INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
NPTEL
NPTEL ONLINE CERTIFICATION COURSE
Mathematical Modeling:
Analysis and Applications
Lecture-05
Numerical Methods to Compute Eigen Values
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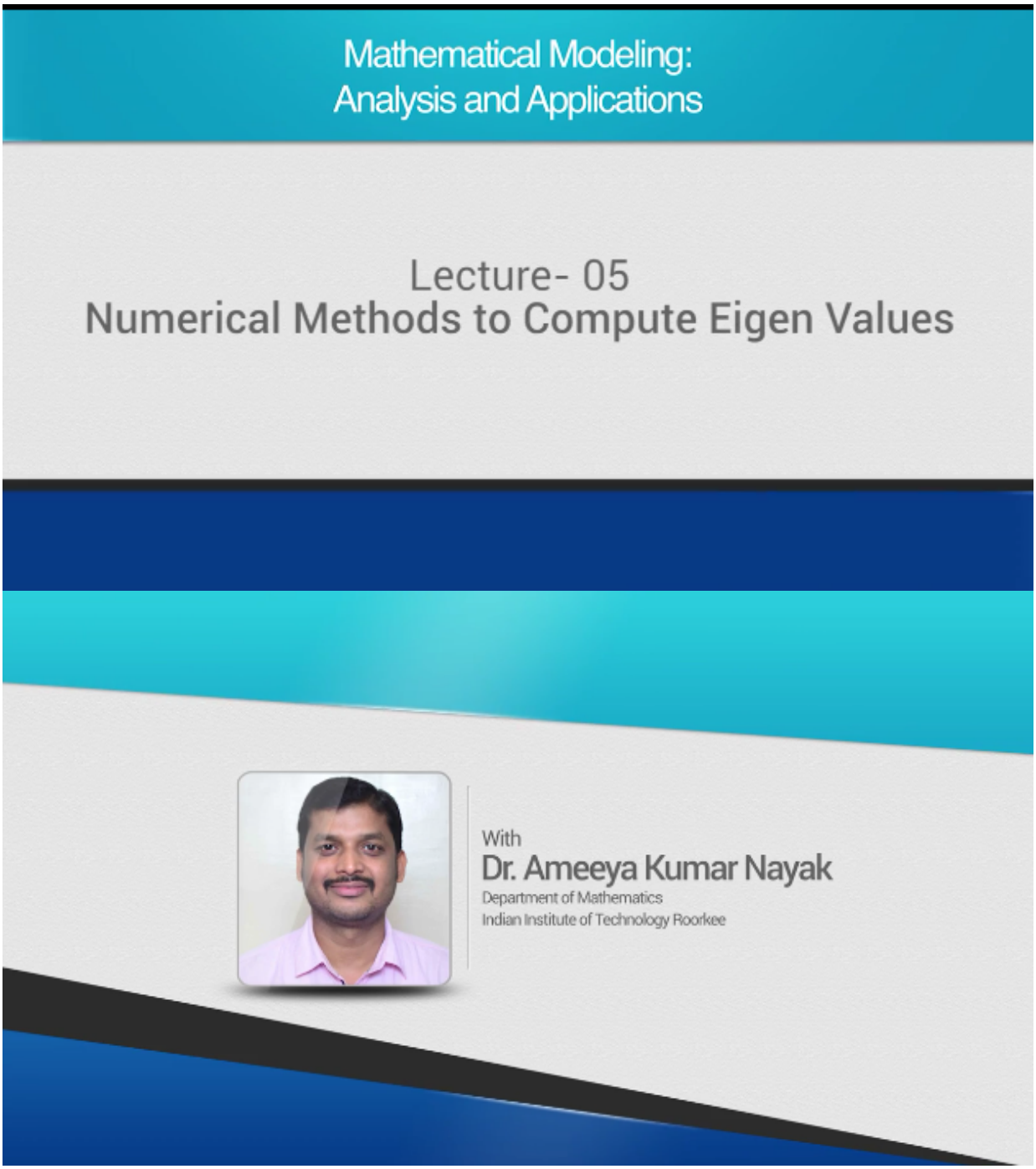


Welcome to the lecture series on Mathematical Modeling Analysis and Applications. In the last lectures we have discussed about linear discrete models based on first order and second order difference equations, and also we have discussed the system of linear difference equations.

Contents:

- Physical Significance of Eigen Values.
- Why Numerical Methods?
- Different Numerical Methods to Compute Eigen Values.
- Power Method.
- LR Method.

So in this lecture we will just discuss about this numerical methods to compute the eigen values, since eigen values is important in aspects of checking this stability, so if you'll just go for this like eigen value computation, so first we have to understand what is the physical significance of eigen values, then why numerical methods has been used for this calculation of eigen values, is it necessary? So thirdly we will just go for like different numerical methods to compute the eigen values, and then we will just discuss here only two methods that is like power method and LR method for the computation of eigen values.

Physical Significance of Eigen Values:

- In the previous lectures, we have talked a lot on eigen values, characteristic equations and stability.
- By the time, you might have realized the importance of eigen values in finding the stability of solution and hence analyzing the behavior of the same.
- Do these eigen values describe something? Or are they mere some numbers?
- Eigen values (and eigen vectors) are very well related with the transformation of a system.
- The need of transformation of system is to make it easier for study and analysis. When you transform any matrix, you need two things:

1. By what amount are you going to transform your system?
2. In which direction are you transforming it?

So if you'll just go for this physical significance of eigen values that we have just discussed in our previous lectures that the eigen values will be used for this calculation of this stability

condition or the characteristic equations can be derived from the system of equations, in terms of matrix notations, and then if the eigen values are calculated then it is just showing whether the system is stable or unstable. By the time you might not have completely realize the things, since we don't know exactly the importance of eigen values in finding the stability of solution exactly, and hence analyzing behavior of this eigen values is important to understand or they realize the importance of the physical behavior inside the system.

So do these eigen values describe something or are they mere same, just representing some numbers, so eigen values and eigen vectors where we are just saying that if eigen values are existing for a system of equations then correspondent to each eigen values we will have certain eigen vectors, maybe it is repeated or maybe it is like distinct, so they are very well related with each other for a transformation of any system. The need of transformation of system is to make it is here for study and analysis, when you transform any matrix you need two things at best I can just say, by what amount are you going to transform your system and in which direction you are transforming, so if you will just consider here these two points then we can just study the behavior of the system which is represented in the form of a matrix.

Physical Significance of Eigen Values:

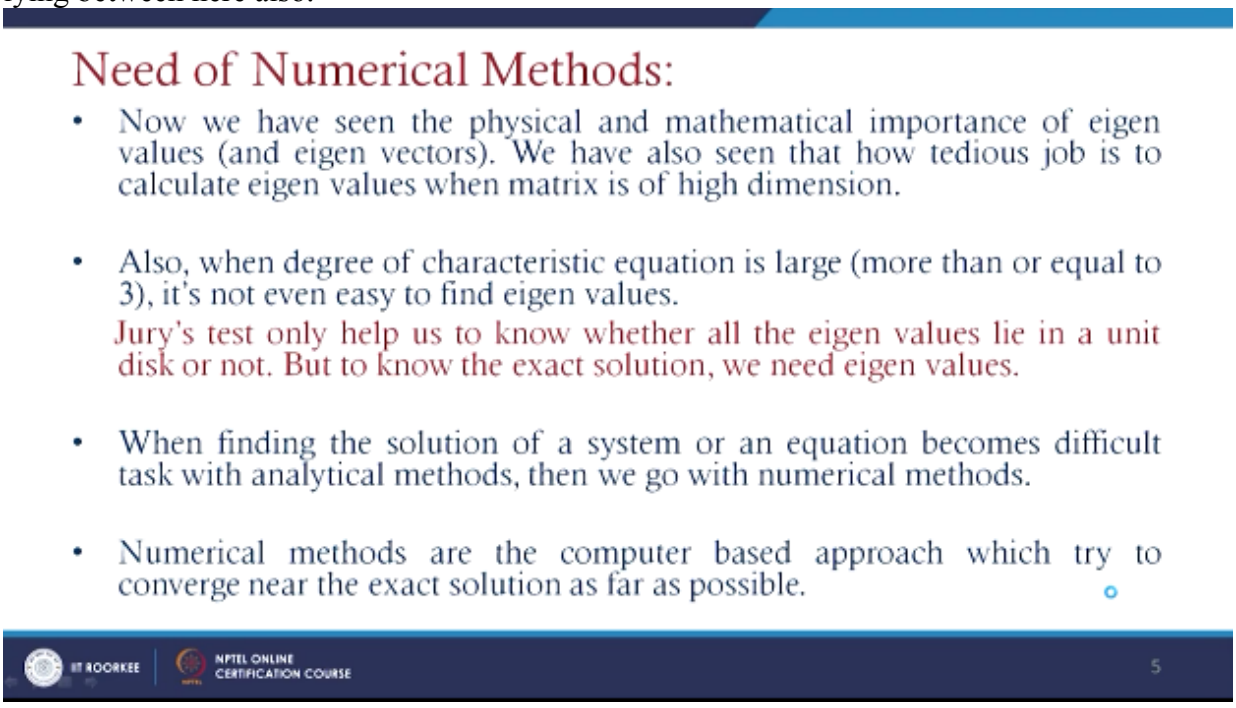
- The answer to your first question is given by the eigen values and to second is given by the associated eigen vectors.
- Suppose you have a 2 x 2 matrix and it's eigen values are λ_1 ,λ_2 and corresponding eigen vectors are v_1 ,v_2 respectively.
- λ_1 gives the amount by which one is transforming matrix in direction v_1 . Similar is the case with λ_2 in direction of v_2 .

If $0 < \lambda < 1$, means system is contracted in the direction of v.
If $\lambda > 1$, means system is expended in the direction of v.
If $-1 < \lambda < 0$, means system is contracted in the opposite direction of v.
If $\lambda < -1$, means system is expended in the opposite direction of v.
If $\lambda = 0$, means system is not at all transformed in the direction of v.

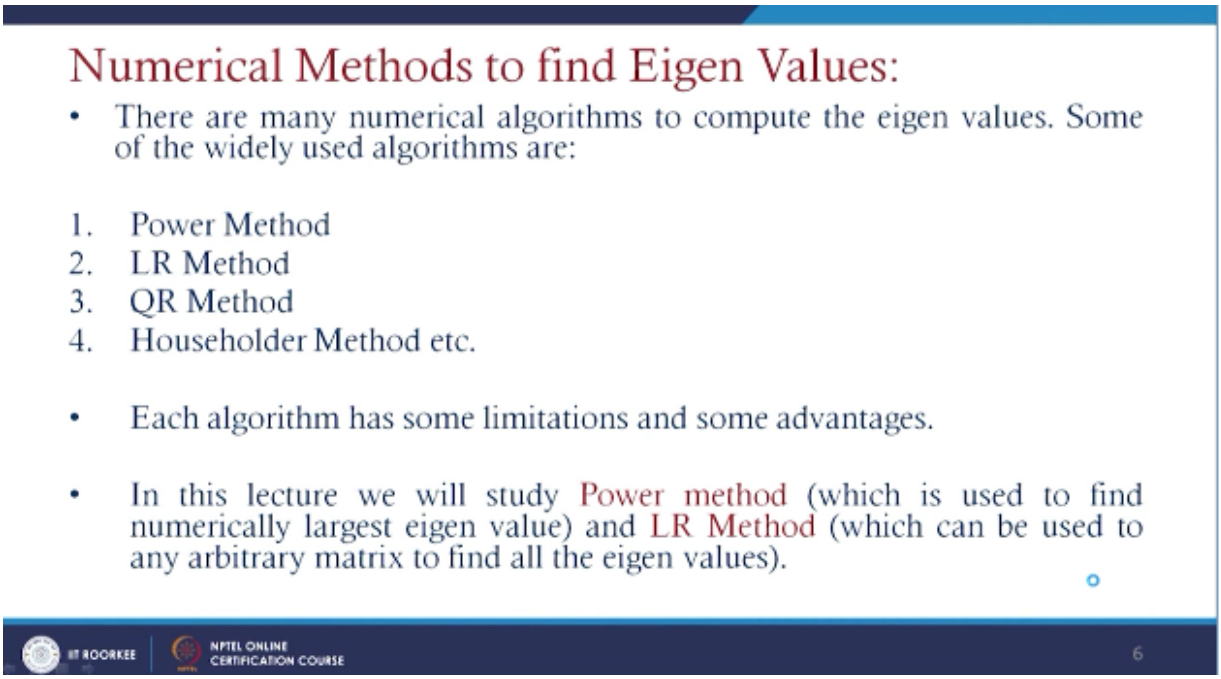
If you'll just go for like little depth in physical sense of eigen values you can just find that the first question will arise that the eigen values is given, if it is given how it is associated with the eigen vectors, suppose if you have a 2 x 2 matrix and it's eigen values are suppose lambda 1 and lambda 2 and corresponding eigen vectors you are just representing as V1 and V2 there, since especially whenever we are just writing this eigen values, so especially it can represented in terms of a vector suppose, V = lambda 1, U1 + lambda 2 V2, this is expressed as linear combines of V1 and V2 also we can just set.

So if you'll just go for like this transformation of the system, so if we will have like two distinct eigen values suppose lambda 1 and lambda 2, so lambda 1 gives the amount by which one is a transforming matrix in direction V1, since V1 is the associated eigen vector with this eigen value lambda 1. Similarly in case 2 with lambda 2 in direction of V2, since especially we can just say that this eigen values they are just transforming the systems in such a way that it just

takes a direction which is the associated vector direction, and if suppose the eigen values, value is lying between 0 and 1 the system is contracted in the direction of V, since it is already a bounded system there, if lambda greater than 1 this means we are just going far from 1 there, means the system is expanded in the direction of V, and if suppose lambda greater than -1 and it is less than 0, then the system is contracted in the opposite direction, since if you'll just define the axis here, and we will have like the values or existing, suppose 1 here, then 0 here, then -1 here, so we are just assuming when this lambda is lying between here, and when this lambda is lying between or when we are just considering lambda lying between here, and also lambda lying between here also.



So this four situations or the conditions we are just assuming to study the behavior of the system whether it is just transforming towards the direction of V1 or V2 or if we are just combining the system then whether it is just behaving towards V or not. So if we are just seeing here lambda is less than -1 suppose, the system is expanded in the opposite direction of V, and if lambda = 0 means the system is not at all transformed in any direction of V, so if we will just go for this numerical methods, since as we have discussed in the earlier lecture that always we will not have like analytical solutions, since this analytical solutions can be possible whenever we will have like quadratic equation, and after that it is not easy to find the values when this degree of the equations are greater than 2. And any mathematical problem associated with this physical significance it is important in the aspects of eigen values and each of this eigen values are associated to the eigen vectors and also we have seen that how tedious job is to calculate the eigen values when matrix is of high dimensions. Also when degree of characteristic equation is large that is more than or equal to 3 it is not even easy to find the eigen values, so already we have discussed the Jury's test in the earlier lectures that it can just help us to know whether all the eigen values lie in a unit disk or not, but to know the exact solution we need the eigen values.



When finding the solution of a system or an equation becomes difficult task with analytical methods, then we are just going for the numerical methods, so numerical methods will provide you approximate solution nor the exact solution, since this is associated with certain order of negligence there, so usually numerical methods are the computer based approaches which try to converge near the exact solution as far as possible, since we are just neglecting here higher order terms, by considering some aspect ratios, so first if you'll just go for like numerical methods for the computation of eigen values there are many numerical algorithms to solve or to compute the eigen values, some of the widely used algorithms for finding the eigen values are like power method, LR method, QR method, and householder methods, but each method has also certain like discrepancies or like drawbacks are there. And in this lecture we will just go for like power method and LR method which is used to find the eigen values and the first method power method is used for this like largest eigen value for a matrix A, or we can just say the smallest value of matrix A inverse, and LR method which can be used to any arbitrary matrix to find all the eigen values.

Power Method:

- The power method is a computer oriented method that finds the numerically largest (dominant) eigen value of a given matrix and the associated eigen vector.
- In order to find sub-dominant, say second largest eigen value, the size of matrix can be reduced by one and the power method can be applied on newly formed matrix.
- Power method is an iterative method (in-direct method) so we need some initial assumption to start the process.
- After each iteration, we will compare the solution obtained with previous iteration solution for convergence. If after some iterations, difference between two successive iteration solution are less than a certain quantity, then the solution is said to be converging.
- This certain quantity relates the accuracy of method for a particular matrix.

So in the power method especially the method is based on like computer oriented method that finds the largest numerically dominated eigen value, sometimes in this method also we are just shifting the points towards the origin or towards the like desired point there, so that is especially called shift of origin points, and we are just shifting this points to find the largest eigen value close to that points.

Example on Power Method:

Question: Find the largest eigen value of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Solution: Assume the initial approximation to eigen vector is $X^{(0)} = [1 \quad 0 \quad 0]^T$.

$$AX^{(0)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

In order to find this like subdominant say second largest eigen value, the size of the matrix can be reduced by 1 and power method can be applied on newly formed matrix. And power method is an iterative method especially it is also a costly method and we need the initial assumptions to start the process and after each iteration we'll compare the solution obtained with the

previous iteration solution for the convergence criteria, and if after certain iterations difference between two successive iterations achieve this degree of accuracy then we can stop this iterations, and that solution is said to be a converge solution, this certain quantity relates the accuracy of method for a particular matrix. So for example if you'll just go for a power method then we can just find the largest eigen value of any system of equations by considering the initial approximation, if the initial approximation has not been given to us, then we can just assume as 1 1 1 or 1 0 0 or 0 1 0 or 0 0 1 there as the initial as vectors.

So especially why it is called like the method to find the largest eigen vector is that, we have to assume that the eigen values are placed in the form like lambda 1 should be larger than lambda 2, should be larger than lambda 3, so like that way we will have to consider here, since whenever we will just find this eigen values so if we are just writing in a like a determinant form that is determinant of A – lambda I = 0 here, then if A is like N x N matrix this will just represent a polynomial of degree N here, and if we are just considering the lambda 1, lambda 2 up to lambda N are arranged in this form and each lambda I's are distinct then this will produce like the eigen vectors are linearly independent or if the eigen values are distinct we will have like N number of linearly independent eigen vectors will assessed with this distinct eigen values here.

And if we can just find that any vector which can be expressed as a linear combination of like lambda 1 V1 + up to lambda N VN here, since lambda 1 is the largest element we have just consider here and we can just consider lambda 1 common here or if we will just pre-multiply this matrix A for this V element here we can just write it as lambda 1 AV1 + lambda 2 AV2 up to lambda N AVN here, and obviously we know that AV1 that is nothing but lambda 1 V1 again, then lambda 1 + lambda 2 and again you will have like V2 lambda 2, so likewise we can just write also, so in the first matrix if you'll just take common suppose lambda 1 here we can just write V1 + lambda 2/lambda 1 V2 up to lambda N/ lambda 1 VN, and if you'll just see here lambda 2/lambda 1 it is like less than 1 here, and if you'll just lambda N/lambda 1 it is also less than 1 here, and consecutively if you'll just multiply this matrix suppose N times, we will have like here A to the power NV, this will just give you like lambda 1 to the power N here, then you will have V1 + lambda 2/lambda 1 whole power N, V2 + remaining elements.

When N turns to infinity then lambda 2/lambda 1 it is a very small quantity and all other quantities it can be just neglected, and we are just remaining here with lambda 1 to the power N, and if you'll just take like AN+1 V here, this will just give you lambda 1 to the power N+1 and remaining factors it will be just there, and if you'll just compare this coefficients we can just find that lambda 1 is the largest eigen value for this matrix here, and based on this approaches we are just assuming that at each level we are just taking the largest element common here, so that's why we are just writing this matrix in the form like AX0 as like 2 -1 0 -2 -1, 0 -1 2 and 1 0 0 here and if you'll just multiply then we can just find this is 2 -1 0 here, and if you'll just take common the largest whatever we have just done for this system here also the same thing we'll just do here that is 2 if you'll just take common it can be written as 1 -0.5 0 which can be written as lambda 1 X1 here.

Example on Power Method:

- By repeating the iteration process for some more times, we will get:

- Clearly, $\lambda^{(5)} = \lambda^{(6)}$ so the largest value of eigen value is 3.41 (up-to accuracy at second decimal place).

$$AX^{(3)} = 3.43 \begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

- The corresponding eigen vector is $X^{(6)} = [0.76 \ -1 \ 0.65]^T$.

$$AX^{(4)} = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

- To find the smallest eigen value, one can use the power method on $[A]^{-1}$.

$$AX^{(5)} = 3.41 \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

And if we'll just multiply this X1 vector again with this matrix A here then we can just find this is the complete vector here, and from that if you'll just take largest magnitude of the vector so it can be just taken common as 2.5 here, then the remaining elements are like 1 -0.5 and 0.2 here, which can be written as lambda 2 X2 here, and this spaces will continue and finally we can just obtained AX3 as 3.43 0.87 -1 0.54 which can be written as like lambda 4, X to the power 4 here, again we have to multiply X4 with the matrix here, which will just give you 3.41, 0.80, -1, 0.61 which also can be written as lambda to the power 5, X to the power 5, again we will just take X5 here and multiply with this matrix A, then we can just find as lambda 6 X6 here, but the difference if you'll just see here that is with this eigen vectors that is 0.76, –0.80 and then -1, -1., 0.65, 0.61, so whatever this accuracy we want to find based on that we can just find that one, accuracy means we can just say that XI K+1 –XI K, it should be less or equal to your desired accuracy that is 1/2 x beta to the power –T here, beta is the base and P is the number of, diseased or decimal places you want to find the accuracy here.

And to find the smallest eigen value one can use the power method that is if you'll just find A inverse the largest eigen value so that is the smallest eigen value of A inverse there, or we can just say that for smallest eigen value of A inverse is 1/lambda here.

LR Method:

- The LR Method was suggested by a Swiss Mathematician H. Rutishauser in 1958.

- This method can be used for any arbitrary matrix to compute all the eigen values. This method generally takes two steps:

$$A_k = L_k R_k,$$
$$A_{k+1} = R_k L_k. \quad \ldots. 5.1$$

where k = 1, 2, 3, … and $A_1 = A$. L_k is unit lower triangular matrix while R_k is upper triangular matrix.

- For large k, matrix A_{k+1} generally approaches to an upper triangular matrix. (**Note:** The determinant of an upper or a lower triangular matrix is simply the product of its diagonal entries.) So the eigen values are given by the diagonal elements.

So then we will just go for LR method here, the LR method was first suggested by Swiss Mathematician H. Rutishauser in 1958, and this method can be used for any arbitrary matrix to compute all the eigen values, and this method generally takes two steps, in the first step we can just write AK = LK RK, and second step we can just write AK+1 this as RK LK, if you like L and R both are symmetric matrices then both these values are equal there, where K = 1, 2, 3, obviously we can just take and first we can just consider A1 = A here, and we are just taking here LK is the unit, lower triangular matrix well RK is the upper triangular matrix. And for large K matrix AK+1 generally approaches to an upper triangular matrix that is the determinant of an upper or a lower triangular matrix is simply the product of its diagonal entries only, so the eigen values are given by the diagonal elements, obviously sometimes if we are just finding only the diagonal entries are there that is nothing but the eigen values of this matrices.

Example on LR Method:

Question: Find the eigen values of matrix [A] given as: $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

Solution: $A = A_1 = L_1R_1$. Assume $L_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix}$ and $R_1 = \begin{bmatrix} \alpha_2 & \alpha_3 \\ 0 & \alpha_4 \end{bmatrix}$

From first iteration, one can find the values of α's as:

$\alpha_2 = 3$, $\alpha_3 = 2$, $\alpha_1 = 1/3$ and $\alpha_4 = 10/3$.

Now $A_2 = R_1L_1 = \begin{bmatrix} \frac{11}{3} & 2 \\ \frac{10}{9} & \frac{10}{3} \end{bmatrix}$. Again find the L_2, R_2 by the same method as followed for L_2, R_2. This process will be repeated until we get the same matrix [A] in the successive iterations. After performing this iteration 10 times you will get,

$$A_9 = \begin{bmatrix} 4.9952 & 2 \\ 0 & 2.0023 \end{bmatrix} \qquad A_{10} = \begin{bmatrix} 4.9976 & 2 \\ 0.0024 & 2.0023 \end{bmatrix}$$

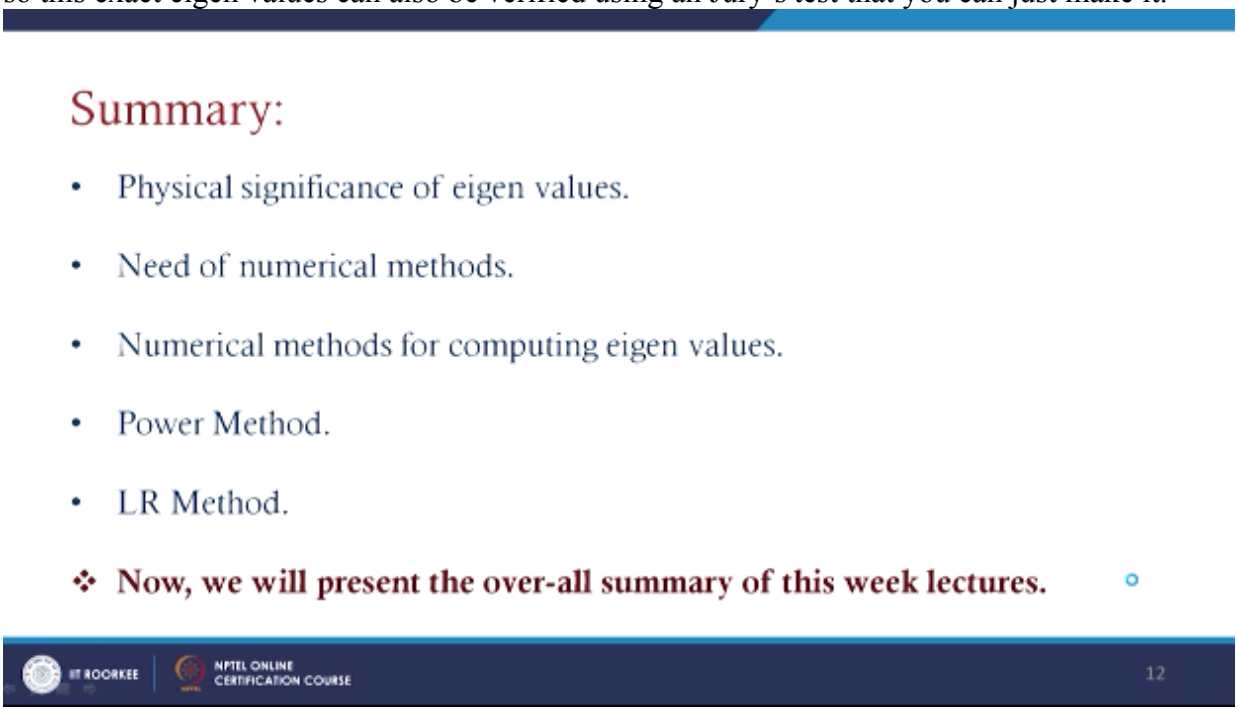
You can observe that A_9 and A_{10} are almost identical and diagonal matrices. Hence the eigen values are 5 and 2. (Verify with exact eigen values and with Jury's test!)

So if you'll just consider one example on LR method, so suppose the example is written like find the eigen values of matrix A which has given as A = 3, 2, and 1, 4, so first we have to assume that A1 = A which can be written as L1 R1 here, so L1 is nothing but the lower triangular entries and R1 is the upper triangular entries, but in this case we are just assuming this problem will be solved by Doolittle's Method and in the Doolittle's Method we can just assume this lower triangular diagonal entries as 1, if you'll just use crowds method then we can use this U diagonal entries as 1 there, but since we are just taking as the left product of this elements here, it is better to go for this Doolittle's Method. As you know that to equate this number of variables with the number of unknowns especially we are just reducing or we are just considering this diagonal entry as 1 here, since neither we'll have like 2N+ 2 unknowns if you'll not consider this diagonal entries of any one of L1 or R1 is here 1 there, so from first iteration if, we'll just compare all the coefficient that is the product of L1 and R1 with this matrix A here, so obviously we can just write here 1, 0, alpha 1, 1 and product of like alpha 2, alpha 3, 0, alpha 4 this equals to obviously you can just write as the 3, 2, 1, 4 here, and we can just take this product like alpha 2 this is nothing but 3 here, and then we'll just take the product of this one, so you'll have alpha 3 = 2, if you'll just product of this one, alpha 1 and alpha 2 this is nothing but we are just going with the product in this form, so that's why we will have like 1 here.

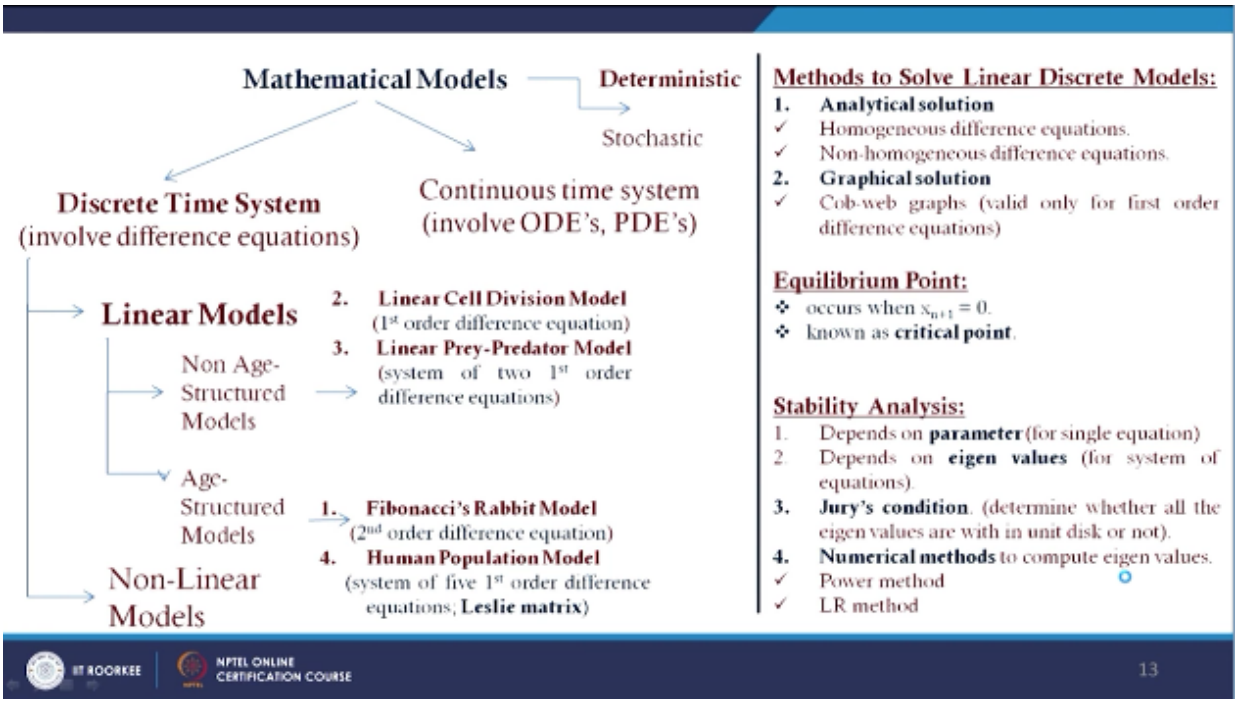
So obviously alpha 2 it is known as 3 here, so that's why alpha 1 = 1/3, similarly if you'll just take product of this 2 here, so we will have like alpha 1, alpha 3 + alpha 4 this is nothing but 4 here, and if you'll just calculate alpha 4 by putting alpha 1 and alpha 3 we'll have this values 10/3.

Now if you'll just go for the calculation of A2 here, so A2 can be written as R1 and L1, so if you'll just take the product simply you can just get the this matrix here, so similarly we will just go for the calculation of again L2 and R2 by the same method and this process will be repeated until we get the same matrix A in the successive iterations, so in this case after performing 10 iterations we are just getting A9 as 4.9952, 2, 0, 2.0023, and A10 as 4.9976, especially if you'll

just see here the degree of accuracy is coming up to third decimal places and 2 is exactly 2 here, and 0 is coming as 0.0024 and this is also coming exactly same, so you can observe that A9 and A10 are almost identical and diagonal, along the diagonal entries, hence the eigen values are, basically we can just consider as a next immediate approximate values as 1, sorry 5 and 2 here, so this exact eigen values can also be verified using an Jury's test that you can just make it.



So in this lecture we have like discussed the physical significance of eigen values, then need of the numerical methods while we are just going for the numerical methods, then numerical methods for computing eigen values, and then we have discussed about like power methods and based on one example then we have discussed about LR method, so now we will just come for this like summary of this lectures, whatever we have just discussed in this week.

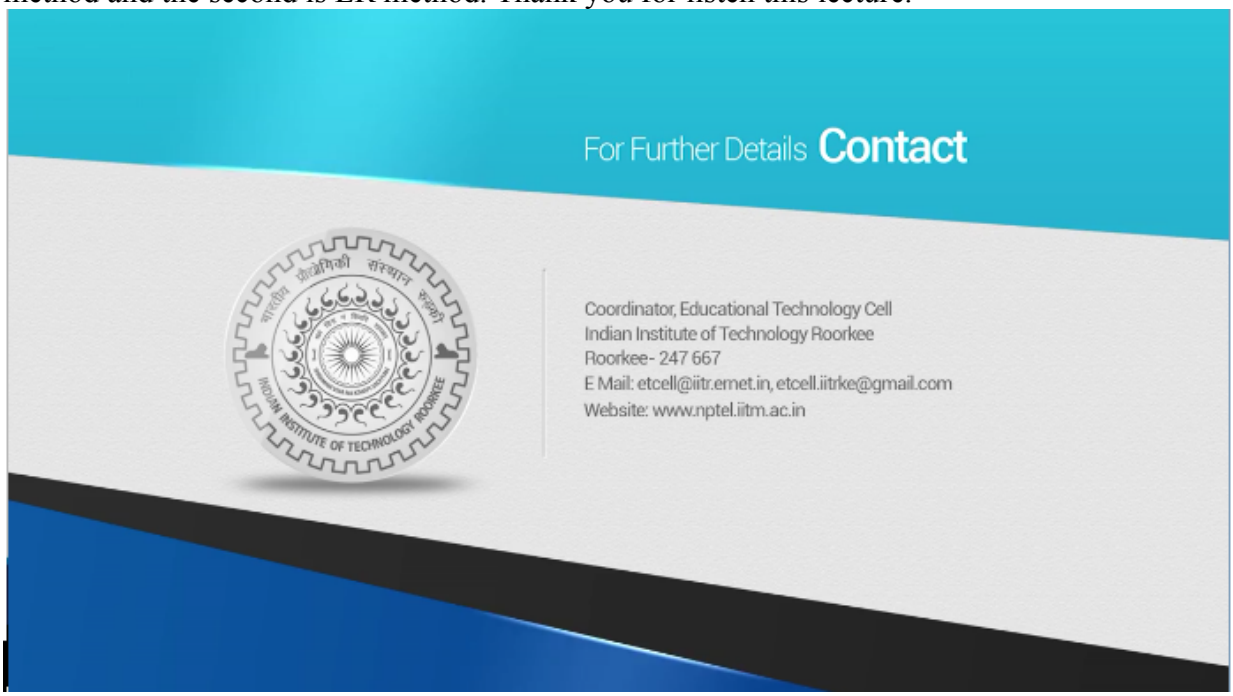


So if you just see here, in the first lectures we have discussed about like what is mathematical methods or like what is mathematical models, so mathematical model is nothing but the abstract model that uses this mathematical languages that we have just discussed, and this mathematical models based on like deterministic and stochastic models we have consider and again we have just define this mathematical models that is in the two forms, it is represented based on like time systems, so first it is called discrete time system that we have discussed in the previous lectures, and then again we'll just go for like continuous time system which will include like ODE's and PDE's, that is ordinary differential equations and partial differential equations, and this discrete time system that involves difference equations, so in the difference systems or the difference equations in discrete times levels we have consider linear models, so in the linear models we have just consider like first order difference equation for linear cell division model, and system of two first order linear difference equations for prey-predator model and that is for like non-aged structured models we have just consider.

And last lecture we have discussed about age structured models, and the age structured models we have consider this Fibonacci Rabbit model in the last few lectures we have discussed that one, that is second order difference equation, and then we have just discuss the human population level based on this age structured models which involves like 5 first order difference equation and which is represented in the form of Leslie matrix, and then we will just go for non-linear models, and basically the methods used to for the solution of linear discrete models are like analytical solution we have discussed already, then homogeneous difference equations and non-homogeneous difference equations that is especially when this migration is coming to the picture that we have just discussed. Then a graphical solution methods also we have discussed based on this cob web graphs, valid only for first order difference equations, then we discussed about how we can just find this equilibrium points or study state conditions, and especially this equilibrium points are caused when we are just considering XN+1 this equals to 0, also sometimes it is known as critical points.

Then we have discussed about the stability analysis based on this eigen values, so sometimes it depends on this parameter, like for single equation depends on eigen values for system of

equations, and we have also discussed the Jury's condition which is used for like higher order equations or determine whether all the eigen values are within the unit disk or not. Then finally we have discussed the calculation of eigen values based on numerical methods, two different methods we have discussed completely by considering certain examples, so first is power method and the second is LR method. Thank you for listen this lecture.



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