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Mathematical Modeling: Analysis and Applications

Lecture-06

Discrete Time Non-Linear Models in Population Dynamics - I

With

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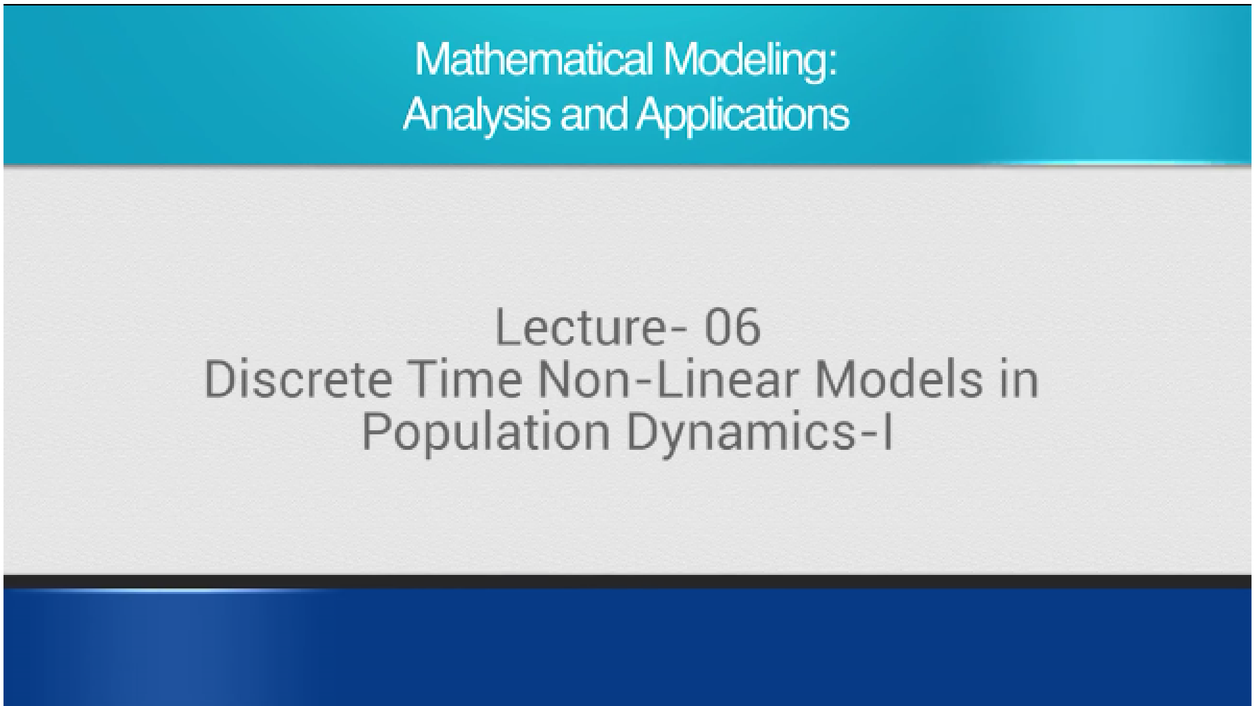
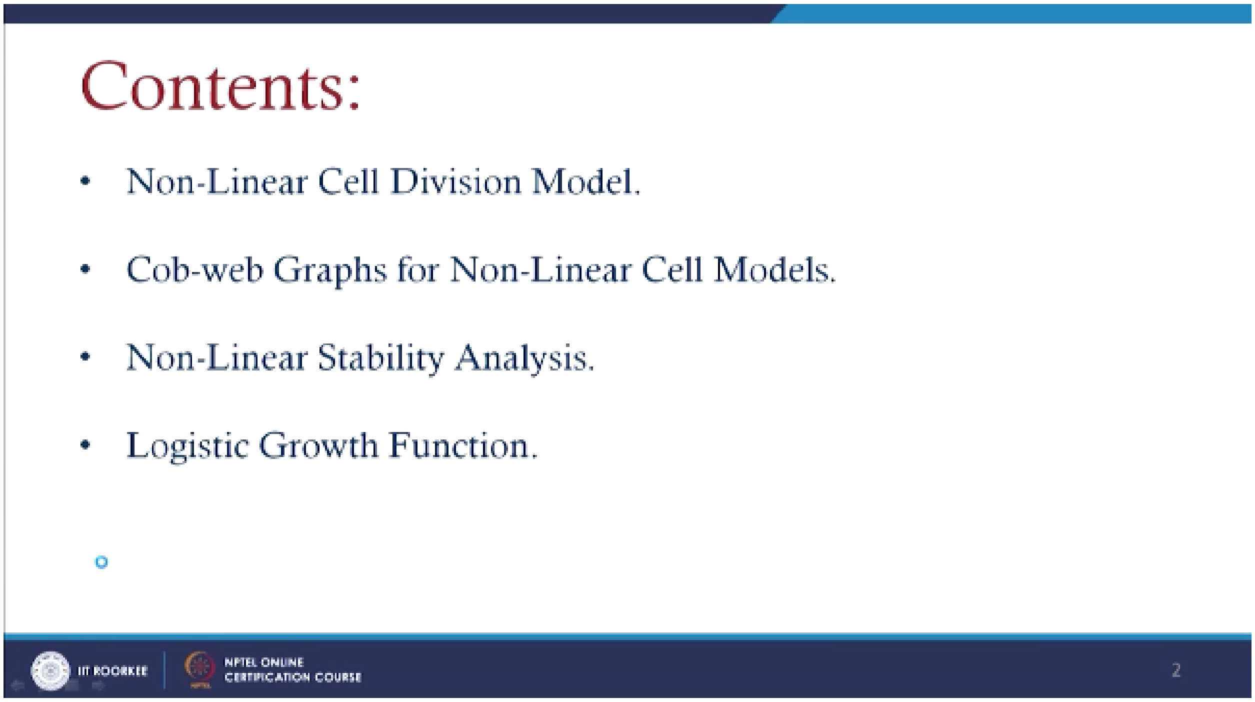
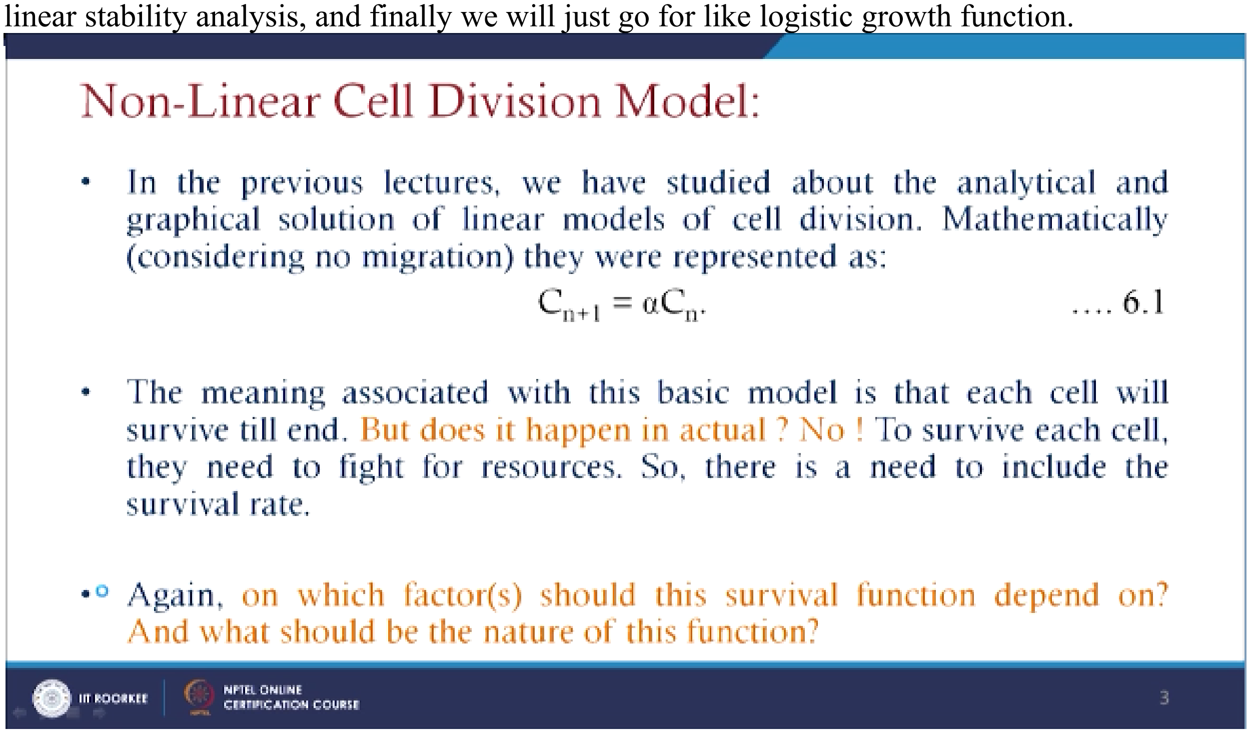


Welcome to the lecture series on Mathematical Modeling Analysis and Applications. In the last lecture series we have just discussed all this linear discrete models based on first order and second order difference equations, considering different examples.

Then we have just discussed the stability analysis and how to find the eigen values and its behavior, considering several examples and several methods. And in this lecture we'll start about discrete time non-linear models in population dynamics.

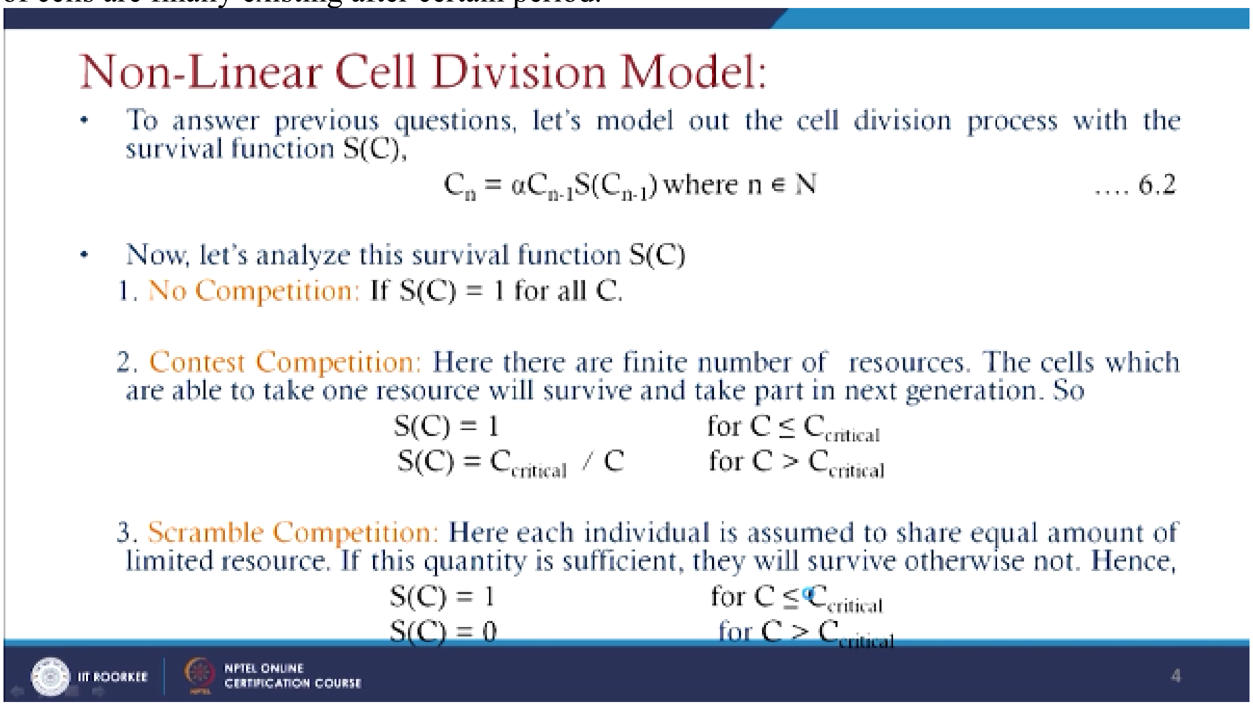


So in the non-linear models, first we'll just consider the cell division model here, and then we will just go for like cob-web graphs for non-linear cell models, then we will just go for like non-linear stability analysis, and finally we will just go for like logistic growth function.



So if you just go for non-linear cell division model here that in the last lecture for linear difference equations we have consider like cell division model also there, and there we have just studied about the analytical and graphical solution of linear models based on the cell divisions, and mathematically if there are no migration they have just represented as CN+1 = alpha CN, and already I have discussed several times that if the migration we'll come to the picture, this cell division model will give you an non-linear model.

The meaning associated with this basic model is that each cell will survive till the end, but does it happen in actually? No, since sometimes the cells are dying due to like unequal distribution of food or deficiency of food or there is a competition between the cells, to survive each cell they need to fight for resources, so there is a need to include the survival rate, how much percentage of cells are finally existing after certain period.

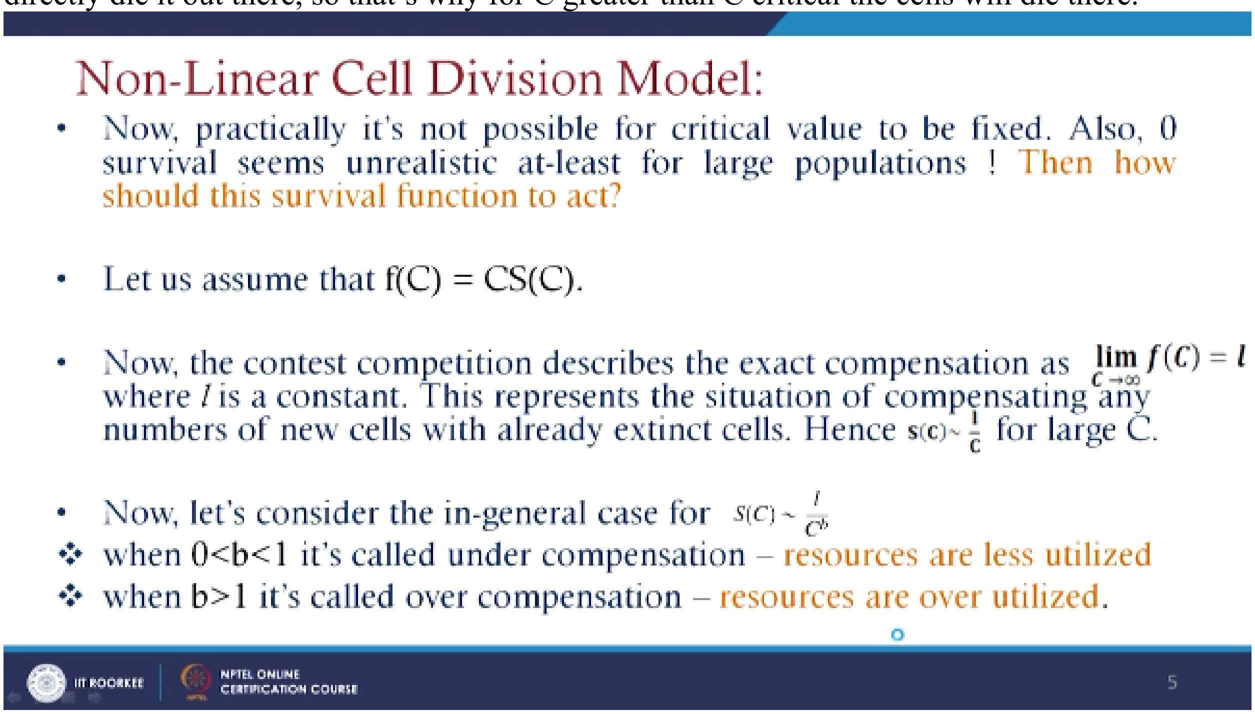


Again on which factors should the survival function depends on, that we have to consider when we are just going for a mathematical model, and what should be the nature of this function? So especially we have to consider like the survival function depending on the factors and what should be nature of this function, and if we want to find the answer of this questions how the function is behaving or how this function should depended on each other. Then let us model out the cell division process with the survival function SC suppose, so since the earlier model if you just see it is just written as CN+1 = alpha CN, directly proportional to the cells there, and for this case we have to consider like a survival function, it should come to the picture, so that's why we can just consider the total number of cell at time level N which can be represented as the proportional dimensional alpha, then CN – 1 whatever the cells are existing N-1 time level, and how this surviving cells factors are affecting this transfer of the total number of cells to the N-th level.

Now let us analyze the survival function SC, no competition, so we'll have like sufficient food for all the cells, so each cell can find the food and there will be no dying of the cells, so that's why in that situation we can just consider SC = 1, and we can just assume CN = alpha CN-1 that is the natural model or the linear model whatever we have just get for all C.

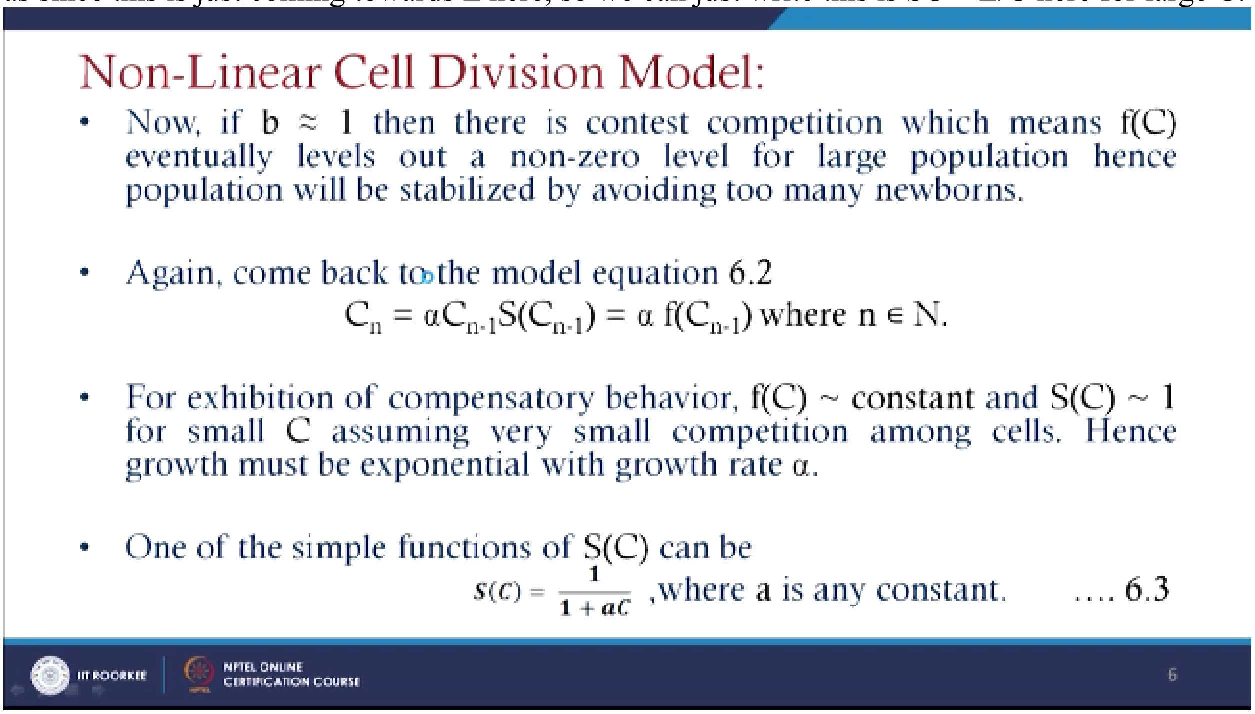
Second case is a contest competition, here there are finite number of resources, suppose we'll have like 100 cells are present, but the food resource is available suppose for 50 cells, then first 50 cells they will just come and they can take the foods then we will have a competition for other cells that they have to like compete this 50 cells to get the food there, so for that we will have, we are just considering that we'll have like resources limitation here, and the cells which are able to take one resource will survive, and take part in the next generation, so we can just consider SC = 1, so first phase that 50 cells will come and take their food, they'll just survive, so we will have like critical parameter, so up to 50 cells we can just define that they can find the proper food without any competition, and we will have then this critical point will exist 50, and after that the cells will not find any food and they will compete with this 50 cells cycle. So that's why that depends on like C critical value with the total number of cells who have got the food they will just try to find the survival there, so that's why we are just writing SC = 1 for C less or equal to C critical up to 50 cells they can find the food, and easily they will survive, for that the survival condition is 1 we are just no competition it is there, but if suppose C greater than C critical this means that after 50 food resources, so then we have to consider like C critical 50 cells whoever got the food division over the total number of cells.

Third condition is that scramble competition, here each individual is assumed to share equal amount of limited resources, and if this quantity is sufficient they will survive otherwise not, this means that so whatever this resource it is available it will be equally distributed with all this cells, and if this quantity is not sufficient then one cell will fight with other to survive over there, so finally if they will have no food then they will die out there, hence we can just consider SC = 1 for C less or equal to C critical, the same condition whatever we, it has been assumed for the earlier case, but in that case the after this like 50 resources of food so all other cells they're fighting with the cells to get the food and to survive, but here there is no option, they will just directly die it out there, so that's why for C greater than C critical the cells will die there.



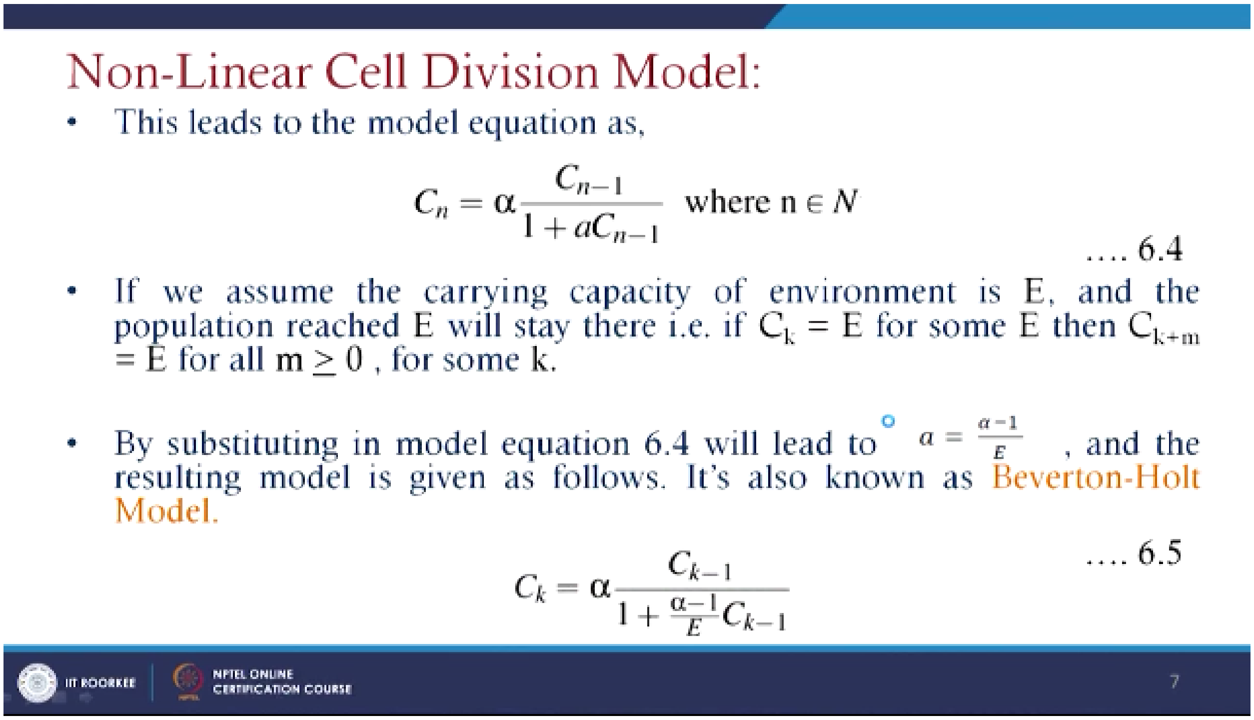
So then we'll just go for non-linear cell division model, practically it is not possible for critical value to be fixed, since the cells will come and they will just take the foods, so we cannot put any barriers or restrictions how they are just consuming the food or how they are surveying over there, also 0 survival since on realistic at least for large populations, then how should the survival function act in that cases, let us assume that suppose the cell function as C into the survival capacity of the cells or the survival function of the cells, or the number of cells which is satisfying the survival conditions.

Now the contest competition describes the exact compensation as limit C tends to infinity, FC = L here, where L is a constant suppose that is a, if we are just taking the limiting condition here, this means infinite number of cells are present then finally if we'll just consider this limiting case will have like a constant level we can just get it over there, this represent the situation of compensating any number of new cells with already extinct cells, hence SC can be approximated as since this is just coming towards L here, so we can just write this is SC = L/C here for large C.



Now let us consider in general case for SC is approximated suppose 1/C to the power B here, that is couple of factors that will just affect to this cell populations, we can just assume a nonlinear model, and that's why we can just assume that SC can be approximated as 1/C to the power B here, so when B lies between 0 and 1 it is called under compensation resources are less utilized, this means that foods will be sufficiently present there, and when B greater than 1 it is called over compensation, resources are over utilized, this means that this is consumed by all the cells at a time, so then if you'll just go for non-linear cell division model, if B is assumed to be 1, then there is a contest competition which means FC we are just assuming here, and which is eventually levels out a nonzero level for large population, hence the population will be stabilized by avoiding too many newborns.

If sufficient number of like cells will be present, so certain restrictions or constants they will find to reproduce again the cells, since due to maybe climatic conditions or maybe due to non-availability of resources of food or certain constants means they cannot find the proper space to like growth of this population levels. Again if you'll just come back to the model based on this equation 6.2, then CN can be written as alpha CN-1, and SCN-1 which can be expressed as alpha, since this is just a complete function here CN-1 as CN-1 which can be written as SCN-1 for N belongs to N.



For exhibition of compensatory behavior, if you'll just assume this function F(C) is constant here, and S(C) is approximated as 1, for small c assuming very small competition among cells, we can find that the growth must be exponential with growth rate of alpha here, and one can use as a simple function of S(C) which is a nonlinear function based on this exponential growth rate of alpha as 1/1+AC here, where A is any constant, directly this function is just constructed, and it has been verified whether this is experimental observations for the cell culture techniques, so this leads to the model equation as if you'll just see in the previous slide, so CN is represented as alpha, CN-1, SCN-1, and if you'll just write all this factors this model can be written as CN = alpha, CN-1/1+ACN-1, where N belongs to N, if we assume the carrying capacity of environment is E suppose, and the population reached E will stay there, that is if CK = E for some E then CK+M = E for all M greater or equal to 0 for some K here, and if we'll just substitute in model equation of 6.4 this will just lead us to like A = alpha -1/E here, and the resulting model is a given as CK = alpha, CK-1/1+ alpha -1/E CK-1, and this model is known as Beverton-Holt model, and if you'll just go for like further analysis of this model, this model can

Non-Linear Cell Division Model:

- This model can further be generalized as:

$$C_k = \alpha \frac{C_{k-1}}{(1 + aC_{k-1})^b} \quad \text{.... 6.6}$$

- This is also known as Hassel Model. Depending on b, it compensates both contest competition (for b=1) and scramble competition (for b>1).

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be written in a generalized form as CK alpha, CK-1/1+ACK-1 B here, so just linearization we have just simplification we have just made it out here, this is also known as Hassel Model depending on B, since it compensates both contest competition for B =1, and scramble competition for B greater than 1.

Stability of Non-Linear System:

- There are relatively very few cases where analytical solution of non-linear difference equations can be computed directly.

- A general non-linear difference equation is in form of:

$$x_{n+1} = f(x_n, x_{n-1}, \dots) \text{ where } n \in N. \quad \text{.... 6.7}$$

- Thus, we must have to be satisfied with the nature of system (stability analysis) or with the computer generated solution (numerical analysis).

- In order to study the nature of system, let's consider a general first order equation (because higher order equations can be converted into system of first order equations. How?).

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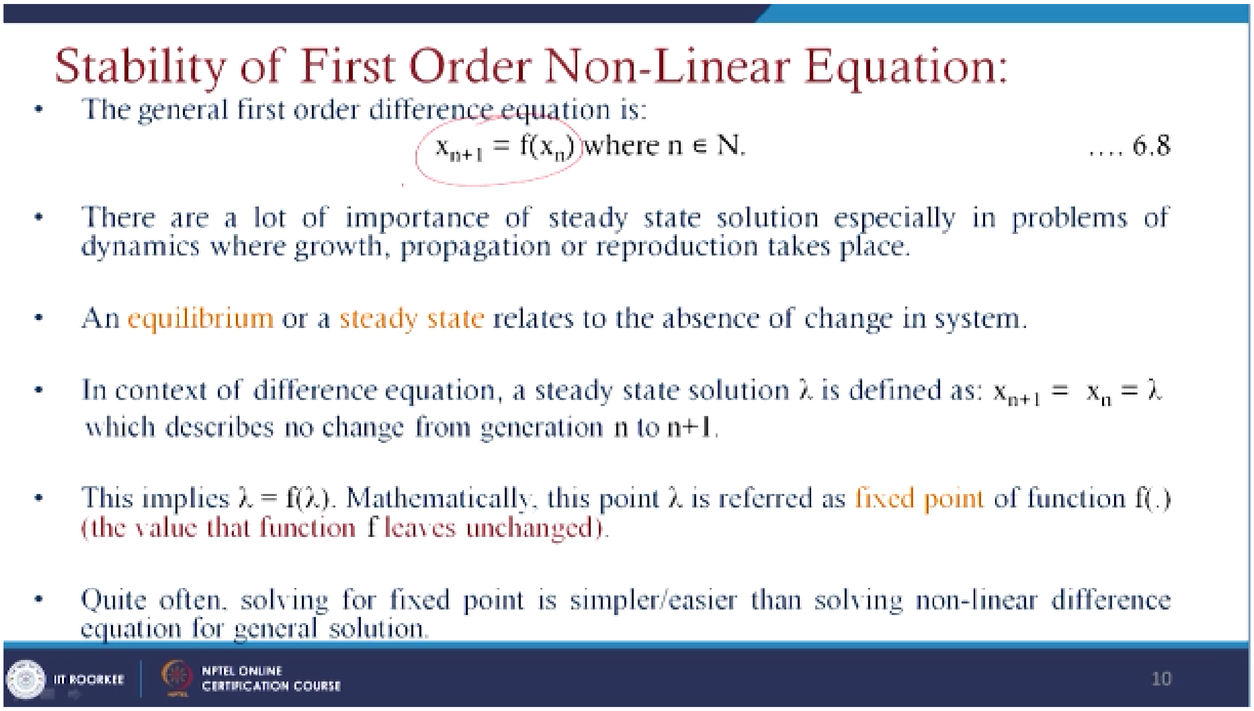
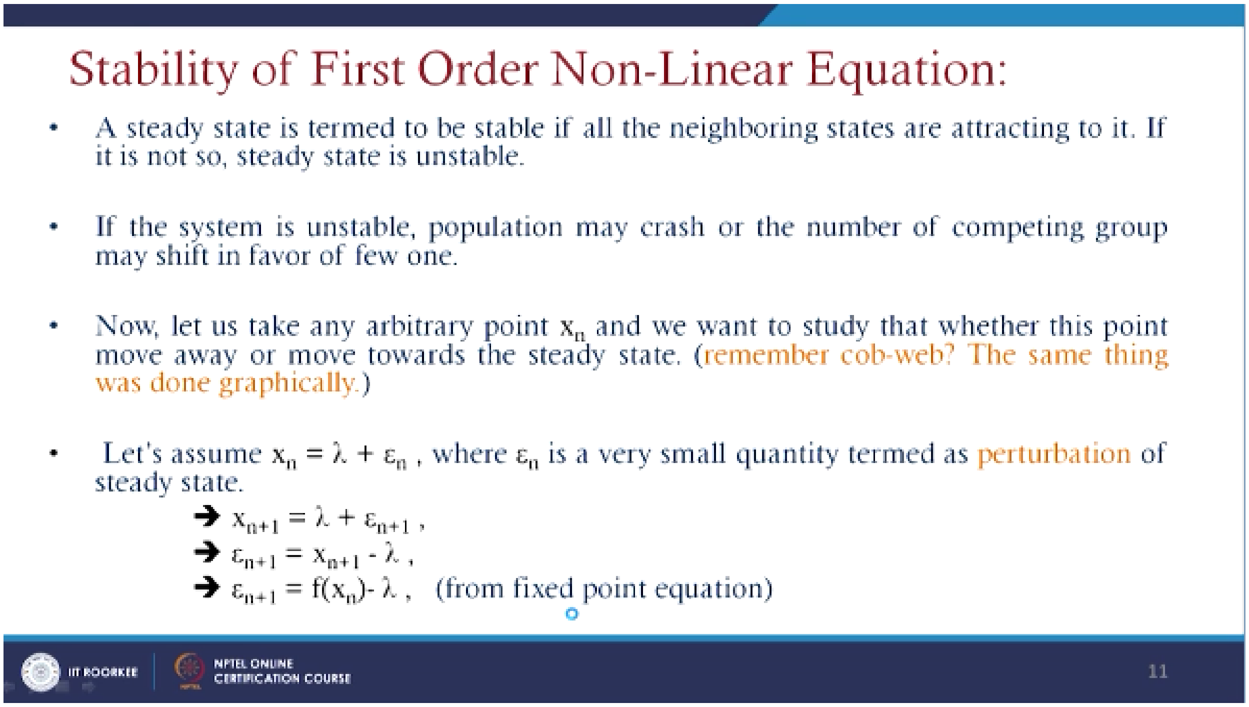
So then we will just go for stability of this nonlinear system, there are relatively very few cases where analytical solution of nonlinear difference equation can be computed directly especially we can just say that it is very few or we cannot find also in case of nonlinear equation the analytical solutions, a general nonlinear difference equations especially it can be represented as

XN+1 as function of X1, XN, XN-1 up to X1 there, where N belongs to N, thus we must have to be satisfied with the nature of the system first that is the stability analysis we have to justify or with the computer generated system that is in numerical computation we have to do to find the solution.



In order to study the nature of the system let us consider a general first order equation, because higher order equations with a N coefficients suppose this can be converted into a system of first order equations, so you have to analyze that in last lecture of that I have just done for numerical methods if you just follow, then you can just find that a method of difference equation it can be transformed by using certain variables to a first order difference equation, so if you just go for stability analysis of first order nonlinear difference equations and then first we will just consider first order difference equation that is in the form of like XN+1 = F(xn) suppose where N belongs to N here, there are a lot of importance of steady state solution especially in problems of dynamics where growth, propagation and reproduction takes place.

An equilibrium or a steady state especially that we have just discussed in our earlier lectures, relates to the absence of change in the system, since when with respect to time there is a no change then we will have a steady state. In contest of difference equation a steady state solution lambda is defined as XN+1 = XN = lambda, whenever we will have like a fixed point solution then we are just saying it is a steady solutions there, that is after that we did not find anything always we'll have the same value at each of the iteration levels, which describes no sense from generation N2 N+1, this implies that lambda = F(lambda) mathematically this point lambda is refer as fixed point of a function, any function, this function takes the value and it's value will not get changed, quite often solving for fixed point is simpler, easier, then solving nonlinear difference equation for general solutions.



And if you just go for like the steady state solutions which is said to be stable, if all the neighboring states are attracting to it, and if it is not so then the steady state is unstable, and if the system is unstable, population may crash or the number of competing group may shift in favor of few one there.

Now let us take any arbitrary point suppose XN, and we want to study that whether this point move away or move towards the steady state, so if you'll just see the cob-web method there, so you can just find that first we are just plotting a graph Y = X and if you'll have this function curve like Y = F(x) which existing there for any function, so first we are just putting this point like X0, then we'll have functional value here, then we are just putting perpendicular, then we are just dropping this one, then we are just moving this way, so again we'll just putting this one and we are just moving towards this origin there, the same thing we can just do for graphical sense here also.

Stability of First Order Non-Linear Equation:

➔ $\varepsilon_{n+1} = f(\lambda + \varepsilon_n) - \lambda = f(\lambda) + \varepsilon_n \frac{df}{dx}|_{x=\lambda} + O(\varepsilon_n^2) - \lambda.$

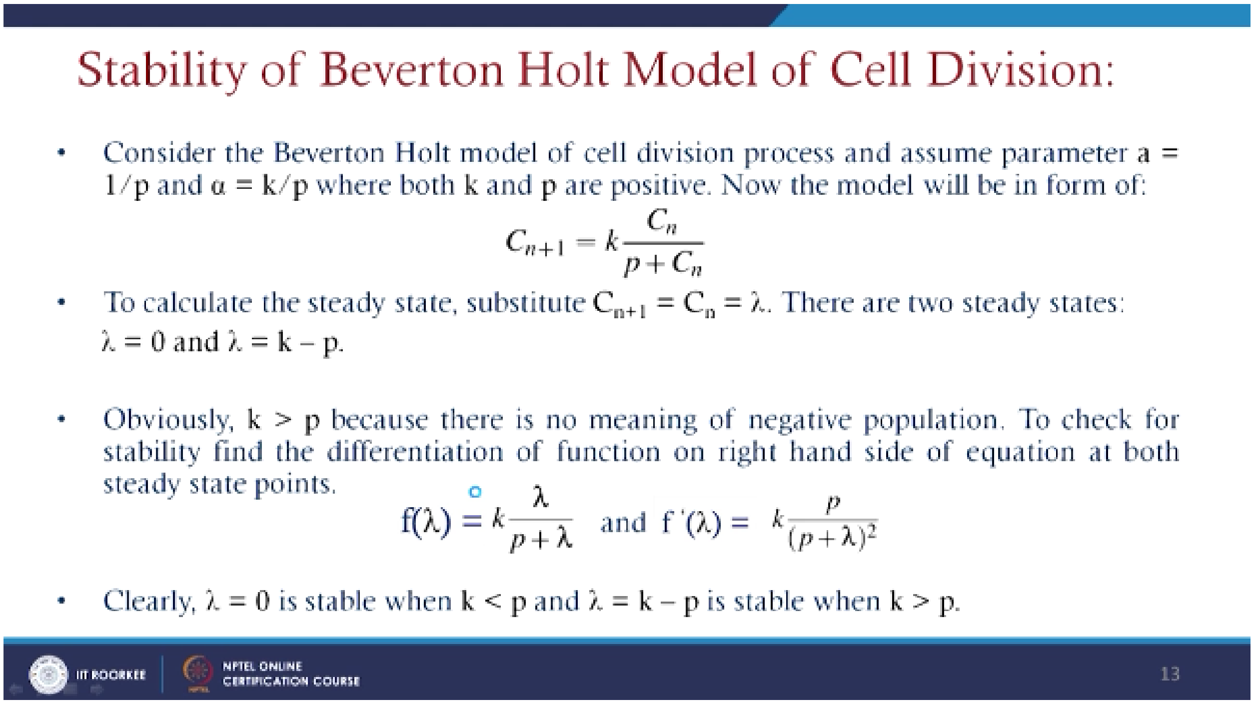
Since ε_n is very small, so we can expand function f by Taylor's series. After neglecting 2nd and higher order terms,

➔ $\varepsilon_{n+1} = f(\lambda) - \lambda + \varepsilon_n \frac{df}{dx}|_{x=\lambda} = \varepsilon_n \frac{df}{dx}|_{x=\lambda}$

- This equation is in form of:
  $\varepsilon_{n+1} = a\varepsilon_n$ where $n \in N$ and $a = \frac{df}{dx}|_{x=\lambda}$
- Recall from previous lectures, the system will be stable if $|a| < 1$ and unstable if $|a| > 1$.

And next if you'll just assume suppose XN can be written as lambda + epsilon N suppose, suppose every term is existing for the calculation of lambda then we can just consider the exact solution XN as lambda + epsilon N sometimes it is called the portal solution and especially here we can just say that epsilon N is a very small quantity which is termed as perturbation of the steady state, and if you'll just write XN as lambda + epsilon N here, then XN + 1 can be written as lambda + epsilon N+1, and the value of epsilon N+1 which can be generalized as XN+1 – lambda and which can be written as like F(xn) – lambda, since XN+1 can be expressed as a function of F(xn) for fixed point equation, so then we can just write this epsilon N+1 as function of lambda + epsilon N – lambda there, since it is a fixed point function and which can be written by Taylor series expansion since function is a continuous function as we have assumed and epsilon N is a very small parameter there, so we can just expand it as like F(lambda) + lambda DF/DX at X = lambda, and so directly if you'll just expand this one so we can just write this one as like F(lambda) + epsilon as F(lambda) + epsilon N, if it is lambda + epsilon N square/ factorial 2, F double dash (lambda) in that form we can just write, and if you'll just see here that F(lambda) it is just written, and this you can be written as epsilon N square here, since higher powers of this F double dash (lambda) or higher order terms it is just involved as a epsilon N square here, since epsilon N is very small so we can expand function F by Taylor series, and if you'll just neglect this, so we can just write here also like –lambda should exist, and higher order terms if you'll just neglect here then we can just get F(lambda) – lambda + lambda DF/DX, and since we are just assuming here lambda is F(lambda) so this can be written as lambda DF/DX at X = lambda.

And this equation is in the form of like epsilon N+1 = A epsilon N here, where N belongs to N and A can be written as like DF/DX at X = lambda. Recall from previous lectures, the system will be stable if a mod A less than 1, and unstable if mod A greater than 1 here.



So next we will just go for stability of Beverton Holt model of cell division, consider the Beverton Holt model of cell division process and assume parameters suppose A = 1/P and alpha as K/P, where both K and P are positive, now the model will be in the form of like CN+1 this equals to K, CN/P + CN, it is the especially the model developed as a nonlinear model here, to calculate the steady state if you just substitute CN+1 = CN = lambda, since fixed points we are just using here, there are two study states, first if you'll just put like lambda = 0, and lambda = K-P here, obviously if K greater than P because there is no meaning of negative population, so population should exist when we are just going for the calculation of this population at different levels, to check the stability find the differences and the function on right hand side of population at both steady state finds, to find that one if you just put here that is a CN+1 as a function of CN here, and then directly we can just write K into lambda/P + lambda, since CN = lambda here, and F desk of CN which can be written as K into P/P+ lambda whole square, clearly if you'll just see here, so lambda = 0 stable, when K less than P and lambda = K-P is stable when K greater than P.

Stability of Beverton Holt Model of Cell Division:

- Now's let's verify the same result by cob-web graphs.
- As one can verify that, all the vertical and horizontal motions are monotonically decreasing, hence the model is stable.
- (Exercise) Observe the nature of cob-web for Hassel model for b=0.5 and 2.

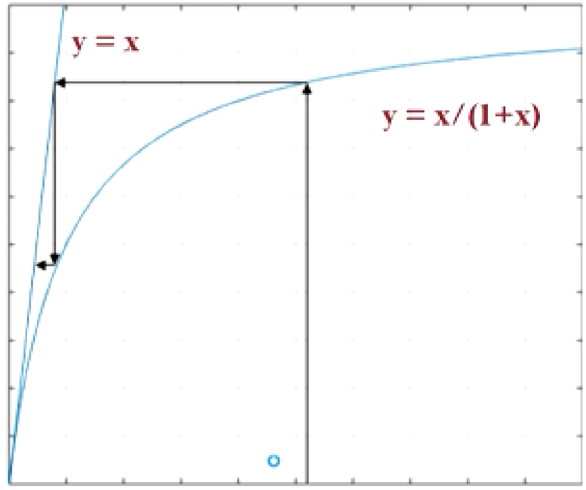


Fig 6.1: Cobweb Graph of Beverton Holt Model

So if you'll just go for this like behavior of the stability of Beverton Holt model of cell division, so we can just verify this result by using this cob-web graphs here, so first if you'll just plot here Y = X's line there, and the function it is just written as Y = X/1+X, and first we are just assuming any point here that is you can just consider as X0 suppose here, then we will have this functional value that is Y0 here, and you are just putting this left moment of this line along this Y = X, then you can just put again this line on the functional value, then we can just move in that form.
So easily we can just verify that all the vertical and horizontal motions are monotonically decreasing, hence the model is stable. So directly if you'll just put here this nature of cob-web model for Hassel model for B = 0.5 and 2, you can just observe that this will be like the stability condition is achieved here.

Logistic Difference Equation:

- We have already discussed several discrete time models in population dynamics. All these model were specific to problem. Can't we think of a model which can govern many classes of systems? E.g. different populations, distinct species, one species at different stage of evolution or development.
- This new model can have variety of meaning depending on the parameters involved in it.
- Now, let's consider a particular type of first order difference equation.

$$y_{n+1} = y_n (r - dy_n) \; . \qquad \ldots .\ 6.9$$

- This equation has 2 parameters r and d. Can we reduce the parameters? YES!. The method used to reduce the parameters is called as non-dimensionalization.
- Why do we need to get rid of parameters? Because, the number of computations are directly proportional to the number to parameters. So, if we are reducing the parameters means we are reducing the computations almost proportionally.

So then we'll just go for logistic difference equations, already we have discussed several discrete time models in population dynamics and all this model were specific to problem, and we cannot think also of a model which can govern many classes of system that is different populations, distinct spaces, one species at different stage of evolution or developments. This new model can have variety of meaning depending on the parameters involved in it.

Logistic Difference Equation:

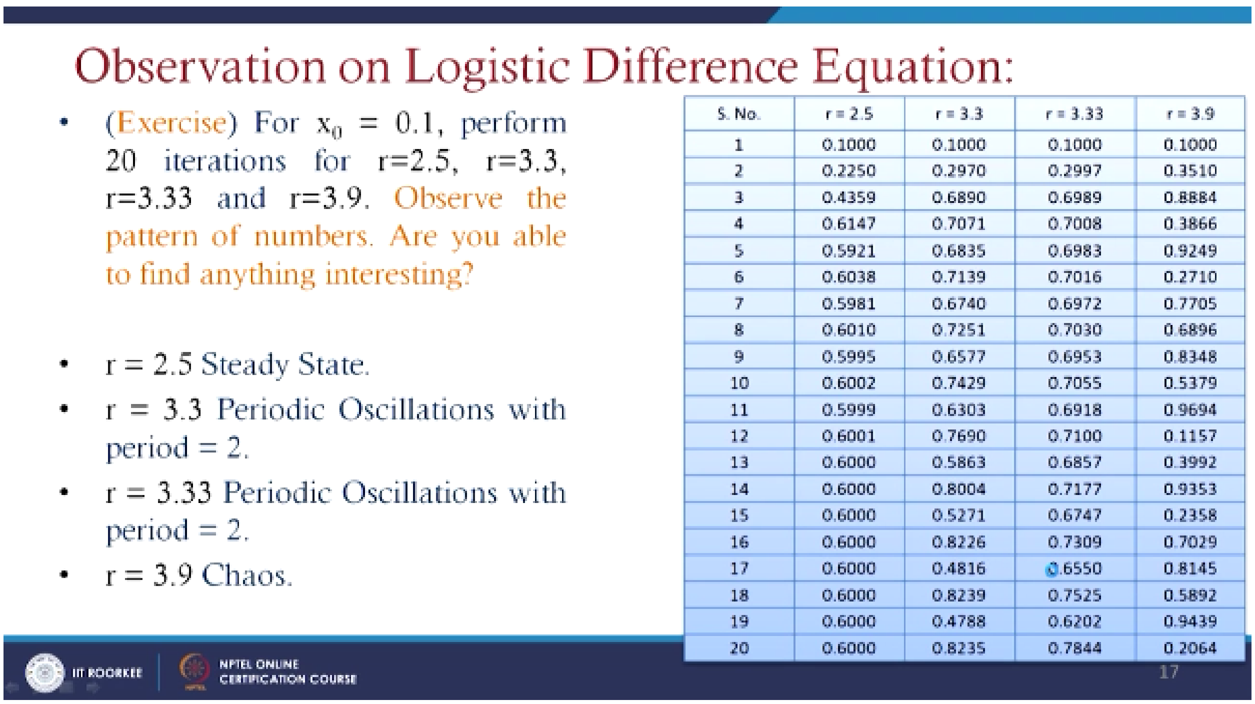
- Now put $x_n = (d/r)\, y_n$. This leads to the following form which is also know as Pearl-Verhulst equation.

$$x_{n+1} = r\, x_n (1 - x_n) \; . \qquad \ldots .\ 6.10$$

- We are left with only one parameter r. To find the steady state solution, let's substitute $x_{n+1} = x_n = \lambda$.
- There are two steady states, $\lambda = 0$ and $\lambda = 1 - 1/r$.
- For the stability, $F(x_n) = r\, x_n (1 - x_n)$ so $F'(x_n) = r(1 - 2x_n)$. At $\lambda = 0$, $F'(\lambda) = r$ and at $\lambda = 1 - 1/r$, $F'(\lambda) = 2 - r$.
- The steady state $\lambda = 0$ is stable for $|r| < 1$ and $\lambda = 1 - 1/r$ is stable for $|2 - r| < 1$ or $1 < r < 3$.

Now let us consider particular type of first order difference equations, suppose YN+1 = YN(R-DYN) here, and if you'll just see this equation involves two parameters R and D here, so can you reduce this parameters? Of course, yes, and the method use to reduce this parameters it is called non-dimensionalization. Why do we need to get rid of these parameters? Since sometimes it is necessary to get rid of these parameters since we'll have like a number of unknown parameters involved in certain equations, and the number of computations are directly proportional to the number of parameters, so if we are reducing this parameters then we are reducing the computations too also, so if we'll just go for this reduction of this parameters, for this equation we can just put here XN as DY or Y, and this leads to the equation as XN+1 = R XN 1-XN, since if you'll just see here or earlier equation it is written as YN+1 as YN(R-DYN).

So if you'll just replace here that is in terms of YN, YN can be written as like R XN/D here, so from this equation directly if you'll just take R common and you will have like or directly you can just replace XN in terms of YN, then you can just find directly the XN+1 = R XN(1-XN) where D is just gone out from this equation, and we are left with only one parameter here R, to find the steady state solution let us substitute XN+1 = XN = lambda here, there are two steady states, so especially we can just say lambda = 0, and lambda = 1-1/R, so for the steady state or for the like stability condition F(xn) it can be written as, if you'll just see R XN(1-XN) here, so F dash(xn) which can be written as like R(1-2XN), at lambda = 0 we can just write F dash (lambda) = R, and at lambda = 1 – 1/R F dash lambda is satisfied as 2 – R here, and the steady state lambda = 0 is stable for absolute value of R, when it is considered less than 1, and lambda = 1 – 1/R is stable for 2 – R is less than 1 or R lies between 1 and 3.


Observation on Logistic Difference Equation:

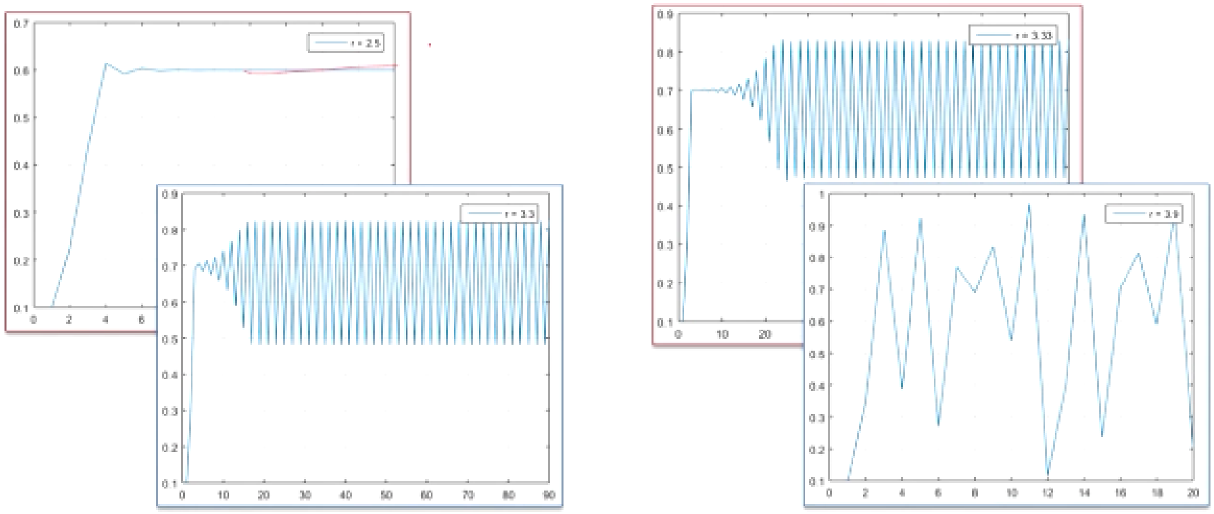
- (Exercise) For $x_0 = 0.1$, perform 20 iterations for r=2.5, r=3.3, r=3.33 and r=3.9. Observe the pattern of numbers. Are you able to find anything interesting?

- r = 2.5 Steady State.
- r = 3.3 Periodic Oscillations with period = 2.
- r = 3.33 Periodic Oscillations with period = 2.
- r = 3.9 Chaos.

S. No.	r = 2.5	r = 3.3	r = 3.33	r = 3.9
1	0.1000	0.1000	0.1000	0.1000
2	0.2250	0.2970	0.2997	0.3510
3	0.4359	0.6890	0.6989	0.8884
4	0.6147	0.7071	0.7008	0.3866
5	0.5921	0.6835	0.6983	0.9249
6	0.6038	0.7139	0.7016	0.2710
7	0.5981	0.6740	0.6972	0.7705
8	0.6010	0.7251	0.7030	0.6896
9	0.5995	0.6577	0.6953	0.8348
10	0.6002	0.7429	0.7055	0.5379
11	0.5999	0.6303	0.6918	0.9694
12	0.6001	0.7690	0.7100	0.1157
13	0.6000	0.5863	0.6857	0.3992
14	0.6000	0.8004	0.7177	0.9353
15	0.6000	0.5271	0.6747	0.2358
16	0.6000	0.8226	0.7309	0.7029
17	0.6000	0.4816	0.6550	0.8145
18	0.6000	0.8239	0.7525	0.5892
19	0.6000	0.4788	0.6202	0.9439
20	0.6000	0.8235	0.7844	0.2064

So if you just go for this observation of logistic difference equations for X0 = 0.1 perform 20 iterations for R = 2.5, R = 3.3, R = 3.33 and R = 3.9, so we can just find that for R = 2.5 after certain time steps we are just getting a steady state here, there is no change of valuation, but in case of 3.3 you can just find a periodic oscillation after certain values it is just obtaining the same values each of the oscillations, the same scenario it is just observed for 3.33 also, but 3.9 you can just find a huge oscillation of values that is nothing but the chaos city has been formed, so that's why we are just written for R = 2.5 after certain iterations we are just obtaining the steady state, and for 3.3 period oscillations with period 2 it is just observed there, and for 3.33 we are just finding this period oscillations of period 2, but for 3.9 it is coming as chaos.

Observation on Logistic Difference Equation:



In a graphical sense if you just see here it is clearly visualized since in this level you are just finding there is no sense of graph here, but here you can just find that the oscillation level is always same here for this two cases, but in this case if you'll just find there is unequal change at each of this levels here, so that's why this graph is specially called your chaos situation.

Summary:

- Different non-linear cell division models.
- Formulation based on survival rate function.
- Non-linear stability analysis.
- Logistic growth difference equation.
- Observation of logistic growth function with a set of parameter values.
- Steady and transient behaviors.

So in this lecture we have discussed like different nonlinear cell division models, then formulation based on survival rate function, and nonlinear stability analysis and logistic growth difference equations, so then we have just studied the observation of logistic growth function with a set of parameter values, then the steady state and transient behavior of the system we have just studied. Thank you for listen this lecture.

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