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Mathematical Modeling:
Analysis and Applications

Lecture-03
Discrete Time Linear Models
In Population Dynamics - II

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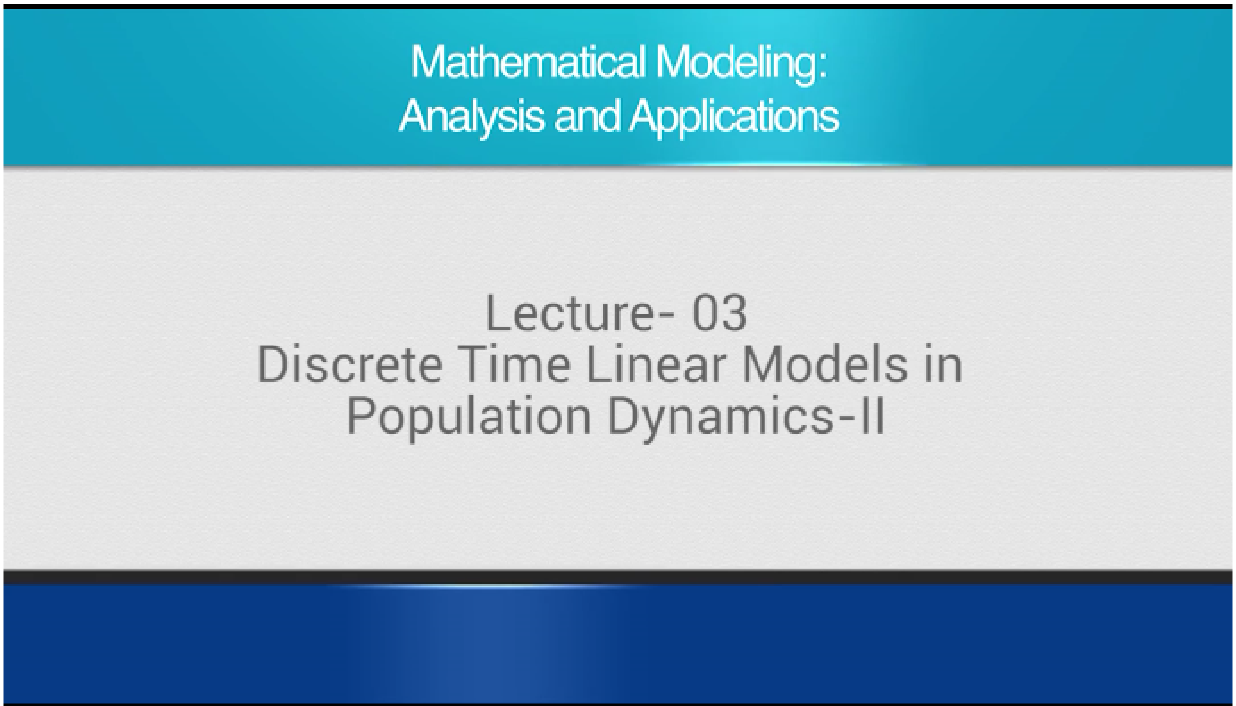
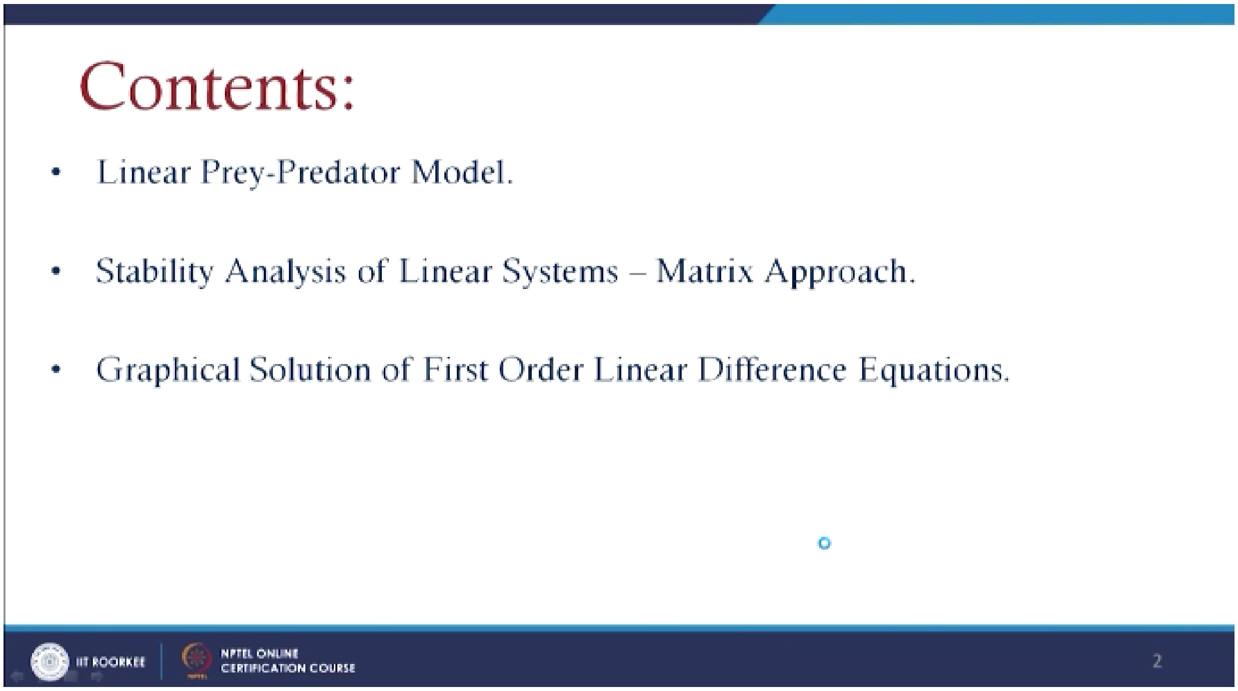
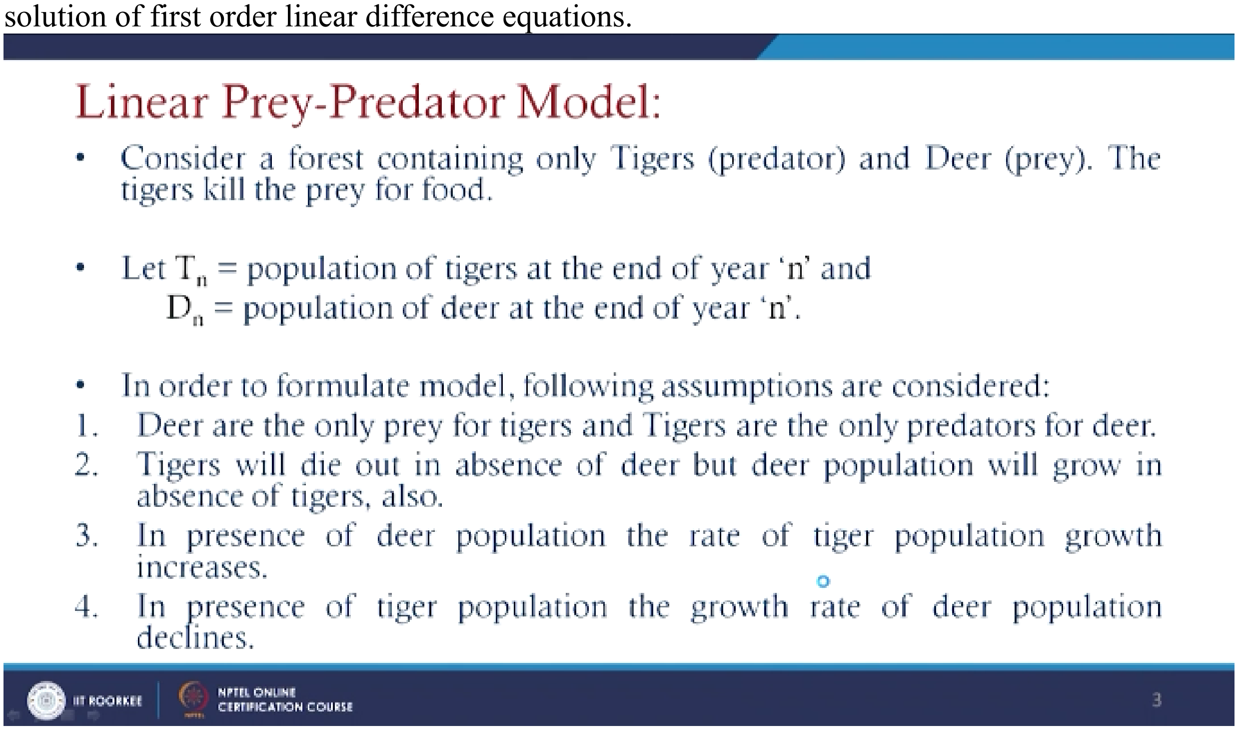


Welcome to the lecture series on Mathematical Modeling Analysis and Applications. In the last lecture we have discussed about linear discrete model, where we have considered also the examples, so linear discrete model means we have just consider either it is of first order or second order difference equations.



In the first order case we have consider the examples of cell division model, and in the second order case we have just consider the Fibonacci rabbit model there, so in this lecture we will just consider that linear discrete model for a system of equations, and in this lecture we will just go for the linear pre-predator model, then we will just go for the stability analysis of linear systems based on matrix approach, and in the last content of this lecture we'll just go for the graphical solution of first order linear difference equations.



So if you just consider this linear pre-predator model we have to consider like a prey, and a predator for this model here, so if you just consider a forest containing only tiger suppose that means predator, and a deer suppose it is like prey there, the tiger kills the prey for food, and suppose at a time instant n if we will just consider Tn is the like, tiger population at the end of year n, and Dn is the population of deer at the end of year n there.

So in order to formulate this pre-predator model here so we have to consider certain assumptions, so the assumptions are like that, first we have to consider deer are the only prey for tigers, and tigers are the only predators for deer, so no dependencies of other factors will be consider for this pre-predator model.

Second assumption we have to consider that tigers will die out in absence of deer, but deer population will grow in absence of tigers also, so this means that when this tiger population will grows off then the deer population will decline, and when this deer population will take the increments then again the tiger will grows off, but certain population levels would be there that in order to existence this like a tiger population or the deer population.

Linear Prey-Predator Model:

- Further, let's assume that
  a : rate at which tiger dies in absence of deer.
  b : rate at which tiger grows in presence of deer.
  c : rate at which deer grows in absence of tigers.
  d : rate at which deer dies in presence of tigers.

- Hence the model equations will be:

$$\Delta T_n = T_{n+1} - T_n = -a\, T_n + b\, D_n ,$$
$$\Delta D_n = D_{n+1} - D_n = c\, D_n - d\, T_n \text{ where, } n \in \mathbb{N}. \qquad \ldots 3.1$$

- In matrix form the model can be written as:

$$\begin{bmatrix} T_{n+1} \\ D_{n+1} \end{bmatrix} = \begin{bmatrix} 1-a & b \\ -d & 1+c \end{bmatrix} \begin{bmatrix} T_n \\ D_n \end{bmatrix}$$

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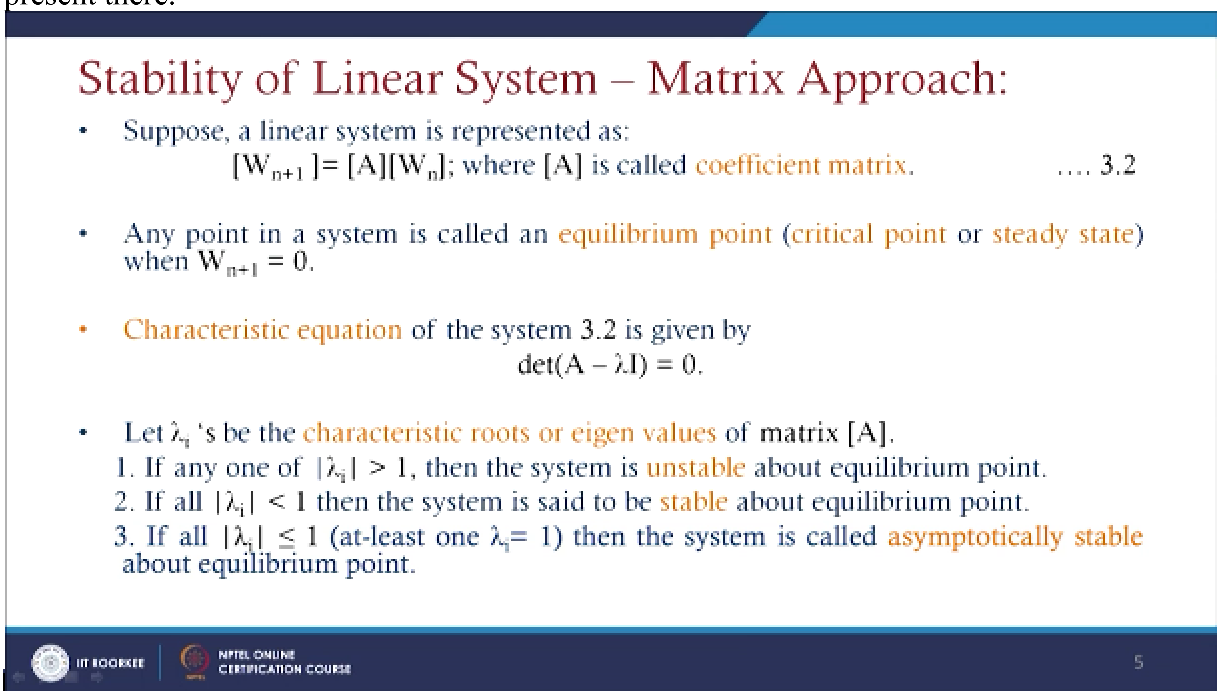
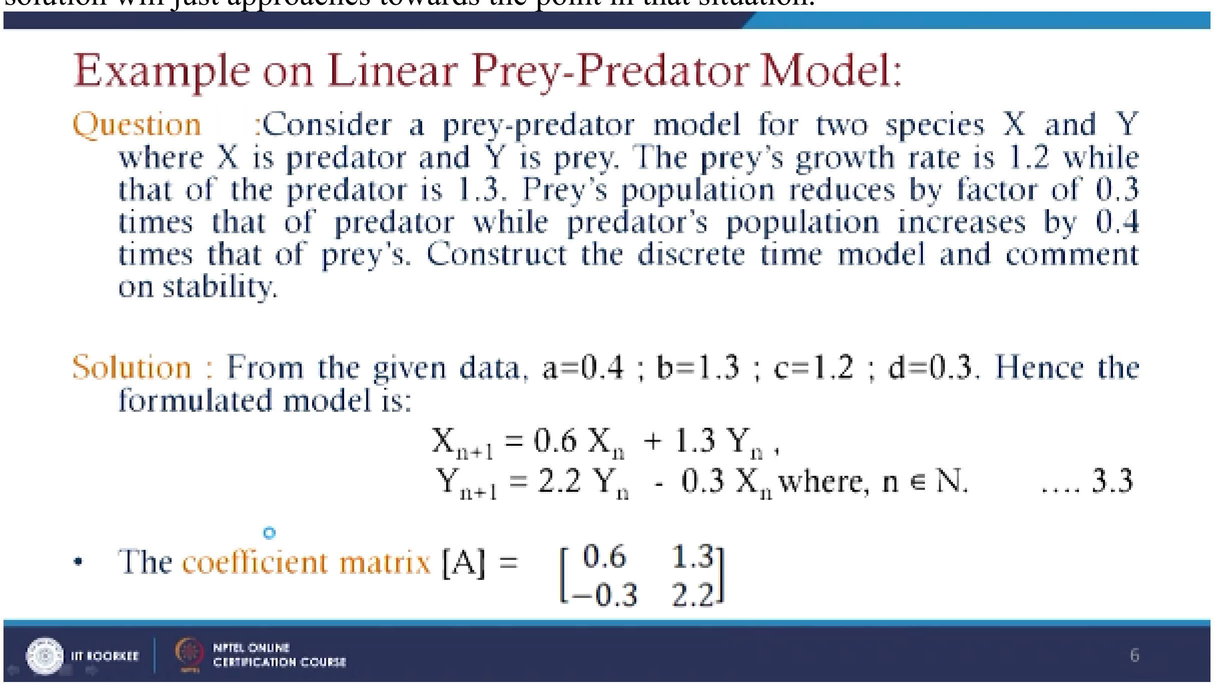
So third assumption if you'll just consider like in presence of deer population the rate of tiger population growth increases, and in presence of tiger population the growth rate of deer population declines that whatever I have just told here, so if you just go for like the construction of this linear pre-predator model here, so further we have to assume that suppose first a, that is the rate at which tiger dies in absence of deer suppose, and b is the rate at which tiger grows in presence of deer, so this means that whenever the tiger is present and deer is present automatically when this tigers will eat the deer's then the deer population will decrease, and the tiger will give birth to other tigers then this tiger population will grows it off. And if you just consider like c suppose is the rate at which deer grows in absence of tigers, so certainly after sometime if you just find that n of tigers are present then this tigers will not find the sufficient food and they will just start dying, and when it will be balanced with again with the deer population there, then again this tiger population will grows it off.

So in that case suppose we are just assuming that d suppose is the rate at which deer dies in presence of tigers, hence at that time level suppose at time level we are just considering as n here, if you just take the difference of like the tiger population at n+1 time steps and n of time steps, then we can just signifying here that on as del Tn here, which is represented as Tn+1 –Tn, and since we are just writing here the rate at which tiger dies in absence of deer as the rate as A here, so that's why this tiger will be declined at a rate A that's why we have just multiplied it as –A x Tn here, and in the absence of this tigers if you just find this deer population will grows it off, so that's why we have just write it as that is b is rate at which tiger grows in presence of deer here, so that is nothing but b Dn here, and if you just see this difference of like deer populations at two different time levels that is at n+1 and n here which is represented as del Tn, as Dn+1 – Dn, so then that can be represented as this deer population will grows it off when this tiger population will decline, so that's why if we'll just consider here rated at which deer dies in presence of tigers that is just –D here, so then this rate of decline of tiger will get increase the deer population there, so if we'll just formulate this model in a matrix formulation here since we have like the quantities as a Tn+1 here, then Dn+1, so these are just like mixed time steps calculated values for tigers and deer's here, and we want to kept this terms in the like left hand side here, then this can be represented as like 1, since Tn it is present in the left hand side if you'll just take to the right hand side we will have this quantity as 1 - a here, and b itself it is present there.



And if you'll just consider here Dn to be come to the right hand side here then that will be take the value as 1+c here, and which is written as 1+c and then –d Tn it is there so that's why this quantity is taken as –T here, and as usual this matrix multiplication values we can just keep it as Tn and Dn there itself, so if you'll just go for like stability of this linear system here, using this matrix approaches so we will have to consider like system of equations there, so that's why it can be represented as a vector elements as a matrix, that is if you'll just represent this system as in the form of Wn+1, this takes like 2 population levels there that is Tn+1 and Dn+1 which can be represented as the complete matrix A, which took this like the coefficient elements of Tn and Dn as 1-a, b and –d, 1+c here, then it can be multiplied with this Tn and Dn that's why we can just write Wn is the factor which takes the values Pn and Dn there, where A is called your coefficient matrix, and any point in a system especially it is called an equilibrium point or a critical point or sometime it is set to be study state, if we will just consider this Wn+1 is 0, and for that if we will just go for the stability of this system we have to find the Eigenvalues of this matrix there, so to find this eigenvalues of any matrix we'll just go for this characteristic equations, so if you just represent this matrix A in a characteristics equation form then we can just write the system 3.2 as a determinant of A-lambda I = 0, suppose this eigenvalues are represented as lambda I here or the eigenvalues of the matrix A and if one of this lambda I value is greater than 1 then the system is unstable.

Already in the last lecture that we have discussed, this means that the system will goes out from the origin so that's why we are just writing this as unstable about the equilibrium point. And if all lambda I is strictly less than 1 then the system is set to be stable, this means that automatically it is just attending towards the origin, whenever we'll just go for higher powers of lambda I, so that's why the point is set to be stable at equilibrium point, and if all lambda I is less or equal to 1 this means exactly 1 value if it is achieving the value as 1 then the system is called asymptotical stable about equilibrium point, this means at 1 it will just fix a line there so that's why spirally it will just come towards like the critical point if it is existing there, then the solution will just approaches towards the point in that situation.



So this is the third point we have just consider, either it is less than 1 or at least one of this values should be equals to 1 there, so if you just go for a example of any linear pre-predator model so suppose for example if you're just considering a pre-predator model for two spaces that is represented as X and Y, suppose if you, in the last slides we have just consider as a tigers and deer's, so if you just consider X as the tiger population and Y as like deer population, so then we can just write X is a predator and Y is the prey, so the prey's growth rate is suppose 1.2, while that of predator is suppose 1.3 here, and prey's population reduces by factor of 0.3 times that of predator, while predator's population increases by 0.4 suppose that of prey's population, so we have to construct a discrete time model and we have to justify the stability based on this approximations.

So from the given data if you'll just formulate this model then like we have just define in the previous slide as like A is the rate at which tiger dies in absence of deer, or b as the rate at which tiger grows in presence of deer, c is the rate at which deer grows in absence of tigers, and d is the rate at which deer dies in presence of tigers. Then if you'll just consider all the coefficients in this problem here or we'll just relate that problem to this problem here, then we can just find A is the quantity, that is the predator population increases by 0.4 times that of prey's that is nothing but the quantity A there, and if you just consider here that is the prey's growth rate is 1.2 while that of predator is 1.3, this means that b is 1.3, and c is 1.2, and finally prey's population reduces by factors of 0.3 times that of predator, that is nothing but d = 0.3 here, hence the formulator model if you'll just write like Xn+1 this equals to 0.6 Xn + 1.3 pi n, since we are just writing that is a negative of 1 – 0.4, so that's why it is just coming as 0.6 here, and specifically b is present independently so that's why it is written as 1.3 here, if you'll just see here that is 1 – a, b and –d 1+c, the same quantities we are just writing here that is 2.2 since we are just writing that one as 1+c there, so that's why 2.2 x Yn it is present there, and -0.3 Xn it is present there over.

Example on Linear Prey-Predator Model:

- The eigen values of the coefficient matrix are: $\lambda_1 = 1.4$ and $\lambda_2 = 1.9$. Hence the solution will be $X_n = c_1(1.4)^n + c_2(1.9)^n$, $Y_n = c_3(1.4)^n + c_4(1.9)^n$. By substituting these equations back in the system, one can get rid of 2 constants out of 4 constants (c_1 ,c_2, c_3, c_4). (Exercise. Verify that $8c_1 = 13c_3$; $c_2 = c_4$.) and with given initial condition one can find the particular solution.

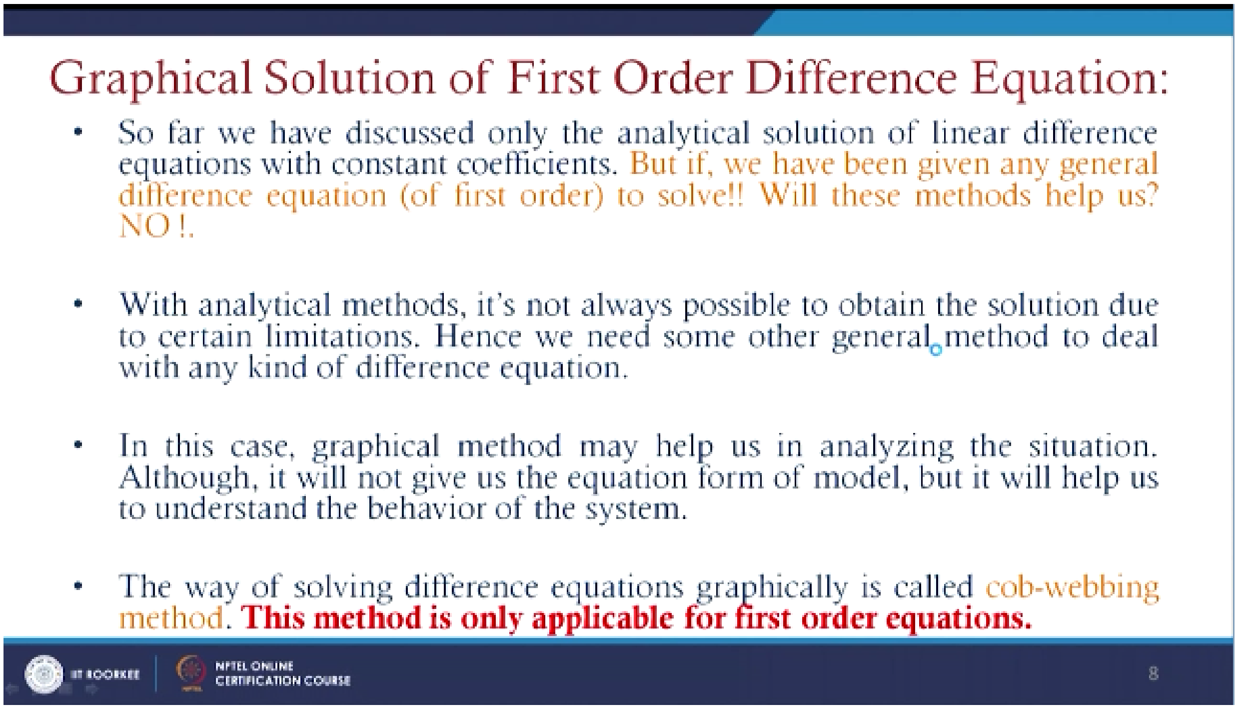
$$\begin{aligned} X_{n+1} - 0.6X_n &= 1.3Y_n \\ c_1(1.4)^{n+1} + c_2(1.9)^{n+1} - 0.6(c_1(1.4)^n - c_2(1.9)^n = 1.3Y_n) &= 1.3Y_n \\ 0.8c_1(1.4)^n + 1.3c_2(1.9)^n &= 1.3Y_n \\ \frac{8}{13}c_1(1.4)^n + c_2(1.9)^n &= Y_n \end{aligned}$$

- The absolute values of both the eigen values are greater than 1, and hence the system is **unstable** about it's equilibrium point (0, 0).
- Now, what do you mean by unstable system here?- This means the process will go on and will never settle. When, predators will increase, prey population will grow on increasing and vice – versa.
- Now, we can analyze the situation better, when one of the λ is less than one and when both λ's are less than one. (Analyze!!)

So if you'll just write this coefficient matrix here, then this coefficient matrix can take the values as 0.6 for the first coefficient here, then 1.3 as the y1 coefficient, then if you'll just see Xn coefficient here that is nothing but -0.3 here, and Yn coefficient that is written as 2.2, if you'll just write this one in a matrix multiplication form then we can just write AX as here Xn Yn, so this is the complete system we can just represent, and to get this eigenvalues of this coefficient matrix suppose then we have to solve this system that is A – lambda I = 0, so you can just put here 0-6 – lambda here, 2.2 – lambda here, and find the determinant, especially we can just write this is an nothing but A – lambda I, this equals to 0 here then we can just write this one as 0.6 – lambda 1.3 – 0.3 and 2.2 – lambda this equals to 0. So if you'll just do this multiplication like we'll just do here 0.6 – lambda x 2.2 – lambda - of - this will just give you + here, 0.3 x 1.3 this equals to 0 here, and we will have a quadratic equation and if you'll just find the roots, the roots are like lambda 1 = 1.4, and lambda 2 = 1.9 here, and the generalize solution as we have discussed in the last lecture that can be written as like complementary function which has been of the form, Xn = c1 x lambda 1 to the power n, c2 x lambda 2 to the power n that is nothing but 1.9 whole to power n here.

Similarly the other solution that is in the form of Yn here that can be written as C3, your first solution that is lambda 1 to the power n, C4 x lambda 2 to the power n here, so now if you just substitute this equations back in the system one can get rid of 2 constants out of 4 constants here, if you'll just see this two variables Xn and Yn this involves 4 constants here, C1, C2, then C3 and C4 here, so if you'll just go for the solution in a like successive iteration method from here, then we can just write this equation that is as Xn+1 = 0.6 Xn + 1.3 Yn here, so we can just write this modified form of this Yn term as Xn+1 – 0.6 Xn as 1.3 Yn here, and in Xn if you'll just replace this as Xn+1 here which can be written as C1 1.4 whole power n+1 + C2 1.9 whole to the power n+1, so that we are just replacing here in terms of Xn+1 – 0.6 into the solution whatever it is just written C1, 1.4 whole to the power n – C2 1.9 whole to the power n here, so this equals to 1.3 Yn here, so which can be written as like this C1 x 1.4 whole to the power n + C2 1.9 whole to the power n and 1 coefficient it can be taken as 1.4 outside, and 1.9 it can be taken outside, and that can be written in a modified form, and in that sense if you'll just find the values that is nothing but 0.8 C1 x 1.4 whole to the power n + 1.3 C2, 1 to the power 1.9 whole to the power n, this equals to 1.3 Yn here.

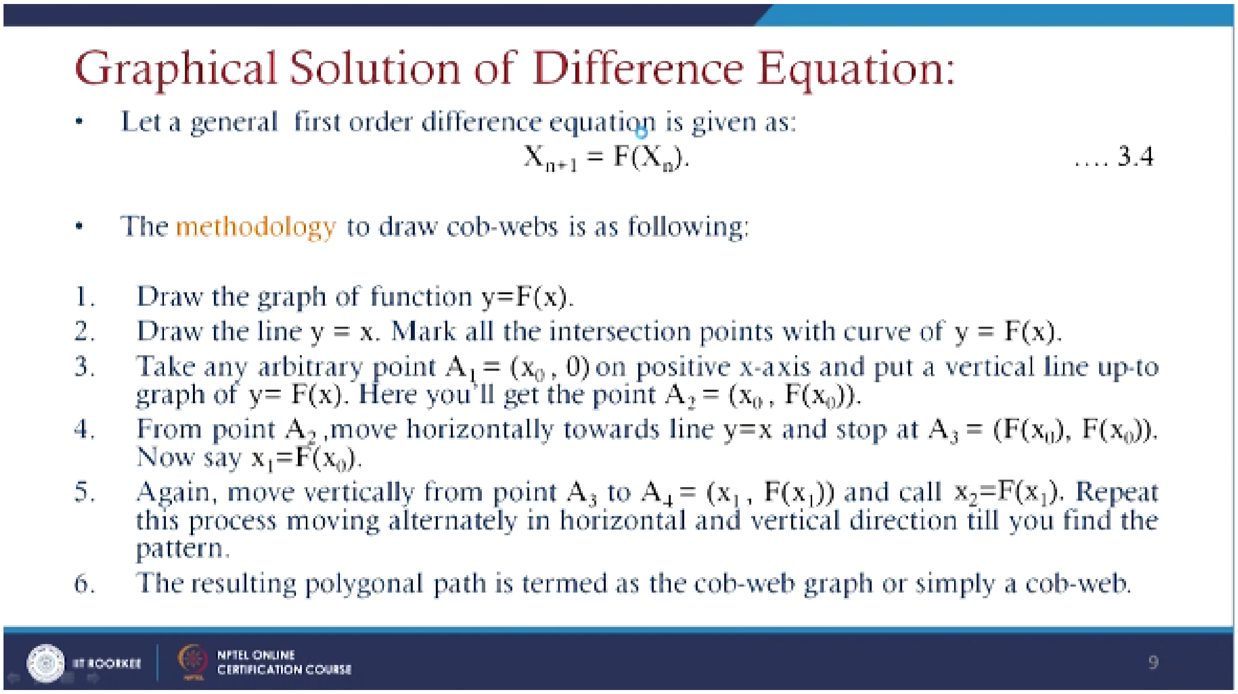
And if you'll just divide 1.3 on the left hand side then we'll have like 8/13 C1 (1.4) whole to the power n + C2 (1.9) whole to the power n this equals to Yn here, and the absolute values of both the eigenvalues are greater than 1, and hence the system is unstable about its equilibrium point 0, 0. And if we will just go for the like, conditional checking of what do you mean by unstable system here, this means that the process will go on and will never settle this means that even if the tiger will be present and the deer will be present, so then both this population will grows it off, even if the tiger will be catch or like kill the deer's and deer's will also be like it is killed by the tigers, but both the populations it will just grows it off, so this is the thing that when predators will increase, prey population will go and increasing and vise-versa similarly if the tiger will dies out then this like deer's will also dies out, so we can at control the situation.



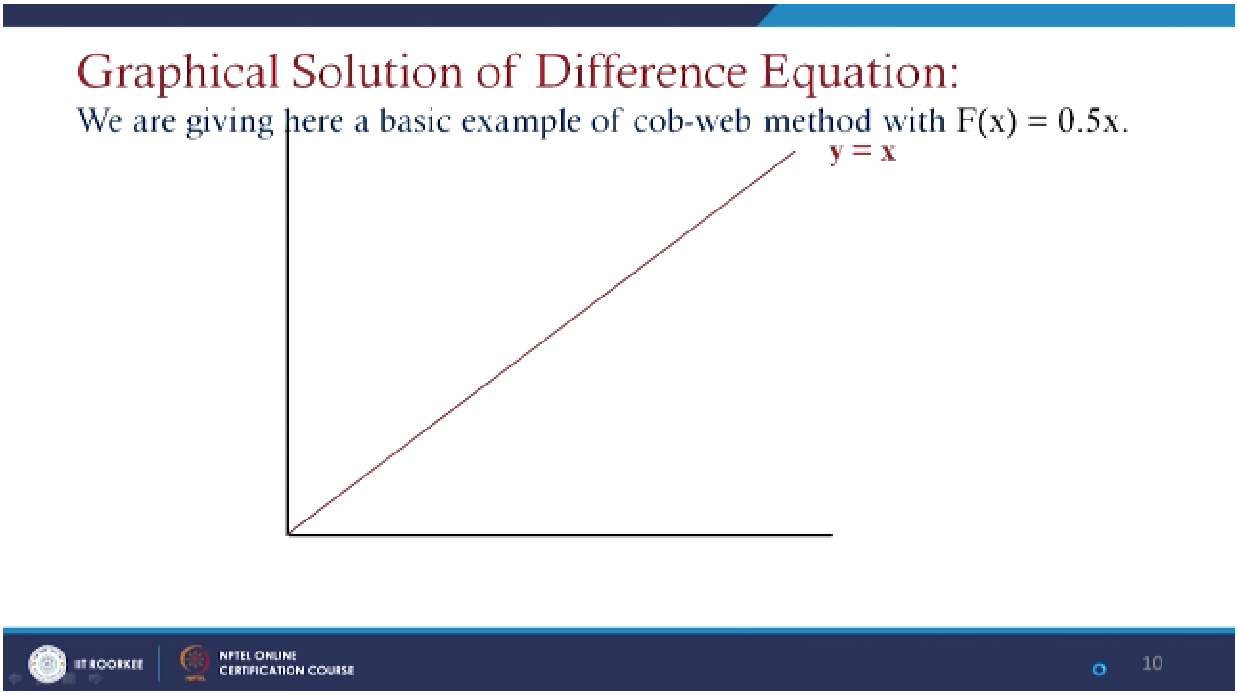
Now if we'll just go for the analyze of this situations in a better way, when one of this lambda I is less than 1 and when both this lambda I's are less than 1, this means that when one of the lambda I is less than 1, then other lambda we don't know what, how it will just performed there, maybe we can just control one variable but other variable how it will just perform we don't know, and when both this lambda I's are less than 1 then we can just say that the system is a going towards like stable condition there, and we will have like certainly definite solution there, so if you'll just go for this graphical solution of first order difference equations, so far we have discussed only the analytical solution of linear difference equations with constant coefficients, but if we have given any general difference equation of first order to solve, will this methods help us? So definitely it is no, so in that situations this methods will not help you out to get it out, so that's why we need a graphical solution method where we can just check the stability whether the solution is moving towards the equilibrium point or not.

So with analytical methods it is not always possible to obtain the solution due to certain limitations, hence we need some other general method to deal with any kind of difference equations, and in this case graphical method may help us in analyzing the situation, although it will not give us the equation form of model, but it will help us to understand the behavior of the system, whether it is just going towards the equilibrium point or towards the origin, or it is just deviating from the origin and going towards the outside.

And the way of solving this differential equations in a graphical sense is called cob-webbing method, and this method is only applicable for first order equations only, and in the previous slide we have also explained here if you just compare this coefficient values in C1 and C2, and we can just relate this variables with this Yn coefficients then at that time we can also establish this relationship between C3 and C4 also, that you can just derive in your form.

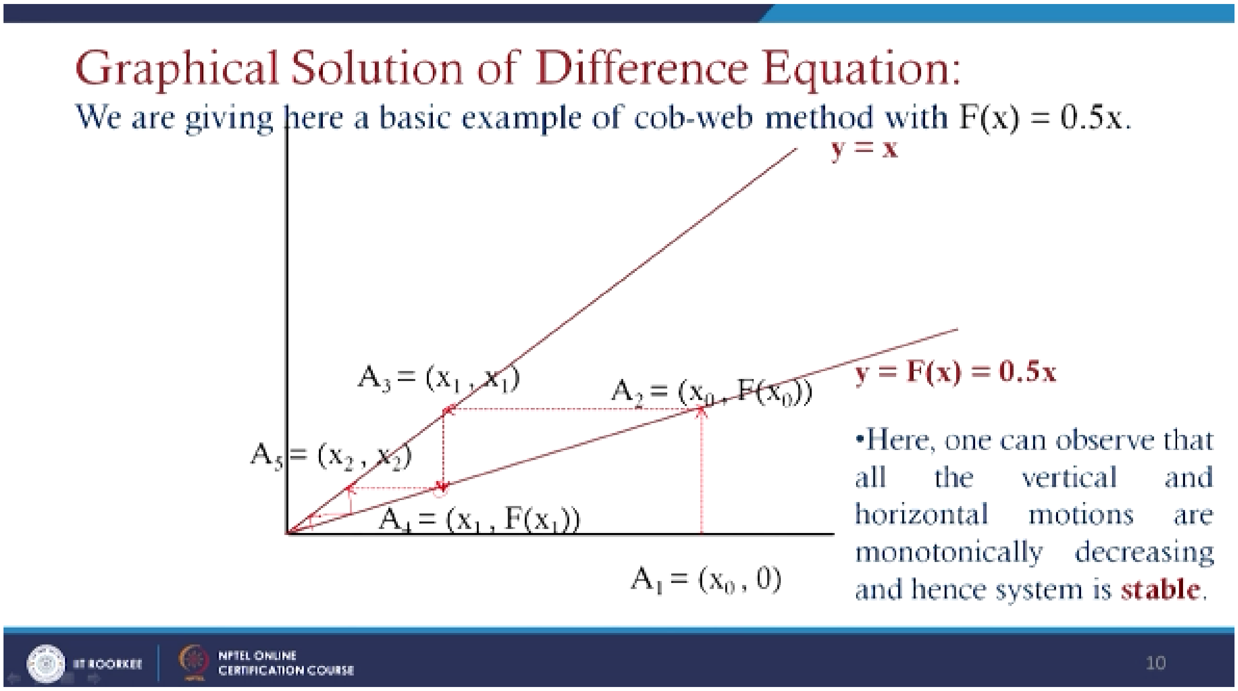


So first if you'll just go for this graphical solution of difference equations, let us consider the first order difference equation which is written as Xn+1 this equals to F(Xn) suppose here, the methodology to draw cob-webs is as following suppose, that is first draw the graph for y=F(x) and draw a line y = x, mark all the intersection points with the curve of y = F(x), and take any arbitrary point A1 suppose, as a point X0, 0 on positive X axis and put a vertical line of to graph of y = F(x), here you will get the point A2 as the point of X0, F(x0) that I just show in the next slide. From point A to move horizontally, if you just see that you will have like graph here and you have this graph like y = X is existing there, then arbitrarily you were just choosing any point here, X0 and you'll have a functional value along this line here F(x0) and the coordinative you can just define as X0 and F(x0), then just draw this line then you can just draw this line along this way, then you'll just put this then you'll just go in that form. And finally you can just reach that, that I'll just explain in the next slide here.



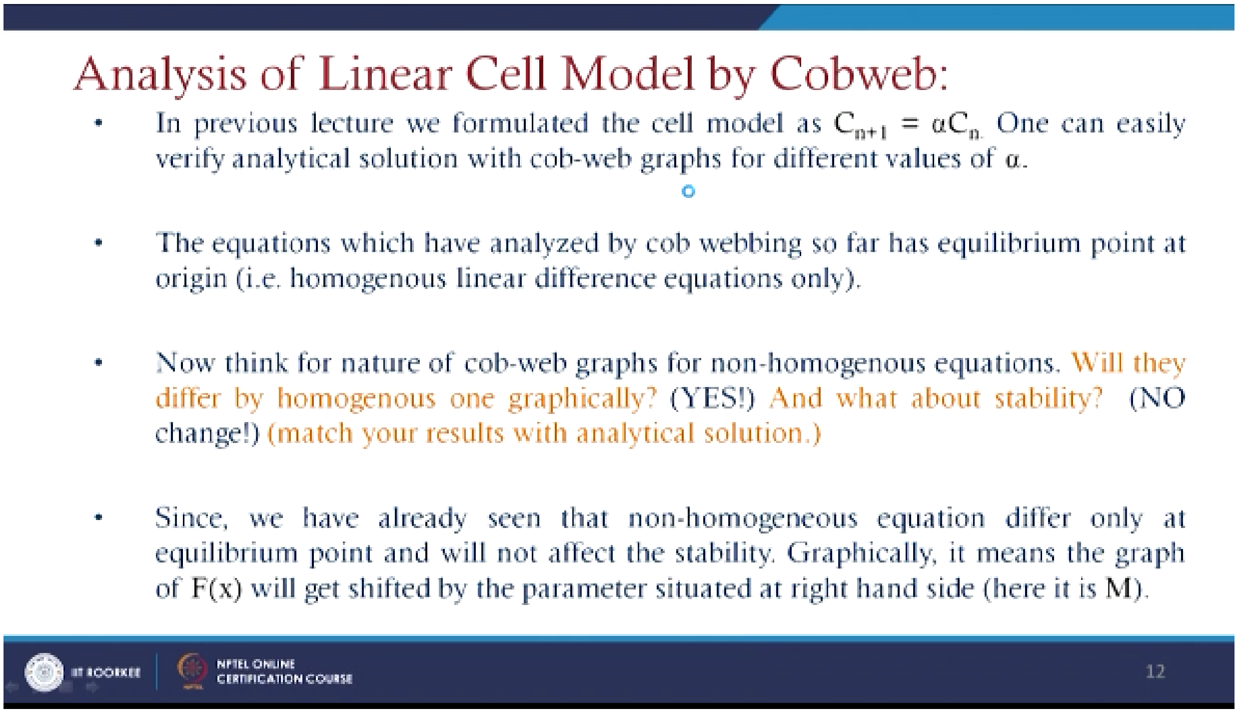
So from this point A to move horizontally towards the line y = x and stop at a A3 = F(x0), now say x1 = F(x0) is there, again move vertically from point A3 to A4, that is the point of x1, F(x1) and call x1 as a F(x1) there, repeat this process moving alternatively in horizontal and vertical directions till you find the pattern, the result in polygonal part is term as the cob-web graph or simply a cob-web, so if you'll just see the slide here, first we are just considering the basic example that is in the form of like F(x) = 0.5x suppose, and if you'll just plot this coordinates here as in the form of like this one here, then directly we are just putting y = x line, and if you just put this line y = F(x) as 0.5x as we have explained in the previous slide, then we are just considering any arbitrarily point suppose x0 is an arbitrarily point on the X's here, then we are just pointing this point on this graph of y = F(x) here.

And next if you are just plotting or putting this like perpendicular line or the functional values along this like y = x line then this line will be represented in this form here, so then we will just put this like functional values of the point along this line that is the functional value here, then again we will just go towards this rotation of this line towards this point, since x1 is known to us then F(x1) we can just find and put this perpendicular line on the y = x line there.



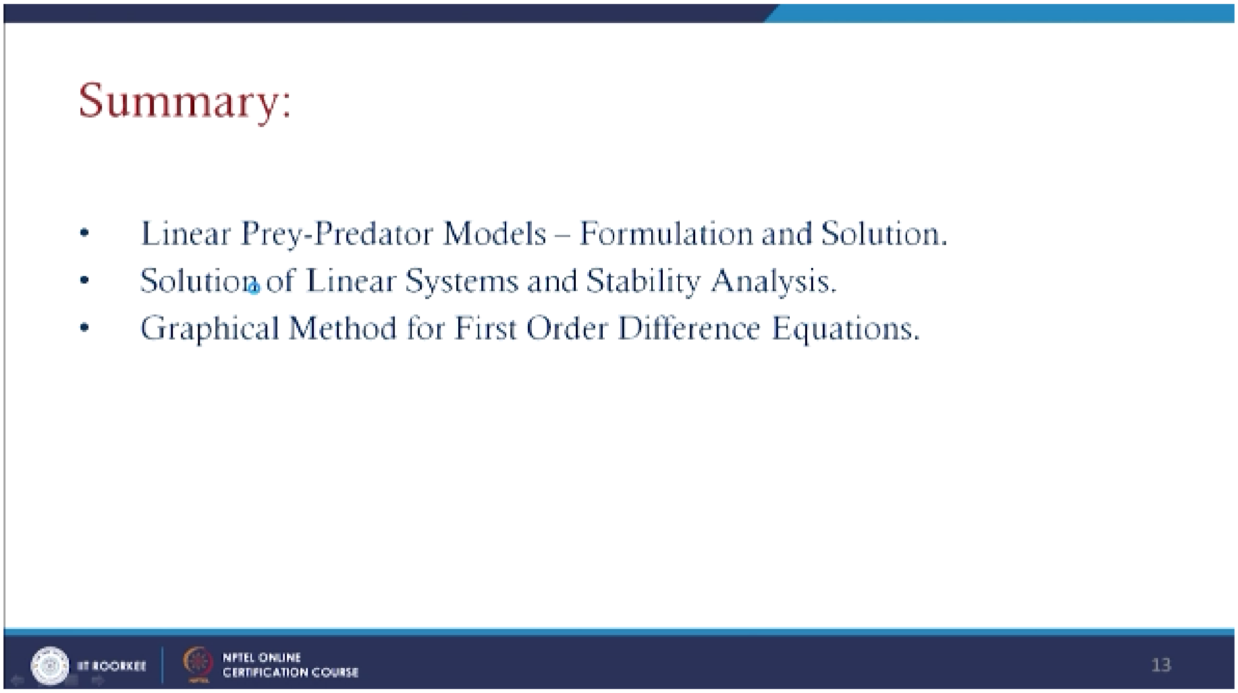
And finally if you'll just plot again this line then this will just go in this direction, then it will just go in this direction and it will just go in this directions, likewise you'll just move towards the origin there. Here one can observe that all the vertical and horizontal motions are monotonically decreasing and hence the system is stable here.

So if you'll just go for like similar solution for a function like F(x) = 0.8x here, and this F(x) = 0.8x it will just lies above the line y = x and if you'll just see here, any point if you're just considering like x0 is the point on this axis then the corresponding functional values along this function y = F(x) will be like x0 and F(x0), then we can just plot the perpendicular line along this line y = x there, and again if you'll just take this line that will just go outer wards towards the origin, hence we don't have like the stable condition here, this means that we are not moving towards the origin or the critical point, it is just going out wards there, hence we can just conclude that all the vertical and horizontal motions are monotonically increasing, hence the system is unstable here, so if you'll just go for the analysis of linear cell model by cob-web

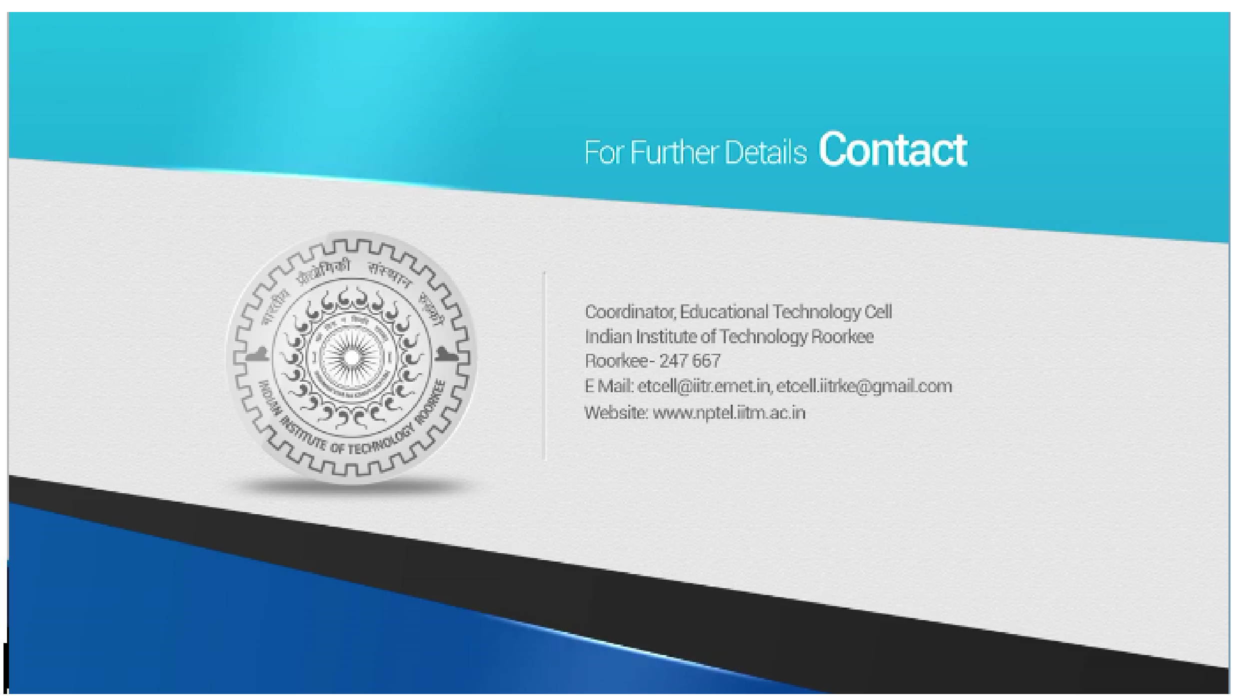


here, since linear model we have just consider cell division model as Cn+1 = alpha Cn, and one can easily verify the analytical solution with cob-web graphs for different values of alpha here, if you just go for this equations which have analyzed by cob-webbing so far has equilibrium point at origin only, that is homogeneous linear difference equations.

Now if you'll just think for the nature of cob-web graphs for non-homogeneous equations, will they differ by homogenous one graphically? Yes definitely it will differ, and what about the stability whether it will just move towards the stable conditions or not? Especially you can just find that it is a no sense, so if you, we can just go for like matching of the results with analytical solution, we can just verify that this non-homogeneous equations it will just differ when they appear in a graphical sense with the homogeneous models.



Since already we have seen that non-homogeneous equation differ only at equilibrium point when will not affect the stability, graphically it means that the graph of F(x) will get shifted by the parameters situated at right hand side here, it is like m if you'll just see that is all the point if you'll just verify then you can just find that this is just a satisfied by the value of m, and then we will just go for the summary of this lecture, that is linear pre-predator models that is formulation and solutions we have discussed here, then we have discussed about the solution of linear systems and their stability analysis, then we have just discussed about their graphical method and first order difference equations. Thank you for listen this lecture.



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Acknowledgement
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