is coming out to be for any differential equation. So here, we will focus in this lecture and in the subsequent lectures also, we will focus on dynamical system as a differential equation.

So here, let us start with what we call as differential equation. So a differential equation is a relation between independent variable, dependent variable and its first or higher order derivatives. And depending on the number of independent variables, we may classify the differential equations in to 2 parts. First one is ordinary differential equation. Second one is partial differential equation.

So in ordinary differential equation, the number of independent variable is only 1. And in partial differential independent variable equation, the number of independent variable is 2 or more than 2. So we will focus on ordinary differential equation. So let yt define a function of t on an interval I where I is some non-trivial interval starting from a to b where b is bigger than a. Now by an ordinary differential equation we define an equation, an equation involving t, yt and its one or more higher derivatives.

So any equation which involves the independent variable, dependent variable and its derivative say at least 1 derivative or say more than 1 derivative which may be first or high order derivatives. And that we call as, that equation we call as differential equation. So here are some examples of ordinary differential equation.

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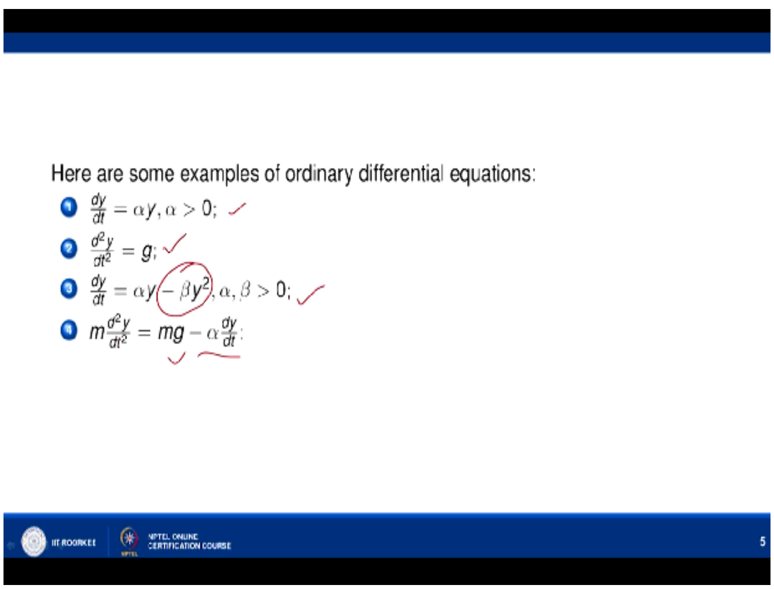

First one is dy/dt=alpha y where alpha is some positive constant and d2y/dt square=g, g is some constant. And dy/dt=alpha y-beta y square where alpha, beta are some positive constants and fourth one is md2y/dt square=mg-alpha dy/dt. So these are some very trivial example of ordinary differential equation. And in fact all these examples have some, originated from some real work problem.

For example, the first problem dy/dt=alpha y basically represents a very simple example of population dynamics where this yt represents the population of a given species at time t. And this simply says that when there is no interaction with the environment, then the population growth can be given by this equation dy/dt=alpha y. And if you look at the second equation, second equation is basically representing the freefall of a particle from a height, some height.

So it means that when your particle is say falling under the influence of gravity and there is no other say friction or there is no other hindrance, then the motion of the particle can be modelled by this equation d2y/dt square=g. Now if you look at the equation number 3, then equation number 3 is slight modification of the differential equation given in 1 where we have considered one additional term that is beta y square and we will see why this beta y square is given.

So this is say modification of simple population model and this model is also known as logistic model. We will discuss more about this model further time of duration. And the fourth is md2y/dt square=mg-alpha dy/dt. If you look at in this, if alpha=0, then it reduces to the differential equation given in 2. And if alpha is non-0, then it will turn out to be this. In fact, it is representing the same, the motion of a particle falling from some height.

The only thing is that now we are also considering that there is some kind of resistance due to air and that resistance is proportional to the velocity of the particle. So here resistance means it is negating the motion of the particle. So in that case, your motion of particle can be given as md2y/dt square=mg-alpha dy/dt. Here g represents the gravitational force. So in second and fourth, here g represents the gravitational force.

So it means that fourth is representing the differential equation which represents motion of particle when there is some kind of resistance due to air. So these are some simple examples of ordinary differential equation. Now we define certain basic things so that we can discuss more about differential equations. So first important part is the order of a differential equation. So what is order of a differential equation? Order of a differential equation is order of the highest order derivative present in the equation.

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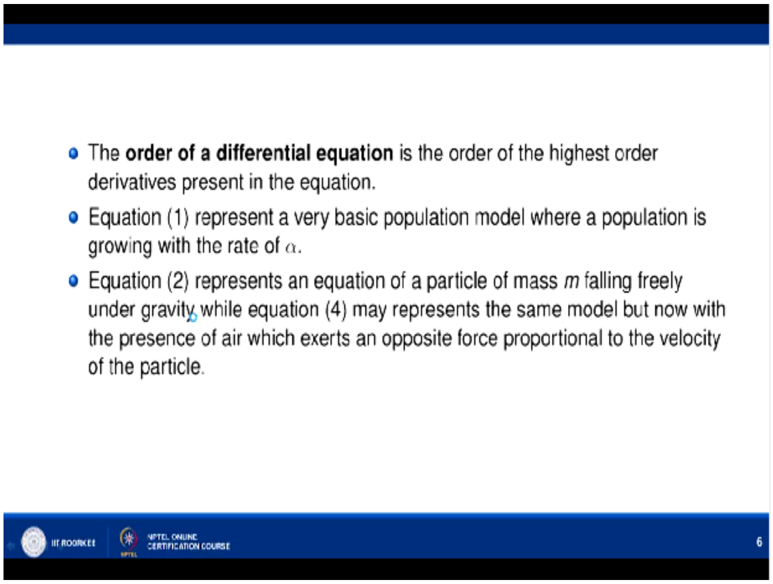

So you consider equation and look at the highest order present in that particular equation. That is considered to be the order of the differential equation. For example, in the first, the highest order derivative present is only 1 that is dy/dt. So this is first order differential equation. Similarly, this equation number third is also first order differential equation. While as your equation number 4 and 2, there are 2 derivatives present.

One is dy/dt. Another one is d2y/dt. And the highest order is d2y/dt. So it means that here, the second order derivative is present. So the order of the differential equation is coming out to be 2. So it means that equation number 2 and equation number 4 is an example of second order ordinary differential equation where as this equation number 1 and 3 is an example of first order differential equation. So first we try to know what is order of differential equation.

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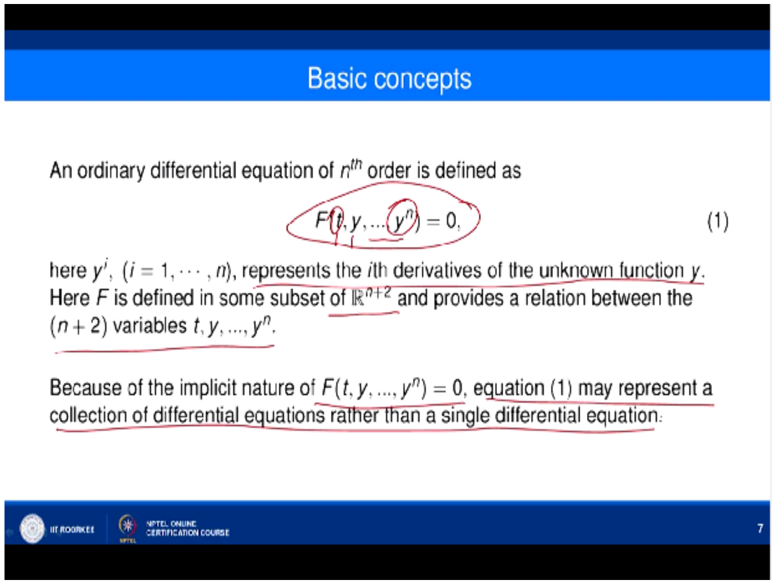

Then look at the basic concept of ordinary differential equation. So as we have already discussed that an ordinary differential equation of nth order is defined as a relation, an equation between t independent variable, y dependent variable and its higher order derivatives 1 or more higher order derivatives. So any relation between t, y, y dash up to yn=0, this is known as ordinary differential equation and the highest order present is given here yn.

So this is an example of nth order ordinary differential equation. Because number of independent variable is only 1, that is t here. So here this yi, where i is running from 1 to n, represents the ith derivative of the unknown function y. And here F is defined in some subset of Rn+2 here and provides a relation between the n+2th variable that is t, y, y dash t, y double dash t up to yn t. So basically this 1 represents the nth order ordinary differential equation.

But there is a small thing we need to consider here that here this function F may be an implicit function of t, y and y dash. So here equation 1 may represent a collection of differential equations rather than a single differential equation. So here because of it may happen that your function F is defined in implicit manner, then this equation number 1 may represent more than 1 differential equation at a time. In fact, here in this particular thing here y is representing a scalar valued function.

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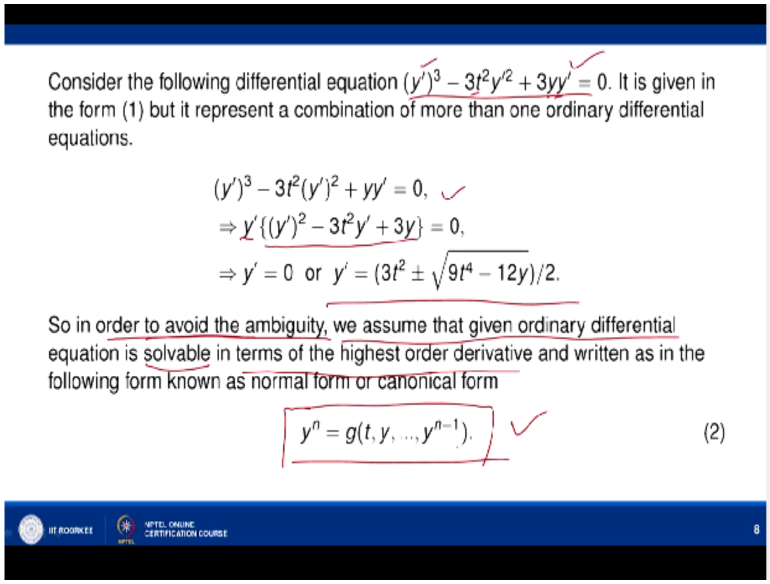

So consider the following differential equation, y dash cube-3t square y dash square+3y y dash=0. If you look at this as a equation, this is a relation between t, y. So t here y, y dash and yes. So basically it is a relation between t, y and y dash. So I can call this as a differential equation. But here you can see that here y dash, this is an implicit differential equation in terms of implicit relation between t, y and y dash.

So what we try to do here. We try to see that this actually represents more than 1 ordinary differential equation. If you simplify this further, then I can write this as y dash cube-3t square y dash square+y y dash, you can take out this y dash common. So y dash*this thing, that is y dash square-3t square y dash+3y=0. So I can write this as y dash=0 or y dash=3t square+-under root 9t to power 4-12y/2.

So basically this simple one ordinary differential equation may give rise to your 3 different ordinary differential equations. So here rather than considering this kind of relation, we assume that to order, to avoid the ambiguity, we assume that the given differential equation is solvable in terms of the highest order derivative. So highest order derivative is here in this y dash, so it means that if it is solvable in terms of the highest order derivative, then we can rewrite the equation number 1 as in this manner, that is yn=gt, y, y dash, up to yn-1.

So here, this, in this way we say that if y is a scalar valued function, then this will represent a single ordinary differential equation. Rather than considering the multiple ordinary differential equation as a one group, we consider the normal form or canonical form. And from now onwards, we assume that whenever we talk about ordinary differential equation, we are talking about the ordinary differential equation given in terms of normal form or canonical form.

So it means that your differential equation is solvable in terms of the highest order derivative and assuming the form given in 2. Now consider, once we know what is differential equation, then we try to know what is the solution of the differential equation. So we define solution of the differential equation as follows.

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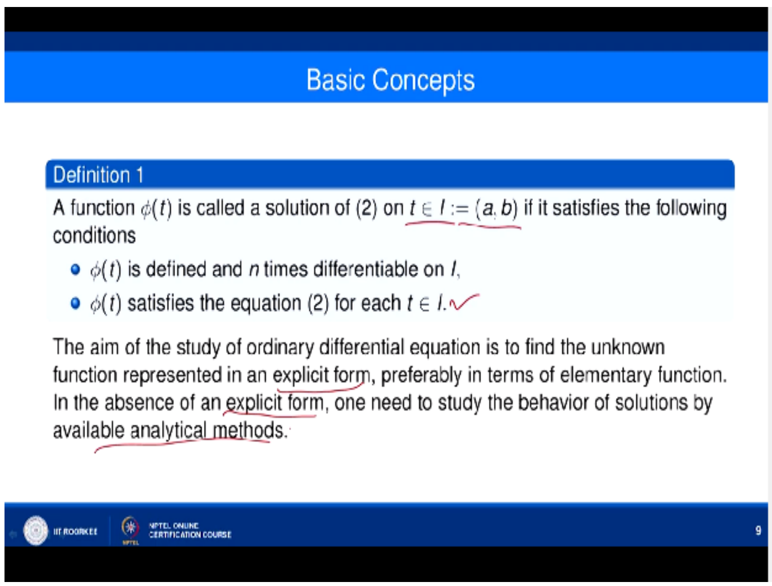

A function phi t is called a solution of equation number 2 given in normal form on an interval I which is given as a, b where b is bigger than a and if it satisfies the following condition that phi t is defined on this interval and it should be n times differentiable on this interval I. And second thing that phi t satisfies the equation 2 for each t=I. So if we are able to find out such a function, we call such a function as a solution of the differential equation given in normal form that is this kind of form.

So this is the definition of solution of the differential equation. And we, in all the problems, we are trying to find out the solution of differential equation defined in terms of this definition. So the aim of the study of the ordinary differential equation is to find the unknown function represented it in an explicit form that it should be, that y is defined in terms of t using some elementary functions.

So what are elementary functions, some examples of elementary functions are these sine function, trigonometric functions, polynomials and logarithmic functions. So it means that our primary aim is to represent yt in terms of t in terms of elementary, using elementary functions. And in the absence of an explicit form, we need to study the behaviour of solutions by available analytical methods.

So in case of when we are not able to find out the solution given in terms of explicit form, we try to focus on the properties of the solution given in terms of implicit form.

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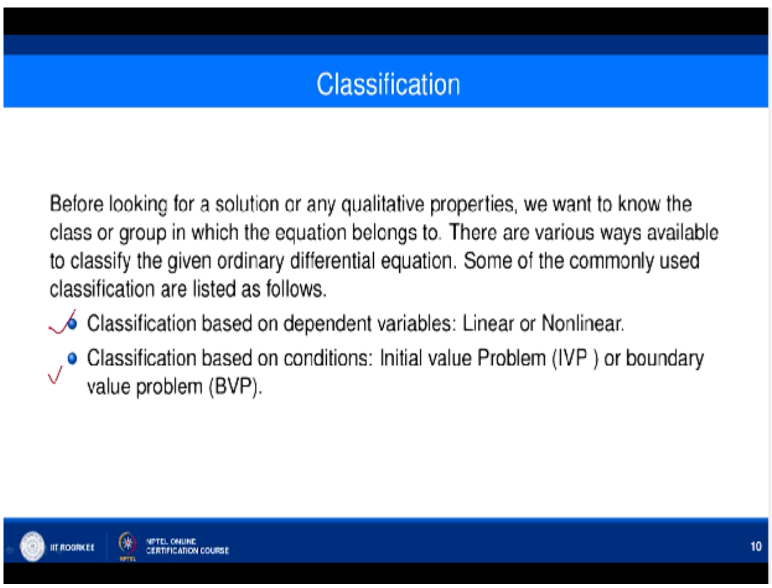

So we know what is the solution and what is the aim of the study. Now to start with the solution procedure, we classify our differential equation. So the classification is basically based on, mainly based on 2 categories. First is classification based on dependent variables that is linear or nonlinear. So the classification based on dependent variable gives you a linear differential equation or a nonlinear differential equation.

And the second one is a classification based on conditions whether it is initial value problem or say boundary value problem. So we will discuss one by one what are these classifications and how we can classify a given differential equation. So first consider the differential equation given in normal form that is yn=gt, y, y dash, up to yn-1.

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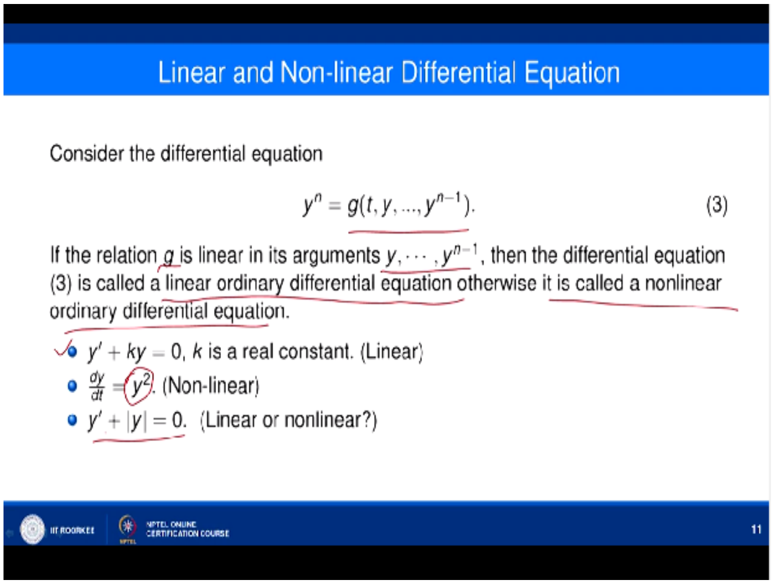

And if the relation g, this relation, is linear in its argument. Argument here is t, y, yn-1 but the linearity or nonlinearity will depend on the dependent variable and its derivative. So if g is linear in y, y dash up to yn-1, then the differential equation 3 is called a linear ordinary differential equation; otherwise, it is called a nonlinear ordinary differential equation.

So for example if you look at this y dash+ky=0, we can see that here your y, y dash coming in a linear manner. So we call this a linear differential equation. And whereas the second problem that is dy/dt=y square, here if you look at, here y square is not linear. This term y square is not linear. So we say that it is a nonlinear differential equation. If we simply say that linearity means that they are the present in the equation in a linear, in the sense that they have only 1 power.

And the coefficient of y or derivative of y may be only the function of t only, the independent variable. But if you follow this kind of definition, then it is very difficult to check whether this third equation, that is y dash_ modulus of y=0 is a linear or a nonlinear differential equation. Because here, it looks that y and y dash are coming in a linear manner. Their power is only 1.

So the procedure that checking that whether the y or the derivatives of y are present in the equation in a linear manner, may not give you a proper definition of linear and nonlinear differential equation. So here to give a proper definition of a linear and nonlinear differential equation, we use the concept of operator and we say that if we define suitably what is the operator based on equation number 3 and if the operator is linear, the corresponding differential equation is also linear.

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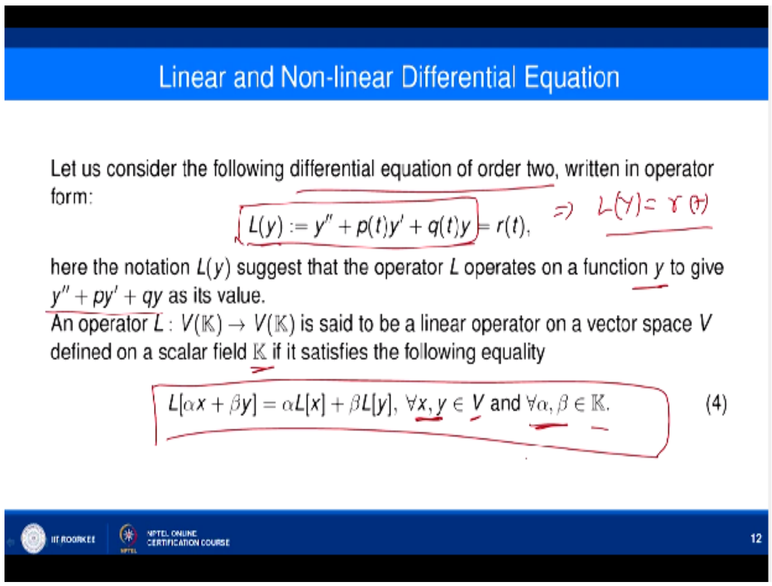

So let us consider the following differential equation of order 2. So let us take example and we taken the example of differential equation of order 2. And define Ly as y double dash+pty dash+qty and your differential equation is this that y double dash+pty dash+qty=rt. So basically I can write this that Ly=rt is your differential equation and where Ly is defined as this that Ly=y double dash+pty dash+qty.

So here once operator is defined, now we check whether this operator is a linear operator or not. If you look at here the notation this Ly suggests that the operator L operates on a function y and gives this value. The output is y double dash+py dash+qy and input is your y. So basically operator is basically what?

An operator L defined on vector space V with the scalar field K to vector space V/K is said to be a linear operator on a vector space V defined on a scalar field K if it satisfies the following equality that L of alpha x+beta y=alpha*Lx+beta*Ly where these x, y is coming from vector space and these alpha, beta is coming from this scalar field. If an operator satisfy these properties given in equation number 4, then we call this operator as a linear operator.

And if it is a linear operator and it is associated with differential equation in this way, then this differential equation that is y double dash+pty dash+qty=rt is a linear differential equation. So here now let us look at here.

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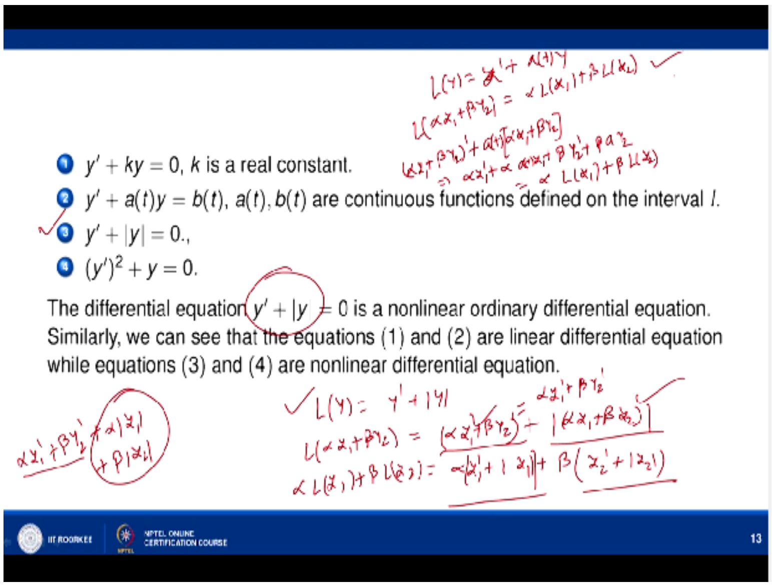

So look at this y dash+mod of y, so you define Ly as your y dash+mod of y. So it is an operator, it takes the input value y and giving the output as y dash+mod of y. And then now look at whether it is a linear operator or not, so look at alpha y1+beta y2 and this is what? Alpha y1+beta y2 derivative + modulus alpha y1+beta y2, right.

And if you look at alpha of Ly1+beta of Ly2, then it will give you alpha y1 dash+modulus of alpha y1+beta*, there is a small problem here. It is, here it is alpha is outside. So here you can write it like this. And beta*y2 dash+modulus of y2 here, right. So alpha*Ly1+beta*Ly2 is given by this. And you can check that though these terms are same that if you simplify, this is nothing but alpha y1 dash+beta y2 dash.

But this term is not equal to, so let me write it here. If we simplify this, what you will get? If you simplify, you will get alpha y1 dash+, here you can get beta y2 dash+alpha*modulus of y1+beta*modulus of y2. So it means that this term is same as this term but this term is not equal to modulus of alpha y1+beta y2. So it means that here, your operator L define in this way y dash+mod of y is not a linear operator.

So it means that the differential equation + mod of y=0 is not a linear differential equation. So we say that this is an example of a nonlinear differential equation. Similarly, we can consider the other example. For example, let us consider the second example. And here you define Ly as y dash+aty. Basically, when we check a linearity, we are checking the linearity in terms of dependent variable and its derivative.

So here now we write Ly as y dash+aty and now we check whether this define a linear operator or not. So you check that alpha y1+beta y2 is coming out to be alpha*Ly1+beta*Ly2. In fact, this is what? This is alpha y1+beta y2 dash+at. Here it is alpha y1+beta y2 and when you simplify it, you will get alpha y1 dash+alpha aty1+beta y2 dash+beta ay2 and this you can write it alpha*L of y1+beta*L of y2, that is your right hand side.

So here your Ly defined as y dash+aty satisfy the property of linearity. So we say that this differential equation defined as Ly=bt is a linear differential equation. So this is the classification based on say a dependent variable and we say that since this differential equation is linear in terms of dependent variable and its derivatives, we call these kind of differential equations as linear differential equation or nonlinear differential equation, if it satisfy the nonlinearity. Now let us look at the classification based on the conditions provided with the differential equation.

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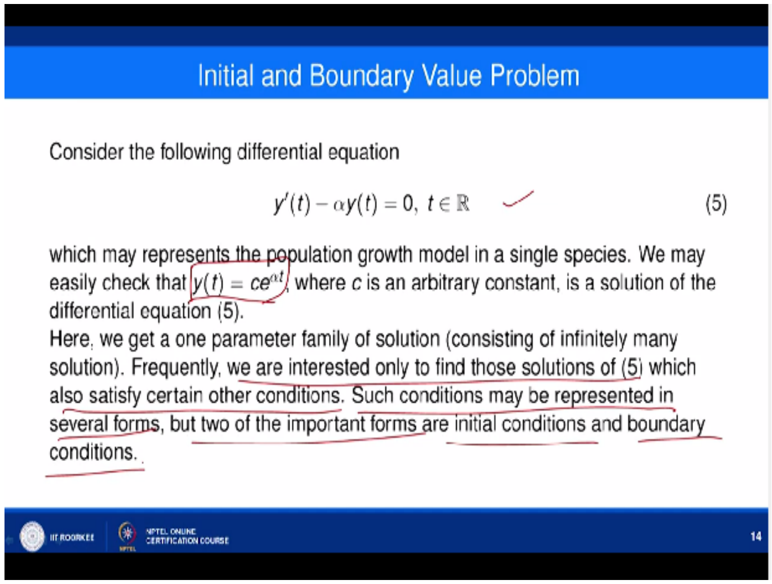

So consider the following differential equation, y dash t-alpha yt=0, t belongs to R. Then this represents the population growth model in a single species as we have pointed out. Now we may easily find out the solution of this. And the solution is given as yt+ce to the power alpha t because it is a simplest differential equation we have already considered. And here c is an arbitrary constant and this will act as a solution of the equation given in 5.

Now here if you look at c is an arbitrary constant. So it means that if we vary our c, we will have different solution of the equation number 5. But our real model problem, basically this equation number 5 is coming through some kind of a real situation, so it means that we are expecting a unique answer to that particular problem. So it means that we have to find out rather than in finding many solution, we have to find out the unique solution.

So for that, we have to prescribe some kind of condition associated with the differential equation. So in this case, we may provide the conditions in terms of the initial condition. So it means that frequently we are interested only to find out those solutions which also satisfy certain other conditions. And such conditions may be represented in several forms. but the most important forms are given in terms of initial condition and boundary conditions.

Now we try to understand what is the initial condition and what is boundary condition. So that we can understand with the help of some example.

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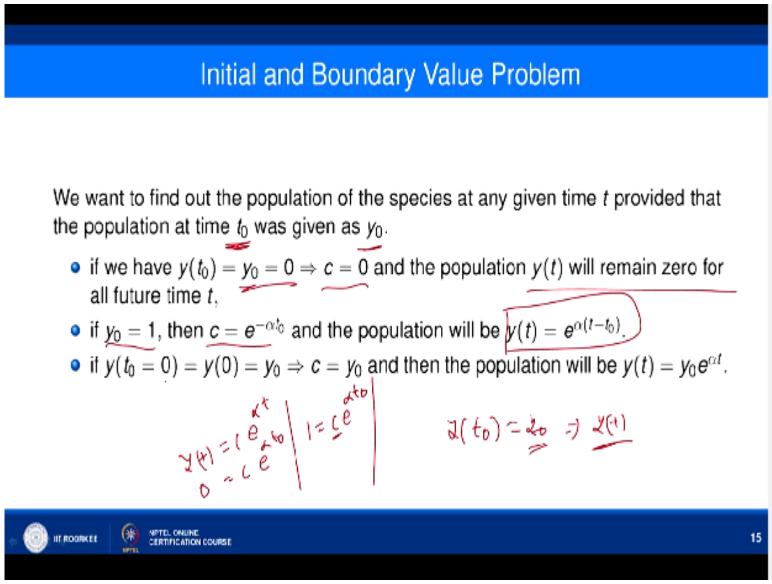

So first let us consider that we want to find out the population of the species at given time t provided that the population at time t0 is given by y0. So it means that initial condition, the condition associated with the differential equation is given in terms of that t=t0, your population is given as y0. So it means that y at t0 is your y0. And now we want to find out the y, population at time t.

And we say that if you look at different value of this y0 and t0, you may have a different solution of the same differential equation. For example, if I take y0 as 0, then we can get c=0. So if I look at yt is basically what? yt=ce to the power alpha t. So it means that if I assume that y0 is 0, that is 0=ce to the power alpha t0, then the only possibility that we will get this equation valid that c has to be 0.

And this case, when the initial population is 0, then yt will remain 0 for all future time t. And if you replace y0=1 and this case, 1=ce to the power alpha t0. So we can find out the value of c that is coming out to be c=e to the power -alpha t0. And your population is given as yt=e to the power alpha t-t0. Now in case of when t0 is replaced by 0 and y0 is given by some value say y0, then you can consider the, we can calculate the value of c as y0 and your population at time t is given by yt=y0e to the power alpha t.

So here we can say that be defining different conditions at the initial point that is t0, we may have different possibility of the solution. So it means that the defining the condition create a lot of impact on the solution procedure. So the condition defined at 1 point or say initial point is known as initial conditions. And the differential equation associated with the initial conditions is known as initial value problem here. Now look at the another example.

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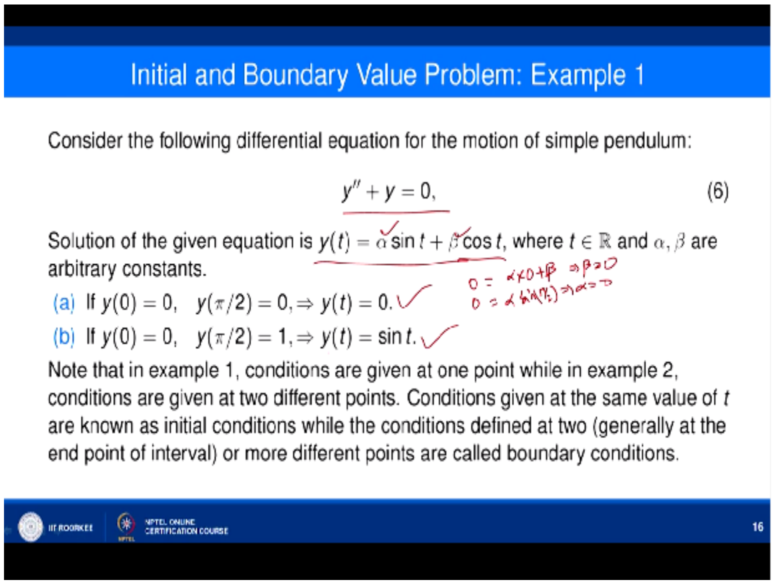

Here we are again considering a very simple example, y double dash+y=0 which may be considered as a example originated from motion of simple pendulum y double dash+y=0. And if we already know how to solve it and we can solve this and we can get yt=alpha sin t+beta cos t. By the way, here I am assuming that you know how to solve some simple ordinary differential equation.

For example, linear differential equation, exact differential equation, or say reducible to exact differential equation or highest order equation involving say constant coefficient. That I am assuming that you know. So here once we know the solution, then if we define condition, since it is a second order, we need to fix 2 arbitrary constant that is alpha and beta. So we need 2 conditions.

So let us define conditions as y of 0=0 and y of pi/2=0. And if you do this, you can easily check that your alpha and beta, both are coming out to be 0. For example, if y0=0, then it is your alpha*0+beta. So this implies that beta=0. Now y of pi/2, this implies that since beta is already 0, so alpha and sin of pi/2 that is 1. So this implies that alpha is also 0. So it means that if we assign these conditions, that is y0=0 and y pi/2=0, then solution is coming out to be yt=0.

But if we replace with conditions like this y0=0 and y pi/2=1, then your solution is now coming out to be non-0 and it is coming out to be yt=sint t. So look at these 2 examples and look at that in example 1, that is this example, here your condition is given at point t=t0 here. Whereas in this example, your conditions are given at 2 different points, that is 0 and pi/2. So it means that condition given at the same value of t are known as initial conditions.

So condition given at 1 point is then known as initial condition. Whereas the condition defined at 2, generally it should be the end point of the interval, or more different points are called boundary conditions, right. So this is the classification based on the conditions associated with the differential equation. So we know what is differential equation.

We know how to define the solution and how to classify the differential equation such as linear, nonlinear, initial value problem, or boundary value problem. So far we have discussed the classification and definition of differential equation, solution and the required information related to this differential equation. And in next lecture, we will continue from this. So here we will stop and we will conclude this lecture. Thank you very much for listening. Thank you.