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Mathematical Modeling: Analysis and Applications

Lecture-04
Discrete Time Linear Age Structured Models

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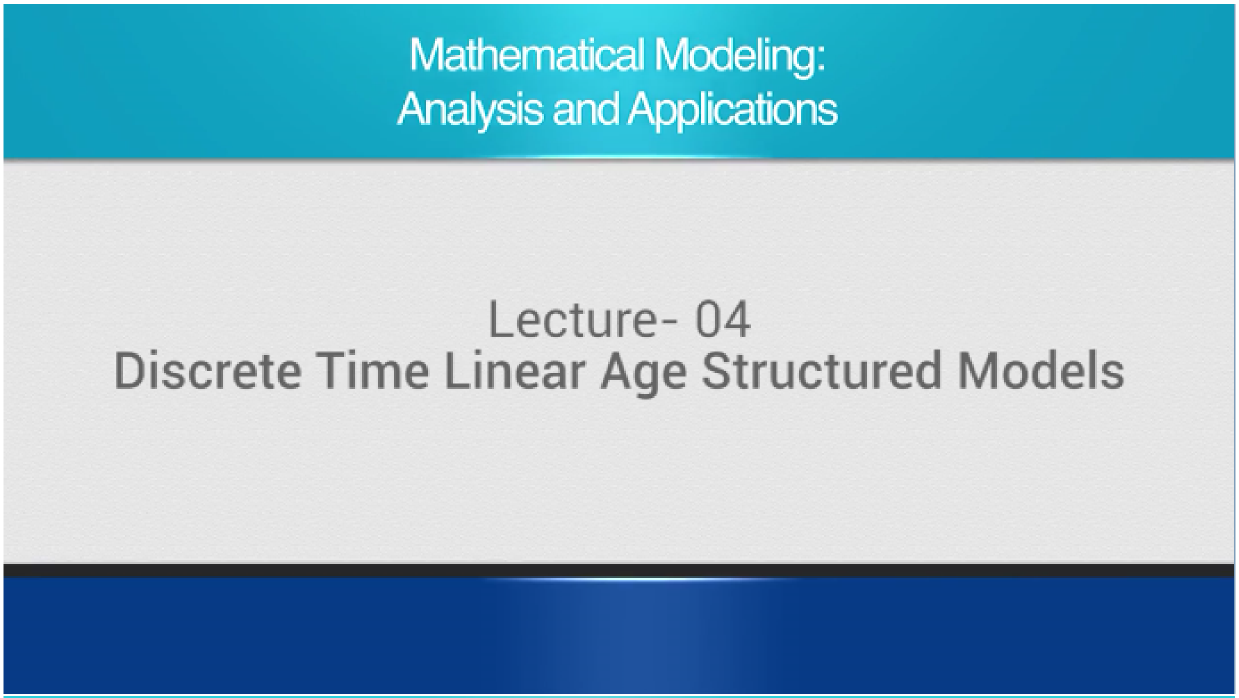


Welcome to the lecture series on Mathematical Modeling Analysis and Applications. In the last lecture we have discussed about linear discrete model for system of equations, and in this lecture we will just discuss about the Discrete Time Linear Age Structured Models.

Contents:

- Bernoulli – Lewis – Leslie (BLL) Model.
- Projection Matrix.
- Leslie Matrix.
- Jury's Stability Test.

And the contents of this lecture includes like Bernoulli Lewis Leslie Model that is BLL model, and in the second we will just go for like projection matrix, and the third stage we will just define about this Leslie Matrix, and in the last section we will just go for Jury's Stability test.

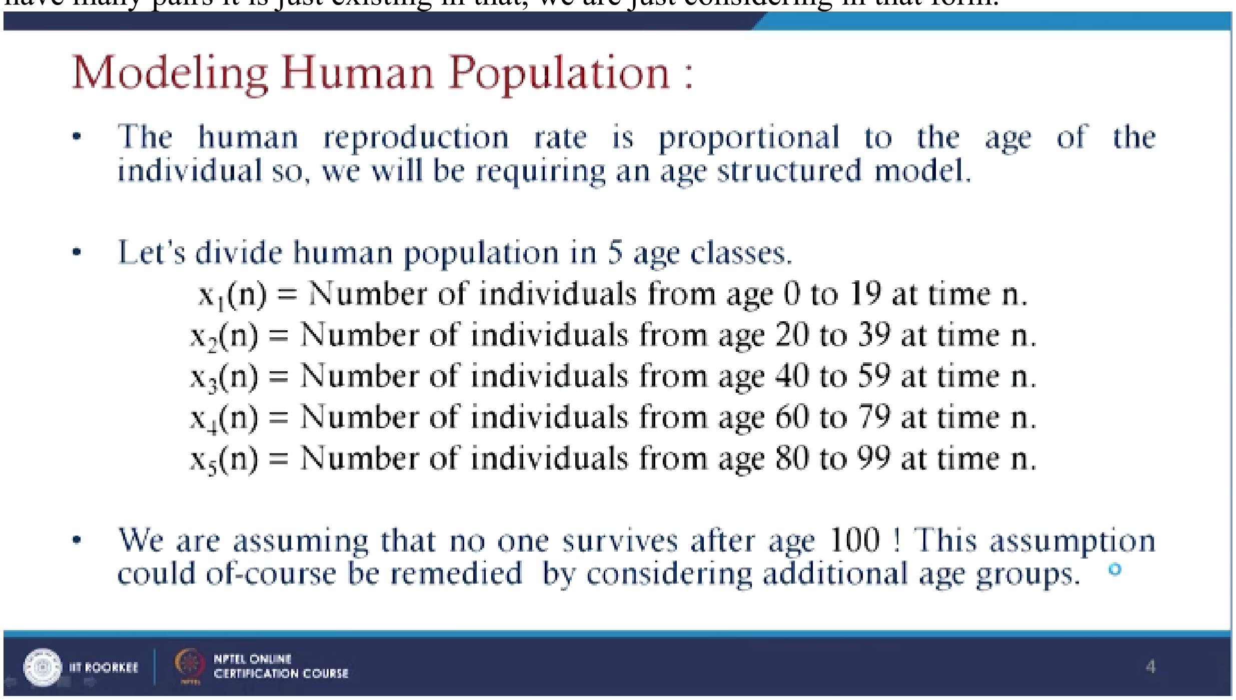
So if we just go for this Age Structured Models this means that if the species involved in modeling are studied after categorizing them in several age classes, then the model is termed as Age Structured.

Introduction to Age-Structured Models :

- If the species involved in modeling are studied after categorizing them in several age-classes, then the model is termed as age-structured.
- Till so far, we have studied 3 models out of which 2 models, linear cell division and linear prey-predator models are non age structured because we haven't consider the ages of species.
- In some species, the amount of reproduction varies greatly with age of individuals. Hence, they can not be formulated using non-age structured models.
- Fibonacci's Rabbit model may be considered as an age structured model because we included the age factor in formulating the model.
- In age structured models, first the whole population is divided in to n age groups. For modeling, separate birth and death rates are considered for each age groups.

In the last lecture we have discussed about pre-predator model where prey's are present and predator are present, but we did not consider the age there itself, since only this tigers are eating this deer's and the deer population is just getting declined or in the second case if we are just considering like if a deer population is declined then this tiger population are also get declined, but we don't know when this like the age that the tigers can give birth to the next tiger or the deer can just give birth to the next deer, but here if you'll just consider like after how many days or how many years then the deer will just get fertile and it can produce the deer's or tiger can be fertile and it can provide, produce extra tigers.

So if you just consider this Age Structured Models then we can have a clear idea about how we can just generalize this model, till so far we have studied three models out of which two models linear cell division and linear pre-predator models are non-age structured, because we have not consider the ages of species there, if some species the amount of re-products and varies greatly with age of individuals, hence they cannot be like formulated using non-age structured models, so Fibonacci rabbit model maybe consider as an aged model, since we have just consider that like one where it is just exiting and after like 2 months they are just giving birth of like one pair again, and then they will be like fertile after 2 more months, so in that structure we can just consider it is like after like two months you'll have like 2 pairs of rabbit and after four months have many pairs it is just existing in that, we are just considering in that form.



And if we will just go for this age structure of models, first the whole population is divided into m as groups, this means that like if a population age is like, that is above certain ages that they cannot produce also any new pairs there, then we cannot also neglect that type of population group also, so for that we have divide like this population into several aged groups, and if we'll just divide like for an arbitrary number n as of groups for modeling then we can just separate birth and death rates in the different as groups based on their like age structures, and here that's why I have just written this statement in age structured models, first the whole population is divided into n as groups, for modeling separate birth death rates are consider for each age groups, and if you just go for like the modeling of human population growth then this human reproduction rate is proportional to the age of the individual, so we'll be requiring an age structure model.

Modeling Human Population :

- The mathematical model of the human population with this age class is given as:

$$
\begin{aligned}
x_1(n+1) &= f_1\, x_1(n) + f_2\, x_2(n) + f_3\, x_3(n) + f_4\, x_4(n) + f_5\, x_5(n) \\
x_2(n+1) &= g_{1,2}\, x_1(n) \\
x_3(n+1) &= g_{2,3}\, x_2(n) \\
x_4(n+1) &= g_{3,4}\, x_3(n) \\
x_5(n+1) &= g_{4,5}\, x_4(n) \qquad \ldots\ 4.1
\end{aligned}
$$

here, f_i denotes the birth rate (over a period of 20 years) for parents in the i^{th} age class and $g_{i,i+1}$ denotes the survival rate for those in the i^{th} age class passing into $(i+1)^{th}$.

- A single set of parents may be in different age groups, we should attribute half of their offspring to each in choosing values for f_i.

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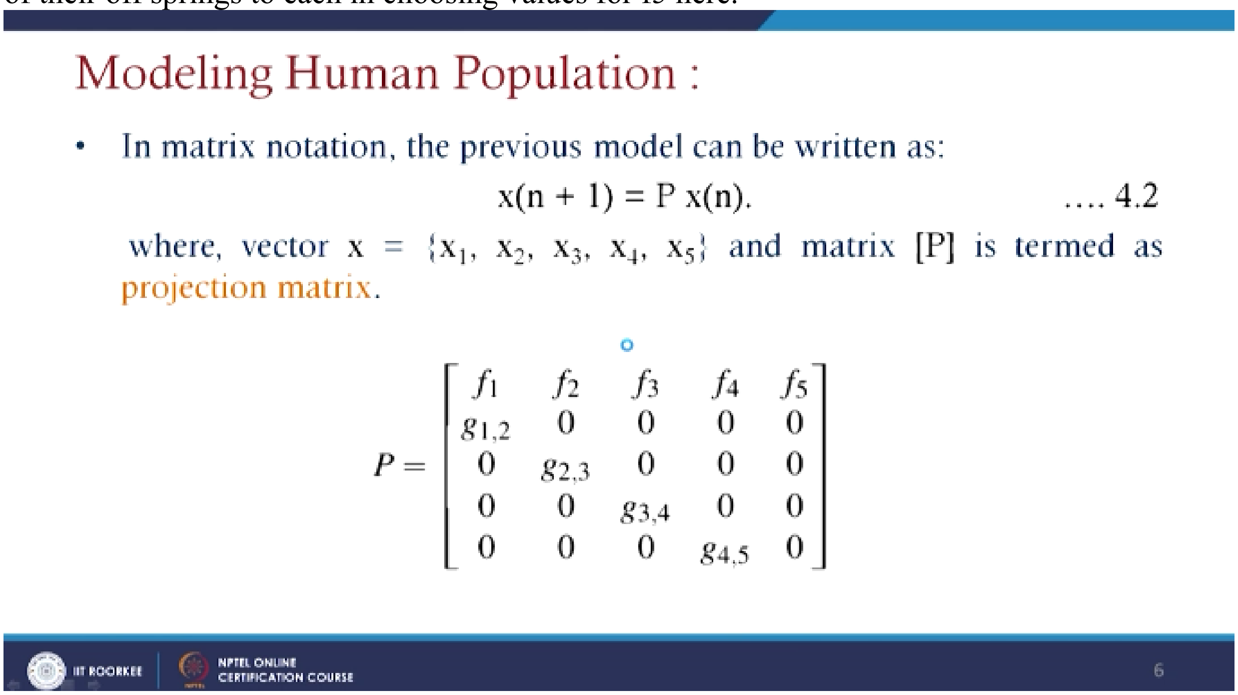
Let us divide the human population levels in to five age classes of course, so suppose x1 is the age class or the number of individuals from ages like from 0 to 19 at a particular time level suppose n, and x2n that is the number of individuals existing from the age of 20 to 39 at that time level, and x3(n) suppose the number of individuals from age 40 to 59 at time n level suppose, and x4 is like number of individuals from age 60 to 79 at that particular level, and x5 is the number of individuals existing from 80 to 99 at time n, and we are just considering that no one survives after age 100, maybe sometimes we can just find that in a, like different countries, peoples are surviving after 100 also, that is sometimes we can just consider it is like unnatural, and if you just consider all this assumptions this can be like, this assumption could be of course be remedied by considering additional age groups, those who cannot like give birth after this age groups so we can just also consider or like this is the age that after that the people will get dies, so that type of constants we can just consider in this class of models here, so if you just go for this like mathematical representation of all this age group of this human population, then we can just write first age group as x1 which is existing between like the age from 0 to 19 here, if you just see sometimes maybe it is 0, we can just consider like f1 x1(n), f2 x2(n), f3 x3(n), f4 x4(n), f5 x5(n), this means that within this 0 to 19 group some people they can also fertile, they can also produce the babies.

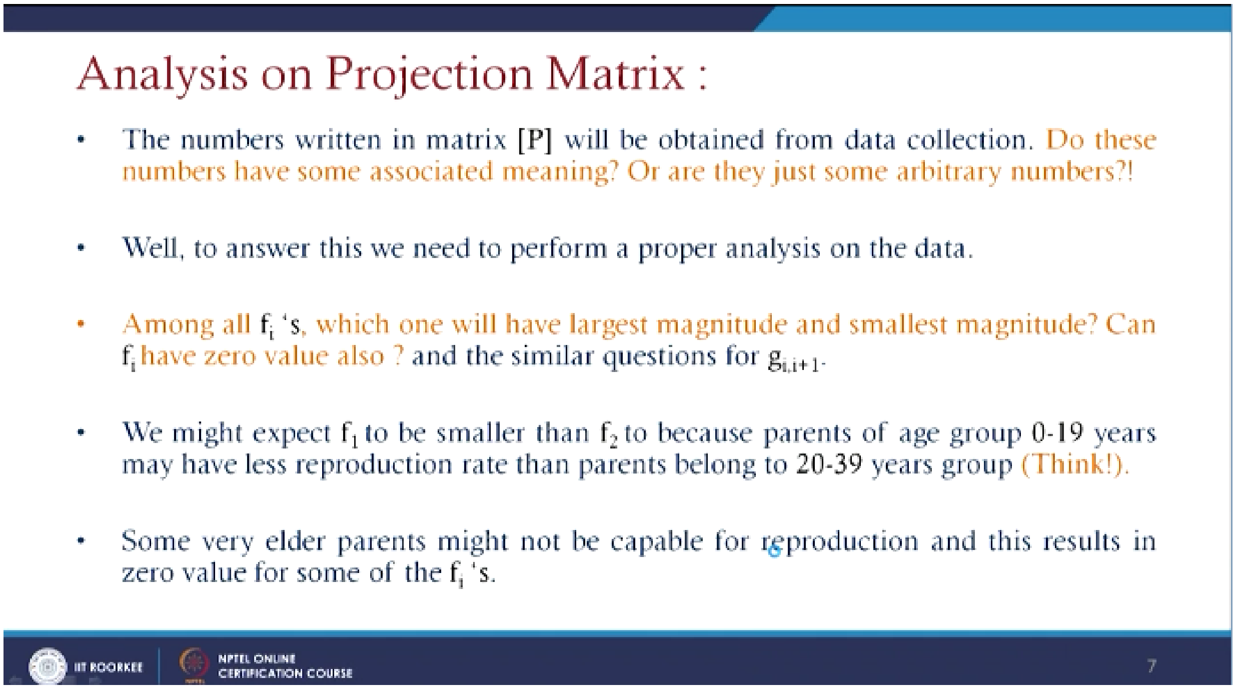
And x2n means those who are like in the age group of, from 20 to 39 they are also producing this or they can like give birth to the children and whose age will lies between 0 to 19, so if you'll just see this x3 that age group also produce the children's and whose age will be lies between like 0 to 19, and similarly you'll have like x5 groups which also produce the babies and which age will lies between 0 to 19.

And second group if you'll just consider like x2 then this population, if you'll just see here that is just to given by g12 x1(n) here, this means that after 19 those who have survived they can be come to the age group of x2 here, and I have just written here f5 denotes the birth rate over a period of 20 years, for parents in the i-th age class and GI+1 that is G1, 2 denotes the survival rate for those in I-th age class passing into i+1, this means that they have crossed the 19 age and they are belonging to the group age of above the 19, so that's why it is just written as G1, 2 x1 here.

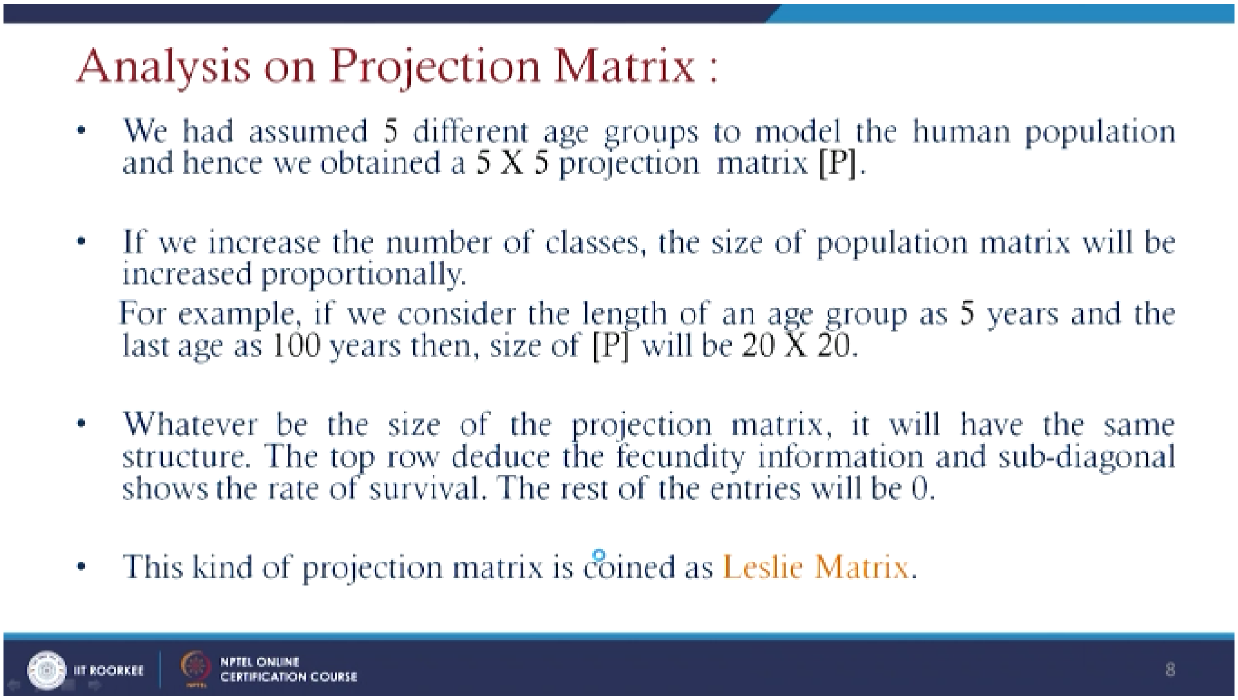
Similarly those who have crossed like x2 age there, they just they did not die and after that they are also surviving, they are in the fertile group of like x3 here that is represented as G2, 3 x2 here, and those who have crossed this x3 age bar then that can be just coming into x4 age group here, and those who have crossed x4 group batches there they can just come up to here G4 5 x4(n) here, and a single set of parents maybe in different age groups which should attribute half of their off springs to each in choosing values for f5 here.



So in matrix notation if you'll just write this previous model, which can be represented as x(n+1) this equals to P x(n) here, if you just write all this variables so that is represented as x1, x2, x3, x4 and x5 as the variable which can be just written as in the form of like x(n+1), then this complete matrix with taking the coefficients as P here, where this vector x as represented as x1, x2, x3, x4, and x5 and the matrix P is termed as, if you'll just see here which takes the coefficients for x1 as f1, f2, f3, f4, and f5 which will be multiplied with x1, x2, x3, x4, and x5 there, and if you'll just see here x1 takes the coefficient G1, 2 and the rest are 0 here.



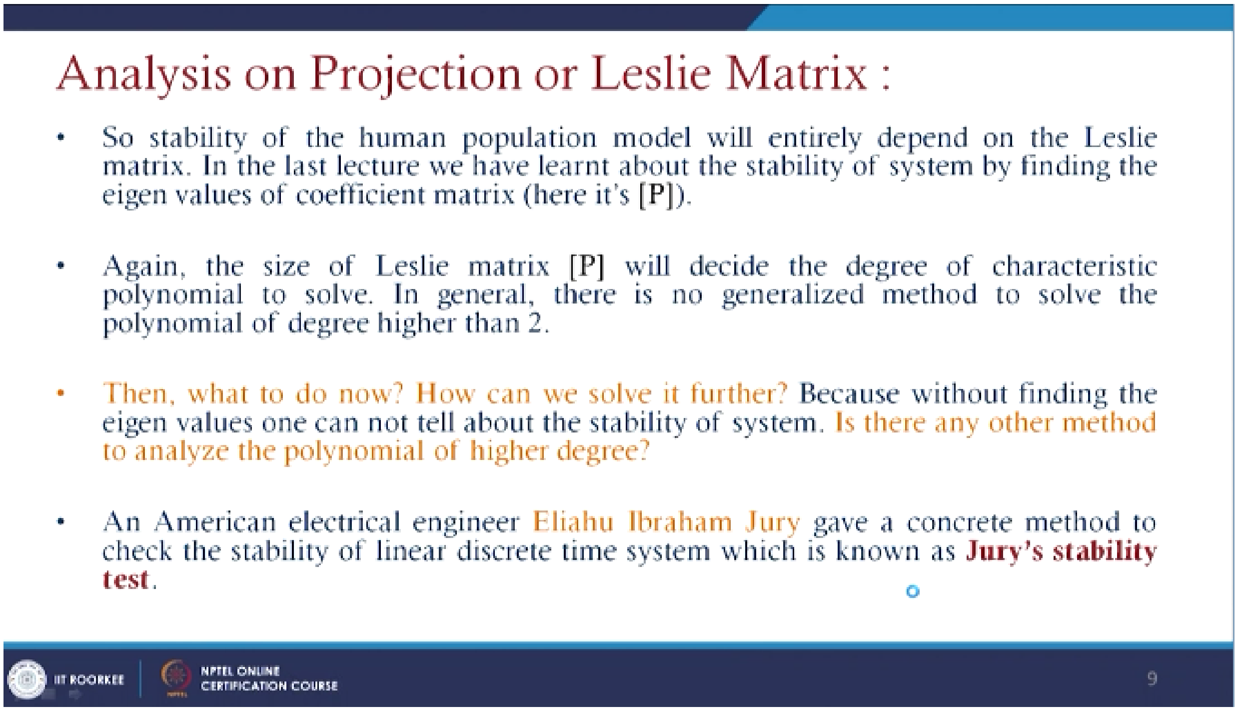
Similarly the next group for x2 here, x3 here if you'll just see this takes the age group of G2, 3 here, rest of the elements are 0. Similarly if you'll just x4 then it takes the age group of G3, 4 and G4, 5 it is existing for like x5 here, if you'll just and which takes the coefficient as x4 here, this takes the x3 here, this is the x2 here, and this is as x1 here, so that's why x5 does not have any coefficients so that is just taking as 0 value, if you'll just go for this analysis on projection matrix the numbers written in matrix p will be obtained from data collection, how many like age groups are existing from 0 to 19, and above 19 how many populations are existing, whatever we have just define for different age groups that we can just know in a direct from the data only, and do this numbers have some associated meaning or they are just some arbitrary numbers, so that we have to also verify.



Well, if we'll just go for answer this we need to perform a proper analysis on the data, among all f5 is that is which one? We'll have largest magnitude and smallest magnitude, can f5 have 0 value also? And the similar questions for gi, i+1, since how many people are surviving like up to 19 or after 39, how many people are surviving? So every possibilities we have to consider when we will just go for, or define this model, we might expect f1 to be smaller than f2 to because parents of age group so 0 to 19 here may be less reproduction rate than parents belonging to 20 to 39 years age group, yes definitely it is true, some very elder parents might not be capable of reproduction and this results in zero value for some of this fi's, and if you'll go for further analysis we had assume 5 different age groups to model the human population here, and hence we obtained a 5 x 5 projection matrix, if you just classify this moderate age groups into like 20, like different age groups, then we will have like 20 x 20 matrix here.

So if we increase the number of classes the size of population matrix will also get increased proportionally, for example if you'll consider the length of an aged group as 5 years suppose, since sometimes we can just find that like different populations like, if we are just going for human it is different, but if we'll just go for like another creatures then we can just find they're like this age groups difference will be like very certain to get this reproductions, so in that class we have to like sub-divide this age groups into very small sections.

And if the length of an age group has 5 years and the last age is suppose 100 years, then the size of you will be like 20 x 20 here, whatever the size of the projection matrix we don't mind, it will have the same structure always, the top row deduce the fecundity information and sub-diagonal shows the rate of survival here, and the rest of the entries will be 0 especially. And this kind of projection matrix is called a Leslie Matrix.



And if we'll just go for Analysis On Projection or Leslie Matrix, so stability we have to go for like human population model and it will just entirely depends on the Leslie matrix here, and in the last lecture we have learned about the stability of the system by finding the eigenvalues of coefficient matrix, since we have just consider that determinant of A – lambda I this equals to 0, and sometimes if we are just finding that lambda I, at least one of the lambda I is greater than 1 then the system is unstable, and if all the lambda I's are less than 1 then we are just saying the system is a stable there. And if at least one of this lambda I is 1, and the rest of the lambda I are less than 1 then we are just saying that the system is asymptotically stable there.

And if we are just going for the size of Leslie Matrix, that the size of this matrix p will decide the degree of characteristic polynomial to solve, in general there is no generalized method to solve the polynomial of highest degree or the degrees greater than 2, since already we have known that the if the system is linear we can have a solution, and two problems may be it is present there, if it is a quadratic equation then we will have a solution, analytical solution but if the degree of this polynomial is greater than 2, we don't have any analytical method or exact solution it is difficult to find thereof, especially we are just going for numerical solution. Then what we have to do now? How can we solve it further? Because without finding the eigenvalues one cannot tell about the stability of the system, and whether there is a any method is existing to analyze the polynomial for higher degrees, so in American electrical engineer named as Eliahu Ibraham Jury gave a concrete method to check the stability of linear discrete time system which is known as Jury's stability test.

Jury's Stability Test :

- Suppose a given discrete time linear system is

$$Z_{n+1} = [M]\, Z_n .$$

where [M] is n x n matrix.

- To find the eigen values, we need to solve for $\det(M - \lambda I) = 0$. This will give a polynomial of degree n. Suppose the polynomial in simple form is represented as:

$$Q(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n = 0, \text{ where } a_0 \neq 0.$$

- To apply the Jury's test, we need to form a table. To create the same, the procedure is given as below:

1. In the first row, write all the coefficients of polynomial starting from highest power of λ.
2. In the second row, write these coefficients in reverse order i.e. starting from lowest power.

And in Jury's stability test first we have to consider a discrete time linear system as in the form of Zn+1 this equals to M Zn suppose, where M is a n by, n system of matrix. To find the eigenvalues we need to determine the determinant of m – lambda I this equals to 0, and if we will just write this determinant of m – lambda I = 0 this will just give you polynomial of degree n there, suppose a polynomial is in simple form represented as q of lambda here, since all the eigenvalues are represented in terms of lambda, so that's why we can just write this one as a) lambda to the power n, and a1 lambda to the power n-1, a2 lambda to the power n-2, an-1 lambda + an = 0, especially we are just considering a0 is not equal to 0 here, since if a0 = 0 then this polynomial will be reduce to like degree of n-1 which creates a problem here.

Jury's Stability Test :

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & \ldots & a_{n-1} & a_n \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_1 & a_0 \end{array}$$

3. For the third row, multiply second row by coefficient a_n/a_0 and then subtract the same from first row. The last term will be 0.
4. For the fourth row, write the first n coefficient of the third row in reverse order.

$$\begin{array}{cccccc} a_0 - a_n^2/a_0 & a_1 - a_{n-1}a_n/a_0 & a_2 - a_{n-2}a_n/a_0 & \ldots & a_{n-1} - a_1 a_n/a_0 & 0 \\ a_{n-1} - a_1 a_n/a_0 & a_{n-2} - a_2 a_n/a_0 & a_{n-3} - a_3 a_n/a_0 & \ldots & a_0 - a_n^2/a_0 & \end{array}$$

5. Repeat the same procedure till a row contains only one non zero element.

For $a_0 > 0$ if all the coefficients in first column of odd numbered rows are positive (if any one of them is 0, then system is **stable** because of unit magnitude eigen value), $Q(1) > 0$ and $(-1)^n Q(-1) > 0$ then all the roots of polynomial lie inside an unit circle with center at origin, which is the required condition on eigen values for **asymptotically stable** system . If $a_0 < 0$, then multiply whole polynomial by -1 to make it positive. If any one of the 3 conditions not satisfied, system is **unstable**.

Since we'll have like n x n system of matrixes and each of these variables are like nonzero, so then we cannot find the exact solution of this polynomial equation. To apply the Jury's test we need to form a table here, to create the same, the procedure is as given as below, first we will just consider that in the first row we'll just write all the coefficients of polynomial, starting from higher power of lambda, and in the second row write this elements in a reverse order that is starting from lowest power, if you'll just see here that the coefficients of this polynomial which is started as a0, a1, a2 up to an here, a0, a1, a2 n-1 an here, if you just start writing in a reverse form then a1 is going to that one, an-1, then an-2, then finally you can just find a1 in this position and a0 is going to the last position there.

For the third row if we'll just multiply the second row by the coefficient of an/a0 and subtract from the first row, then we will have like third row here. And the last term especially if you'll just multiply in that form then you can just find since we're just multiplying here an/a0, and subtracted from AN there, so then I'll have the 0 element in the last entry here.

And for the fourth row write the first n coefficients of the third row again in the reverse form, so if you'll just write here the third row, the third row representation will be like a0 – an is multiplied by an/a0 so it will just give you AN square/a0. Second if you just see here a1 – an is multiplied here, so an-1 x an/a0, third element will be a2 – an2 x an/a0. And the last element if you just see lambda n – a0 x lambda n/a0 so a0 a0 it will be cancelled, so an – an this is just giving you 0 value there.

And if you just write in a reverse form except the 0 element, then we will have like, this will be the first element here, and next element it will just come in this form here, and here the last element will goes up to as a0 – an square/a0, and again we'll just repeat the procedure till a row with a single entry or single nonzero entry should exist, and if anyone of the entries will be negative so at that position we can just say that the system is unstable. And for a0 greater than 0 if all the coefficients if you just consider here, in the first column of odd number rows are positive, this means that the odd number rows if you just see 1, then 3, then 5, likewise we can just consider and if all this odd number rows are positive, if anyone of them is 0 then the system is stable here because of the unit magnitude of eigenvalue here, and this is nothing but that is discrete rule, this is the inverse of discrete rules that to find this roots of number of positive roots or negative roots of any polynomial, that is nothing but if you just consider here as a q1 is greater than 0 and -1 whole to the power n, q(-1) is greater than 0 then all roots of the polynomial lie inside a unit circle, already we have known this one with centered horizon, which is the required condition on eigenvalues for asymptotically stable system here.

Jury's Stability Condition for Quadratic Equation:

- Let's consider a quadratic equation $Q(\lambda)=b_0\lambda^2 + b_1\lambda + b_2 = 0$, where $b_0 > 0$.
- Jury's stability table will look as:

1. b_0	b_1	b_2
2. b_2	b_1	b_0
3. $c_0 = b_0 - b_2^2/b_0$	$c_1 = b_1 - b_1 b_2/b_0$	0
4. $c_1 = b_1 - b_1 b_2/b_0$	$c_0 = b_0 - b_2^2/b_0$	
5. $d_0 = c_0 - c_1^2/c_0$		

- So in the 5[th] step only, we got a single non-zero number d_0. Since $b_0 > 0$ the condition for stability is $c_0 > 0$, $d_0 > 0$, $Q(1) > 0$ and $Q(-1) > 0$.
- Without loss of generality, say $b_0 = 1$ (for simplicity). Then the conditions for stability turned out are (Exercise!):

$$|b_2| < 1 \text{ and } |b_1| < |1 + b_2| \qquad \dots 4.3$$

If at least one of this a0 is less than 0, then multiply whole polynomial by -1 to make it positive, and if any one of the three conditions are not satisfied then the system is unstable here, so if you just go for the testing of this Jury's stability condition for a quadratic equation here, then we can just write a quadratic equation as q(lambda) as b0 lambda square + b1 lambda + b2 this equals to 0, especially in that case we have just also mentioned here a0 should be greater than 0 or a0 is not equals to 0, we have just consider in this case as b0 is greater than 0 here.

And first if you just write all the coefficients that is in the form of b0, b1 and b2, and in the reverse way if you just write that can be written as b2, b1, b0 here. First we will just multiply this element by b2/b0 and subtract from the first row here, so if you just do that one we can just find that the last element as 0, and here we can just write as b0 – b2 x b2/b0, so that's why b2 square/b0 here.

Similarly the third element if you'll just see here that is b1 – b1 x b2/b0, so this element is written in this form, again we'll just write in a reverse form here so first c1 can be written as b1 – b1 b2/b0 here, and then again c0 will come to the position of the second element here. Last element it will just taken as, again we will just considered as the difference by multiplying this one, so last element it will just come as 0 here, so that's why the last element d0 is written as c0 –c1 square/c0 here. So in the fifth step we are just getting here a single nonzero enter number for d0, since b0 is greater than 0 especially the condition for stability is c0 greater than 0, since three conditions we have to consider here that is 1, 3, and 5, odd number rows, so first condition it is just to given b0 is greater than 0, we have to consider that c0 must be greater than 0 and d0 must be greater than 0.

So without loss of generality we can just say that, suppose b0 = 1 suppose, then the condition for stability turn out are S, like if you just consider directly b0 as 1 here, then b0 – b2 square it should be like greater than 0 we are just considering here, so based on that we can just say that b2 is less than 1 and b1 should be less than modulus of 1+b2 here.

Jury's Stability Condition for Cubic Equation:

- Let's consider a cubic equation $Q(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0$.
- Jury's stability table will look as:

1. 1	p_1	p_2	p_3
2. p_3	p_2	p_1	1
3. $q_1 = 1 - p_3^2$	$q_2 = p_1 - p_1p_2$	$q_3 = p_2 - p_1p_3$	0
4. $q_3 = p_2 - p_1p_3$	$q_2 = p_1 - p_1p_2$	$q_1 = 1 - p_3^2$	
5. $r_2 = q_1 - q_3^2/q_1$	$r_3 = q_2 - q_2q_3/q_1$	0	
6. $r_3 = q_2 - q_2q_3/q_1$	$r_2 = q_1 - q_3^2/q_1$		
7. $s_3 = r_2 - r_3^2/r_2$			

- So in the 7th step only, we got a single non-zero number s_3. The condition for stability is $Q(1) > 0$, $Q(-1) < 0$, $q_1 > 0$, $r_2 > 0$ and $s_3 > 0$. After solving for same we will get simplified form as:

$|p_3| < 1, |p_1 + p_3| < |1 + p_2|$ and $|p_2 - p_3p_1| < |1 - p_3^2|$ …. 4.4

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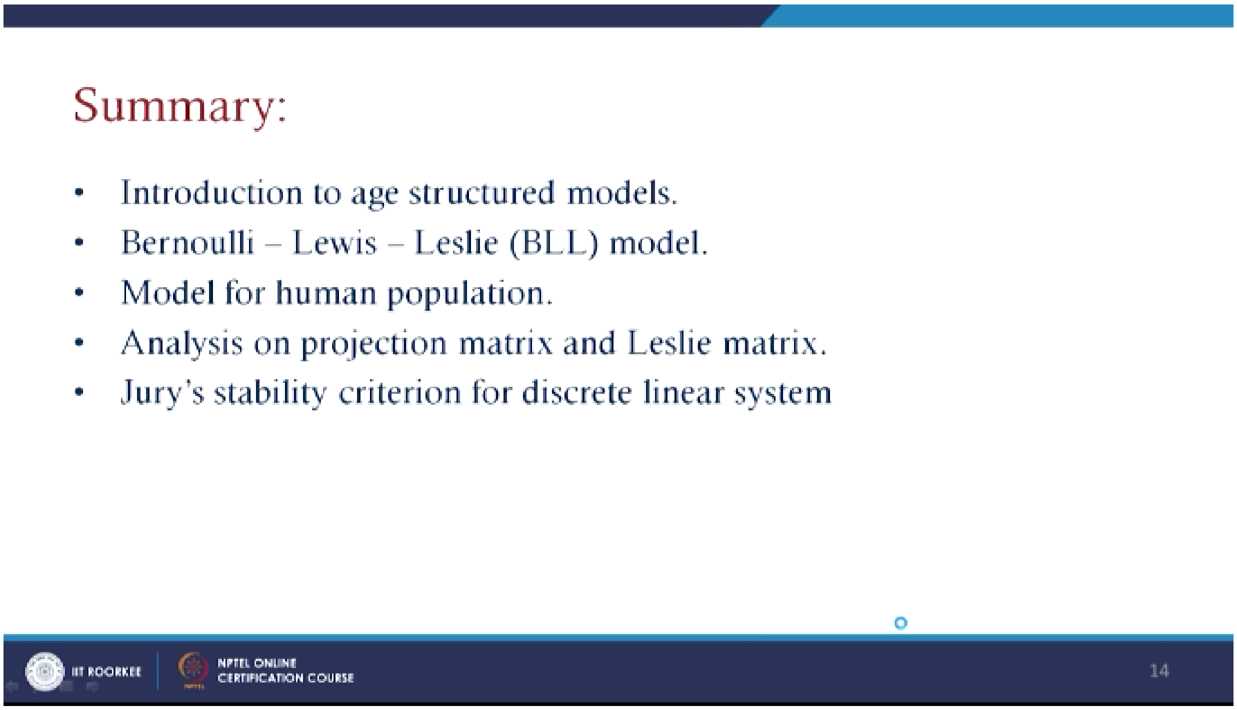
And if we'll just go for like Jury's test for polynomial of a degree 3 here, then the cubic polynomial it can be written as like lambda q + p1 lambda square + p2 lambda + p3 this equals to 0 here, and in this case if you just see the coefficients are written as like 1, p1, p2, and p3 here, and in the reverse form if you'll just write this one, this can be written as p3, then p2, then p1, then 1 here, then we will just take the difference by multiplying suppose p3 – you will have like 1/p3, so then you can just find that 1 –p3 x p3 so it will be like 1 – p3 square.

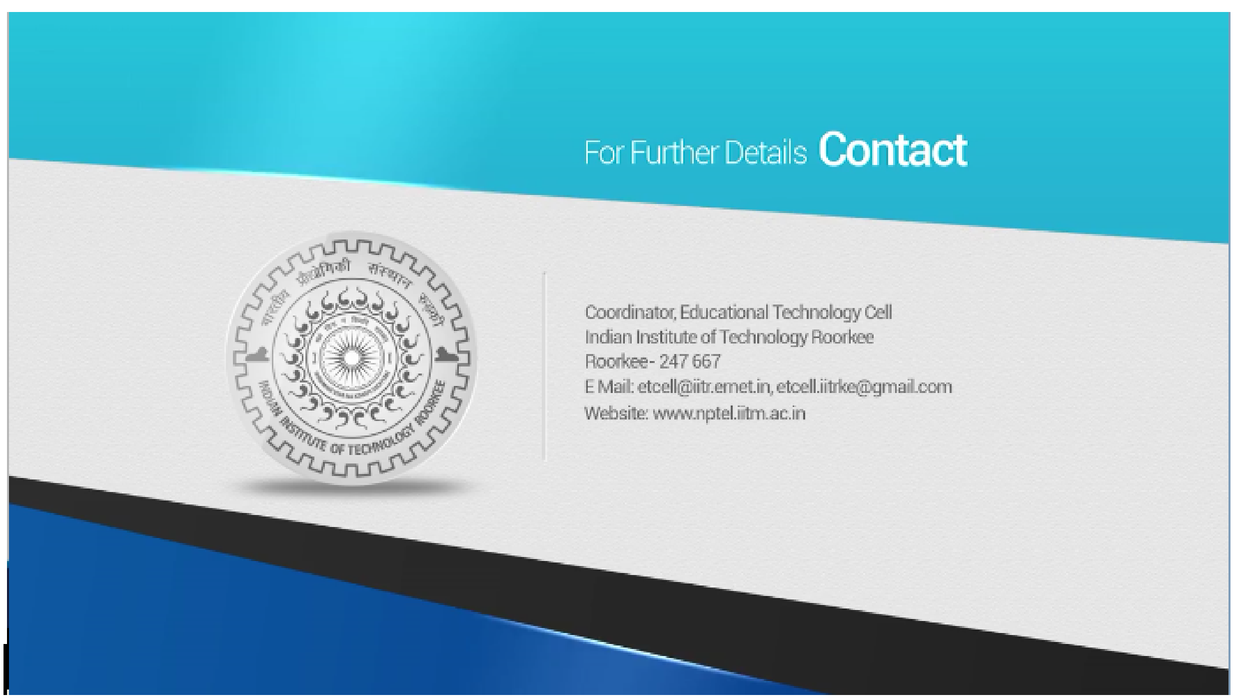
Similarly p1 – p1 p2, so this one can be written as p2 – p1 p3, so in the reverse form if you'll just write this can it goes to here, this will goes to here, and this will just go here, and again we will just apply this same multiplication then we will have like r2 = q1 – q3 square/q1, r3 can be written as q2 – q2 q3/q1, and this one will goes to 0, so again we will just write this one in a reverse form, so this is the element and this is the element, and finally if you'll just subtract them then we can just have the element like s3 equals to this one.

Now we will just consider like odd or elements here, that is 1, 3, 5, and 7, so in the 7 step only we got a single nonzero number s3 here, and the condition for stability is q1 is greater than 0 and q(-1) is less than 0 here, and for that we have to consider like q1, since 1 it is already 1 there, so q1 should be greater than 0, r2 should be greater than 0, and s3 should be greater than 0 here.

After solving all this conditions we can obtained that p3 should be less than 1, and absolute value of p1 + p3 it should be less than absolute value of 1 + p2, and p2 – p3 p1 it should be less than absolute value of 1 - p3 square.



So in this lecture we have discussed about this like age structured models, and for the age structured models we have just consider like Bernoulli Lewis Leslie model and how this human population we have just consider to represent this like your Leslie matrix there, then we have analyze the projection of this matrix and the Leslie matrix, then to test the stability we have used Jury's stability criterion for discrete linear system here. Thank you for listen this lecture.



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