INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
NPTEL
NPTEL ONLINE CERTIFICATION COURSE
Mathematical Modeling
Analysis and Applications
Lecture-20
Continuous Time Prey-Predator Model
With
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Welcome, to the Lecture series and mathematical modeling analysis and applications in the last lecture we have discuss the competition model, where we have consider like, Lotka Volterra equations and in that model we have discuss like, the stability analysis based on the null-clients and statutory matrix and like phase diagrams and also we have discuss the comparison of different models they are like, they are for the stability analysis how we are just using this phase diagrams are null clients whether we can have a like, qualitative analysis with the exact solution that we have also made and this is the final lecture for this one and here will just discuss this continuous time reactor model.

And in this model also, the same

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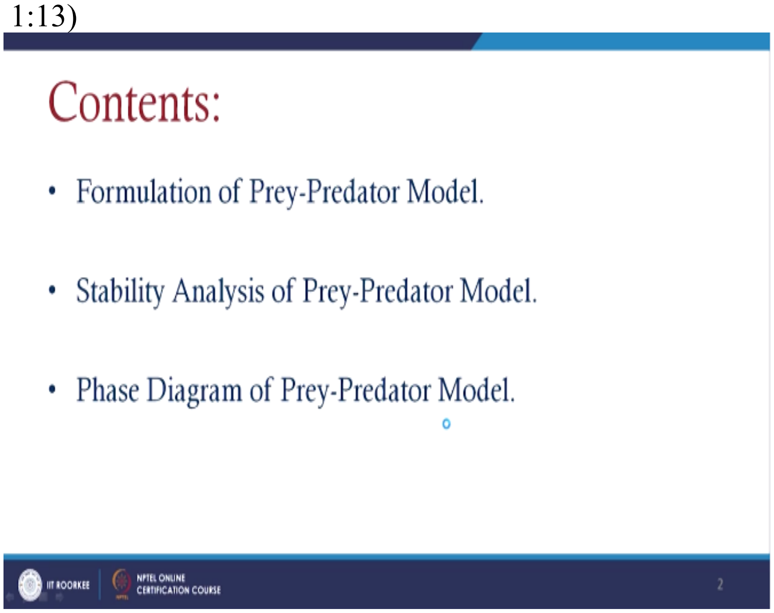

Prey-predator Model, we will just use but, somebody think that will just do adding some terms then, we will just go for this stability analysis of this modified model and finally, we will just go for the phase diagram since, in the last

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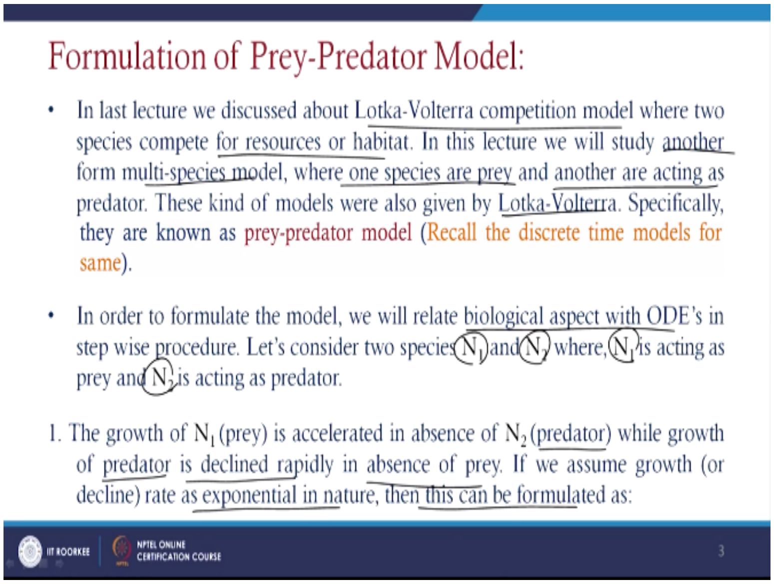

Lecture, if you see we have discuss about Lotika-Volterra competition model like, if more than 1species, if is present either it is like, inter specific or intra specific they will just compute each other for they like, resources are like, survival over there, so two or more species if it is present, then especially we are just using this competition model, so they can just compute for resources or habitat. So either one of them, if it will justified they can just, we can just say that is like a competition model and in this lecture we will study another form multi-species model, where one species are prey and another species acting as predator.

But these kind of models were also, discussed by Lotka and Volterra, specifically they are known as a prey-predator model, if you just recall this descript time models for the same you can just find that, we have used to like, different forms of a mathematical formulation there, so in order to formulate this model we will just relate the biological aspect with ordinary differential equations in step wise procedure.

Let us consider two species N_1 and N_2, where N_1 is acting as prey and N_2 is acting as predator and the growth of N_1 that is prey is accelerated in absence of N_2 or the predator this means that, if predator is not present there, then this prey will grow it of like, geometrical form or in the expansion form it can just grow it of initially since, like a initial resources are the food is available to them, then directly there is no decline of that population then they can just grow it of while the growth of the predator then if predator is present there, they can just eat the prey, they can just decline this prey population.

And this predator population if it suppose, sufficiently this prey will be reduced then this predator is decline rapidly in absence of prey. Initially in the last lecture also, I have consider like, one example like, suppose deer population if it is presented and some tigers are there, so then when this like deer population will may use then the tiger will eat the deers and they can just grow rapidly and once this population of deer will be like less, then again this tiger population will be less since, this tigers will fight each other to get this deer for their food.

And if we are assumes growth or decline rate as exponential in nature, then we can formulated this model

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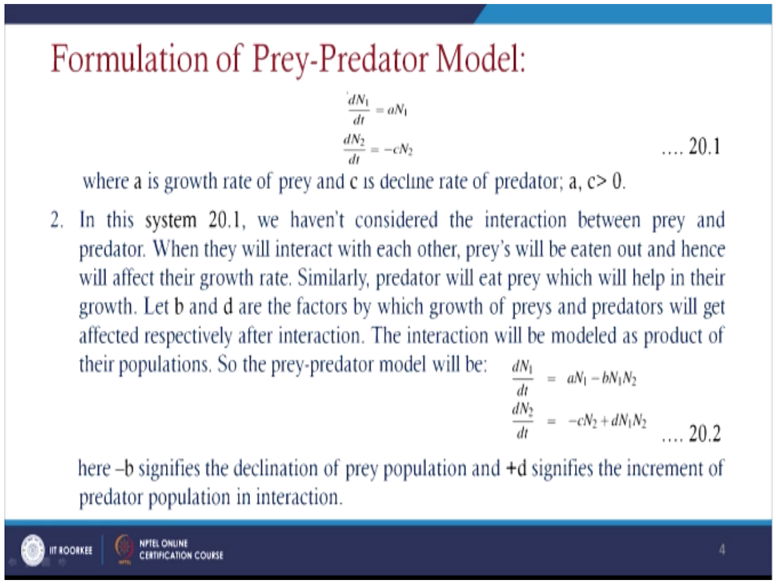

Like, dN_1/dt= like, aN_1 here, since, directly if you just get this solution it will represent N_1=we can just write as aN_1t, so expansion is grow it off with respective time if you just see and if you can just find here, that a means, we have considering with the growth rate of prey and C is decline rate of predator here, and if you just consider here, predator, predator means like, suppose N_2 is a predator here, and if you just considering that they are also decaying exponential in absence of prey then we can just write this model as dn_2/dt as $–CN_2$ here, where already we have consider this like growth rate of prey or decline rate of like, predator it can be consider as positive, they cannot be negative.

So, in this system if you just see we have not consider the interaction between the prey and predator this means that both the equations are independent to each other and if you just consider like, interaction between prey and predator then preys will be return out and hence, will affect their growth rate and similarly predator will be eat prey which will help in their growth so that is why? .We can say here b and d are the factors like, which the growth of preys and predator will get affected respectively after interaction.

And if you will consider this interaction will be modeled as a product of their population. Since, individually if you just consider like, one tiger just they are eating like, two deers at a time, then individually we have to consider like, this predator population or prey population they are affected by each other, so that is why? We have just consider that when this predator population they will just a grow it of then it will just show a decline of this prey population.

Similarly, if this prey population will be like sufficient then this predator population they will just grow it off, so that is why? We have just considered there is a decline rate of prey here, and there is like, postiveness of population growth in predator here.

And here, we have just written here –b signifies the declination of prey population and +d signifies the increment of predator population in interaction. Since, if you just see that sufficient amount of deer and sufficient amount of tiger if it is present there, so no other like, predators are available there, only tigers and only deers if it is present then sufficiently they can just like, tiger can use this deers and their population can grow it off, but in the mean while you can just find that this deer population will get reduced, so that is why? –b and this is + d, so

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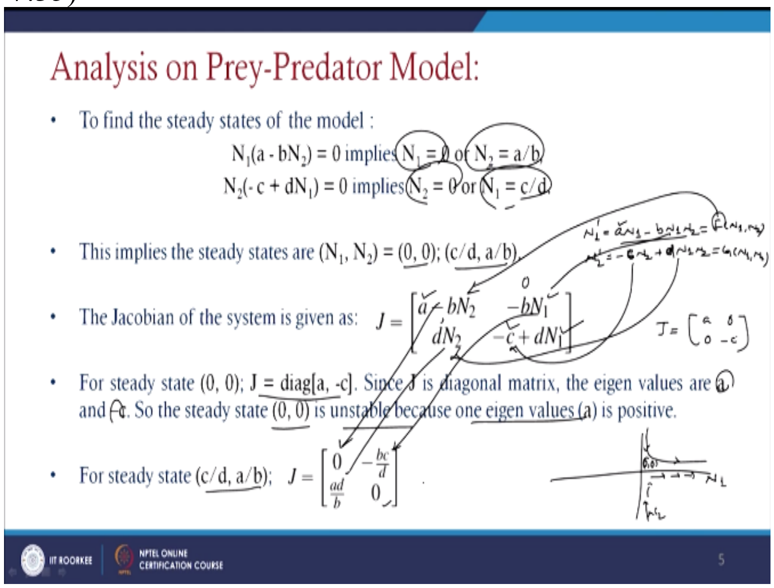

If you just to find here, find this study states of this model, if you just see that we have consider here, this model as d and 1=0, and dN/dt this = 0 so then we can just find that, if you as just take common here directly N_1 X a-b and N_2 this =0 and similarly if I just take common N to here, I can just write –C+ dN_1 this = 0 here, obviously I can just write here either N_1=0 or like, a-b N_2=0 which will just provide you here, like,N_2= a/t and similarly, if I just consider this second equation here, then that will just you to provide you either N_2=0 or –C+dN_1=0 here, and which will also provide you like, N_1=c/d, and to achieve this critical points especially we will have this points like, N_1=0 and N_2/d and N_2=0 or N_1=c/d here, so obviously the steady state points are like 0,0 and C/d, a/b here, and if you just formulate this Jacobean and if you just formulate this Jacobean here, then we have to take this differentiation with respective to first N_1 for this equations.

Since, it is nothing but we can just write FN_2 and second one we can just write this one as the CN_1 and N_2 here, so for this Jacobean first we just take the differences of an F restricted N_1 then second element we can just consider the differentiation with respective N_2 so, that is why? if you just write directly the model here, and then we will have like, first one that is in the form of N_1 is which is return as aN_1-bN_1 N_2, and second one if you just see it is return as a N_2 which is nothing but –bN_2+ this C, this is d here, N_1 and N_2 stay as –CN_2+dN^1

So, if just take this differences considering here, this is FN_1 and N_2 and this is a FN_1 and N_2 and this is C N_1 and N_2 here, first I am just considering this differentiation of F with respective N_1, then I will have this value like K-b and N_2 here, from this equation similarly, I just take this a differentiation of F with respective N_2 here and then I will have like, -b N_1 from this term here, similarly, we can just differentiate C with respective N_1 here, I will have this term as a b N_2 here, similarly if I just take the differences on of C with respective N_2, I will have –C+d N_1 and for this study state if I just put directly as a N_1=0 and N_2=0 here, then I will have this values from this equation I will have a only and from this equation I will have –a only and rest of this quantity is

0, so that is why? I will have this Jacobean S like A and this is the 0, then this is 0, this is –C here, so that is why? I have just written for study states 00, Jacobean represents diag element as a and –C here,

Since, J is the diagonal matrix here, and this is like, diagonal matrix so the diagonal is nothing but only the diagonal entries here, so the diagonal entries are nothing but aN-C so, that is nothing but the eigen is here, and for this study state if you just see here, one eigen is positive and negative that represents this is unstable state so this study state is 00 is unstable because one of the eigen value a is positive here, this means that, if you just see the phase diagram we can just find that, one of this value since –C it is present there like, N_2 axis so, it will just to push to this 00 point but this X axis if you just see you can just directly put here, as like, this is suppose N_1 like this is N_2 X axis directly I can just find that, these level, you will have like, this errors will go out.

But, in this level you will see this will just move to out wards the 00. Here, so hence like, phase diagram it will be like, common, this way and then it will just go out wards here, combinely sine, this it is negative it will forces towards the 00 point and since it is positive it will just take away from this points here, again if you just go for this study state analysis for this point here like C/d a/d here, then directly we can just find that if you will have like, a-b and N_2, b N_2 means if I just put directly N_2 as a/d here then it will just cancel it out it will have like a-a so that is why? It is just giving you 00 values here.

Similarly, we will just put like, N_1=C/d I will have this value –b X c/d here, similarly if I just put here N_2 has a a/b here, I will have like, d x a/b and if I just put here N_1=C/d so then – a-a it will just cancel it out. So this will just provide you as a 0 value for this term if I just put C for this like, Eigen values here.

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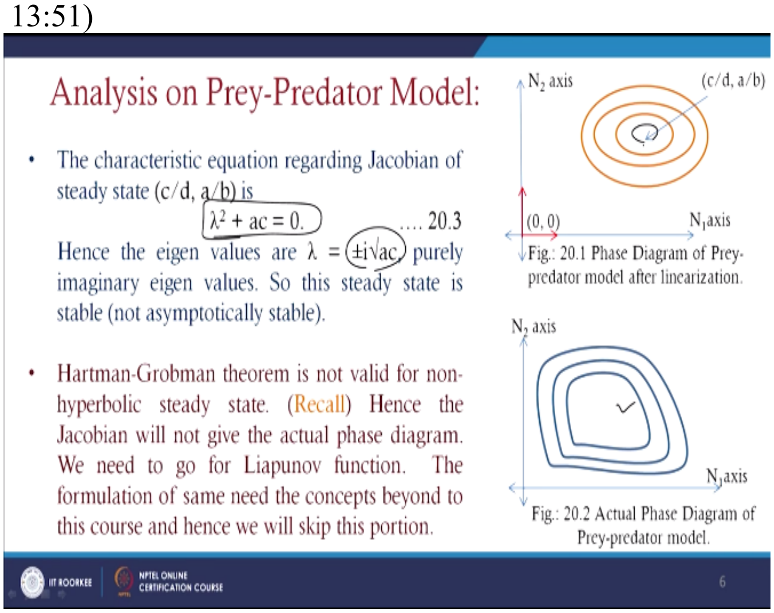

Then this eigen value is nothing but if I just put here as a – lambda, - lambda here, and it is determining =0, and then it will just give you like λ square + ac=0, this is nothing but the correct value, the characteristic equation at the study state points C/d and a/b here, and if you just see this eigen values so they are nothing but this like, a hyperbolic values like square root of ac here, since, it is the complex root so we will have like, - sign here, since eigen values is prey imaginary and we will have like, the study state is stable here, and if you just see here, that the real part of eigen value is the 0 here, it means that represents say non-hyperbolic functions or we can just say non-hyperbolic study state by analysis of Hartman and Grobman theorem hence, if in this state this Jacobean not give the actual passé diagram we need to go for the Liapunov function.

And the formulation of same need the concepts beyond this course and which will just try to skip this exam here, since, they are say sums of this function we have to consider then we will just consider like 0 values and the 0 point and outside that point will have like >0 value and then will just go for this derivative values where it can just have a like stable value or where it will have unstable value that, it is not way of phase session here, so if you just go for this phase analysis or the phase diagram for this eigen value for this C/d and a/b point here, we can just find that 00 point definitely if you will have like, no prey, no predator, then will have a unstable conditions.

And if you just find suppose λ =+ - I square total of ac already we have discuss in the earlier lecture that it will just form this electric measure, or this phase diagram will just replace this electric phase there, this means that, it just follows like, when there is a predator population they will have a like growth rate of level then certainly this predator population they will just grow it up also, and after that we can just find that there it will be a decline of a prey population so then again we can just find there is a decline of predator population so that is why? It is just following this electric phase.

And exactly if you just see this actual phase diagram we can just find that this figure represents the actual diagram of this predator model here, and for this C/d and a/b point we can just find that this represents the stable point for this predator model here, and if you just go for

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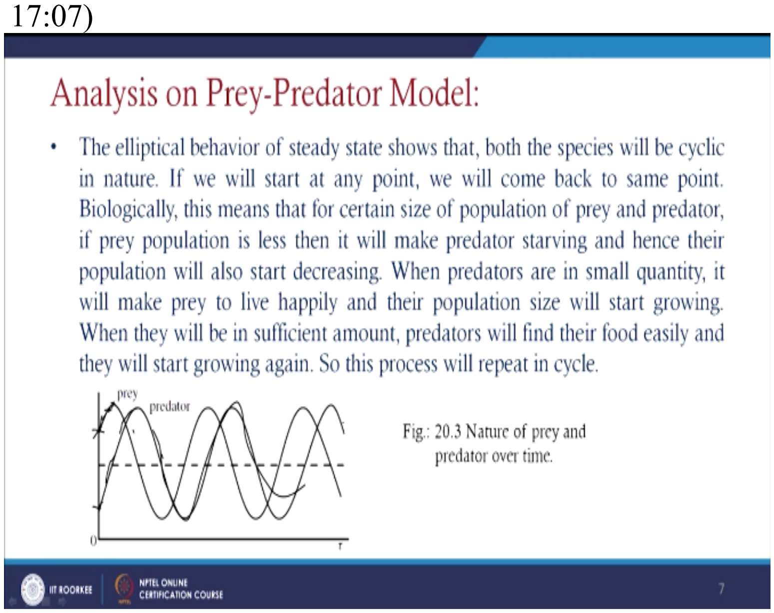

The further Analysis of this Prey Predator model, the elliptical behavior already I have discussed in the previous slide that study state shows that, both the species will be cyclic in nature since, if we will start at any point, we will just come back to the same point, biologically, will just visualize then we can just find that, this certain size of a population prey and predator if you just

consider at a level, if prey population is less, and it will just force to the predator population study, and hence their population will start decreasing and when predators are small in quantity we can just find that it will make prey to live happily and their population will grows it off.
And then when they will be in sufficient amount then predators will find their food sufficiently and they will start growing again. So we will have a like, reputation of this cycle, so if you just visualize a graphical sensor we can just find that, if certain amount of prey if it is present there, and certain amount of predator if it is present there, definitely we can find that initially less predator if it is there then this predator will prey will just grow it off
Since, if it less prey so that is why they can just eats less so that is why? this prey population they can just grow it off and finally we can just find that prey population it will be maximize suppose then this predator population it is just growing it off and after one level you can just find that the predator population has like, greater than the like a ring capacitive, they can just eat sufficient prey then this prey will decline.
But, either level you can just find that this predator population they will again grow it off and after certain level if you will have the less prey suppose then this predator population again they will just get it decline so, that is why? They will just follow like cyclic nature.
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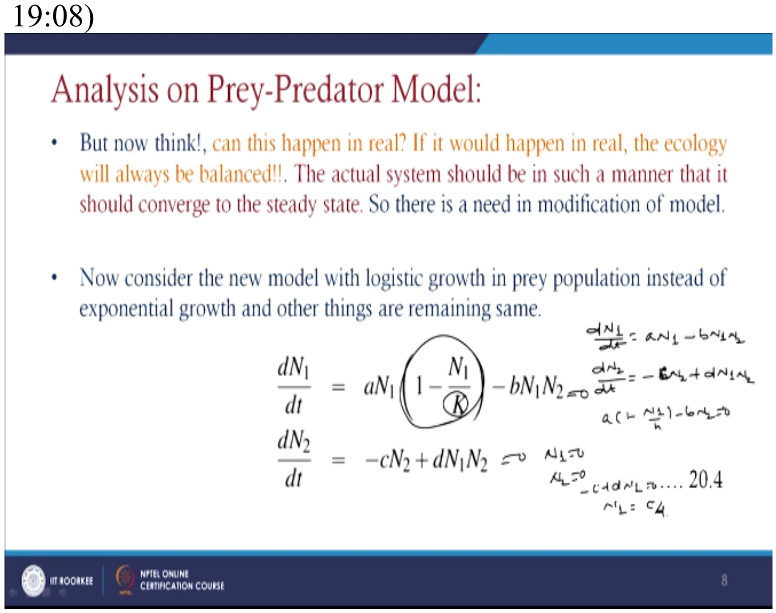

And if you just go for further analysis of this Prey Predator model, now you just think that can this happen in the real? Especially it would happen in real, the ecology will always be balanced, so the actual systems should be in such manner that it should converge to the study state since if you just see in this diagram here, always it just coming towards at definite point there, so there is a need a special modification of this model. Since, especially if you just see this model here, so this ecology should be balanced either we can just find there will be no population of prey or there will be no population of predator so that is why? so we have need some modification of this one now if you just consider this a new model but by this logistic growth population instead of exponential growth population then we have to like, modify this model like, dN_1/dt this can be written as like, we will have a maximum curing capacitive that is K here, so we have along with this equation form of $K-N_1/K$ here, this is the only modification owe have just done from this earlier equations if you just see, in the earlier equation we have the terms like, dN_1/dt this is nothing but aN_1 that is the exponential growth we have consider – this decline rate will be due to this predator population.

Similarly, we have this predator population has a $dN_1/$ dN_2/dt this is nothing but $-bN_2$ this C here, this d of a N_1 N_2 this predator population grows due to like, prey presents and we are just modifying this terms, we have just define in this model if you just see in this cyclic process will have a like, maximum curing capacity here, so for that we will have just added this term that is $1-N_1/K$ or you can just write it as $K-N_1/K$ here, which can just balance this population of prey in presence of predator.

So then if you just for like, further analysis of this modified model instead of this exponential growth if you are just considering here the logistic growth model, then we can just write this one =0 and this one =0 here, then we will have a first term as $N_1=0$ for this study state points, second one we can find it as 2=0 here, and the next one we can just find same 1 as like $-C+dN_1=0$ we will have N_1=like, C/d similarly if I just take a the first term $N_1=0$ then I will have remain form as a X $1-N_1/K-b\ N_1N_2=0$.

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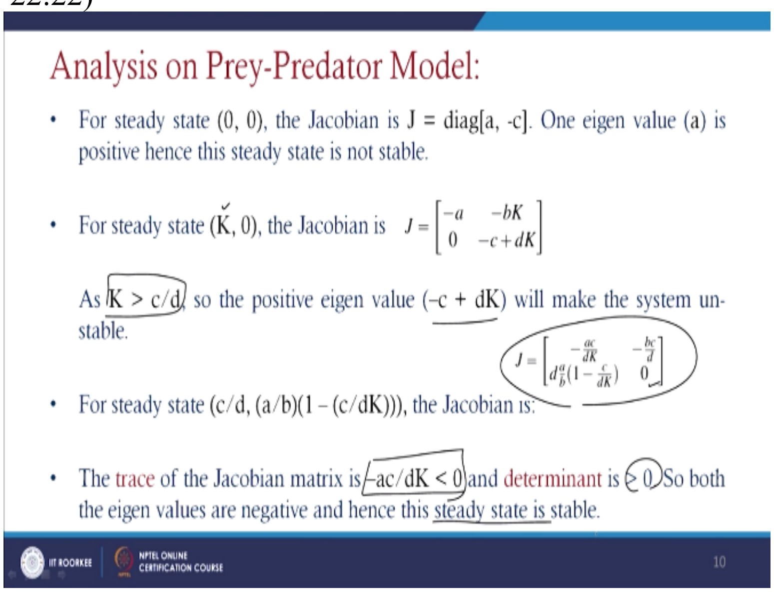

If you just see this something, it has been written again and if you just see here, $a-aN_1/K$ $K-bN_2=0$ so this you just provide if you just combinely consider all the points here, we will have this simultaneous equations that is the steady state points are 0 next one is suppose K is 0 here, and last point is c/cd and a/b, X 1-c/d K observe that it coordinates of steady state must be positive so if you just give here $K>C/d$ so C/d is nothing but it is just represents the ratio of like, decline of predator in absence of prey.

And it should be >0, and if you just consider this Jacobean model so then this Jacobean model if you just consider this equation that this is suppose F N_1 and N_2 here again I have explained in the earlier slides so this can be written as in the form of g N_1 N_2 here, and if I just differentiate with respective N_1 so that will just provide you the term this one and if I just differentiate this F with respective N_2 here, that will just to provide this term here, and if I just differentiate this g term with respective N_1 here, I will have a terms as dN_2 here, so dN_2, and if I just differentiate this one with respective N_2 here, I will have the term –c+dN_1 here, -c+dN_1 and for this steady state 00, if I just see the Jacobean is nothing but this same one I can just get it out here, since, N1=0 and N_2=0 and N_1=0 I will have the remain terms a-c in the diagonal positive.

So, one of the eigen value if it is positive then will have like, unstable situation in that steady state point and for this steady state if you just see here, since, we have consider K is the population for a prey here, and predator population is 0, the Jacobean it will just take this value as –a-bK0 and –c+dk, and if you just consider the maximum curing term capacity, this is greater than like rate of decline of this predator population in the absence of prey so then the positive eigen value is –c+dk will make this system unstable here, and if you just find the Jacobean for this steady states c/d, a,b X 1-c/dK the Jacobean can be return in the form of like, -ac/dK-dc/d and a/d1-c/dK and last value is 0 and if you just take this preys of this Jacobean matrix is nothing but the sum of the diagonal elements, which is nothing but –ac/dK+0 here, which is < 0 here, and if you just see this determine of this matrix then this determine is >0 and hence, conclude that both eigen values are negative, hence this steady state is representing as stable point over here,

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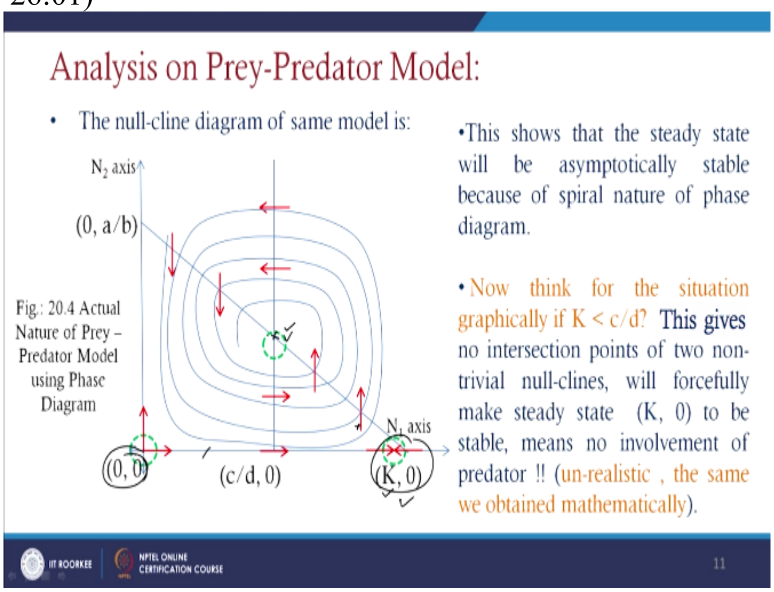

And if you just go for this Analysis on Prey Predator Model here, and if you just plot the null clines for this functions here, so we will have the steady state points is 00 here, and we will have this steady state point K_0 and another steady state point if you just see here, that is nothing but, C/d and a/b X N-C/dK here, so that is nothing but this point over here, and we can just find that the null clines here, that is just intersecting each other and this point also, and as you have discuss in earlier lecture that it will also give you a steady state point here, and if we can just consider that K_0 is a stable point here, then we can just visualize that if there is a sufficient amount of presents of prey is there, there is a no predator.

Then, we cannot find that the decline rate of like, since, we have consider as a C/d is < K here, and in that scenario, we can just think that a rate of decline of predator population in absence of prey that is less, compare to presence of prey here, so that is why? this is on a stable point and this is giving no intersession points of two null clines and if you just consider also, sometime

suppose, if an mathematical point of way if you just generalize the equation then if you will consider suppose K is inside suppose, like, in this interval this means that the C/d is >K suppose, in that scenario we can just also find that this is also unrealistic situation.
Sine, C/d represents this like, rate of decline of predator population in absence of prey but here prey is present there, and K is present then we will have a like, unstable situation so, that is why? we can just say that this is giving more intersection point of two nontribal null clines will forcefully makes steady state K_0 to be stable, means no involvement of predator here so this is a un realistic and mathematically we cannot have 0 here, and if you just see here, all of the points like, different points, if you just see here, that below this point is if you consider then it will just take this phase that the value will go upwards.
And if you just see this physical scenario also, always it will just formulate a cyclic procedure to come towards this conversion point here.
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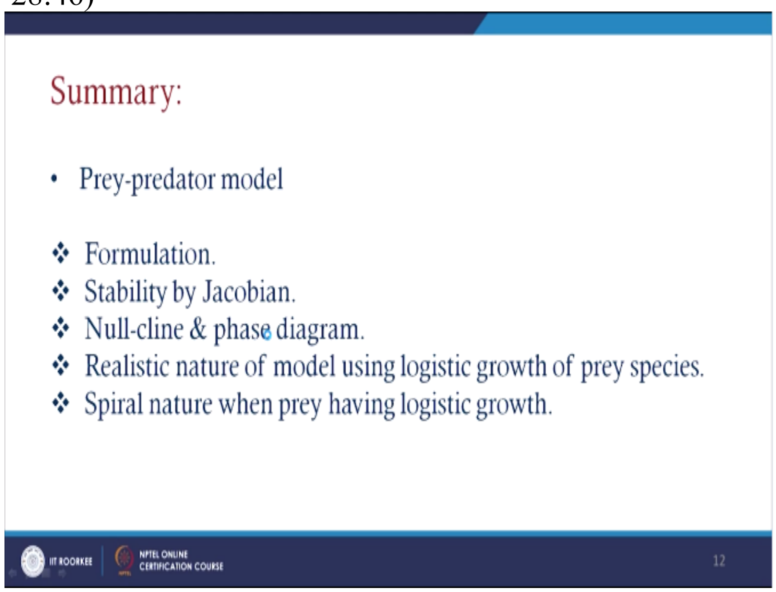

And in this model, we have discuss like, the modified predator model, where like, two species are present that is prey and predator then we have discuss about this stability using this Jacobean matrices then, we have discuss like, null clines and phase diagram for this model and realistic nature of this model using this logistic growth of prey and species, this spiral nature of this prey having logistic growth, and if you just analyze the
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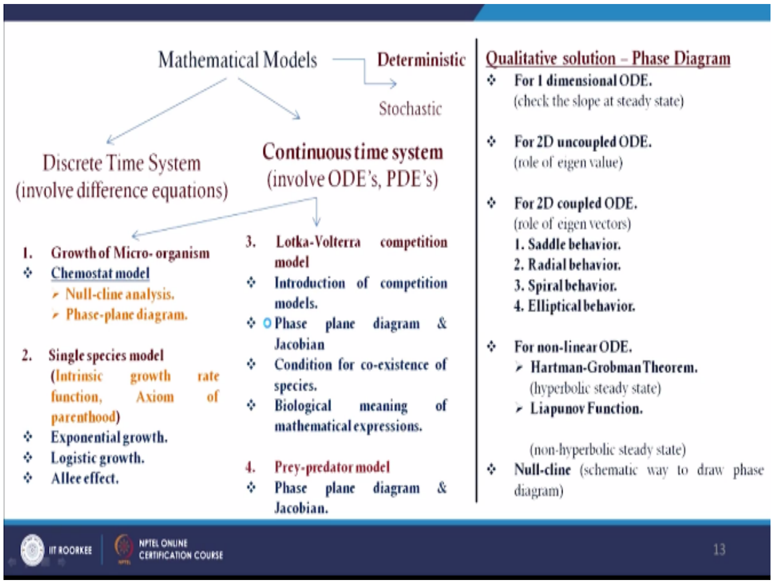

Total lectures here, then in the first mathematical models we have just discuss like, deterministic and stochastic process and in that models first we have discuss like, Discrete time system which involves like, difference equations then in the second phase we have started like, a continuous time system, which involves like, ordinary difference equations like, partial difference equations and in that we have discuss like, growth of micro organism based on that we have develop this chemostat model and we have analyze that model using null clines in analysis and phase plane diagrams so, first we have discuss about this single species model and it can be like, intrinsic growth rate function or axiom of parenthood, then we have consider for this single species model in exponential growth then the logistic growth model and finally were have just consider as the special case model is called allee effect.

And if we have just discuss in to two species model that deals with Lotka and Volterra competition model, where we have just discuss about the introduction of competition models for like, species more than 1 and then the analysis has been made through phase plane diagram and Jacobean to find this stable nature for this system, and this condition for co-existence of species and in each of this co-existence are like, competition models we will try to visualize this mathematical phases or the diagrams or this analysis through this biological meaning and we try to establish this relationship of this mathematical expressions with this biological like, scenarios and in the last phase we just discuss with phase plane diagram and Jacobean for predator models.

And if you will also analyze this one in qualitative phase, then you can just find that this phase diagram analysis we have made for ordinary differential equations to check the steady state their also we have just to find this slopes whether this slope it is just presenting a postiveness or negativness based on that we can just say that whether it is as a stable solution are not and for the 2D system we have discuss for ODE for like, coupled system and uncoupled system and how this eigen value is behaving in case of like, this stability is choosing or measure of this system in a 2 D coupled form and in the coupled form we have just analyze the nature of the eigen vectors there, how they are just like, behaving inside the system and based on that we are just try to analyze this like, stability or unstability nature of this system.

And then the 2 D coupled system we have just define this eigen values to find this saddle behavior, Radial behavior spiral behavior, Elliptical behavior of this system then for non-linear ordinary differential equation we have discuss the Hartman Grobman theorem and we will have like, hyperbolic measure or like, non- hyperbolic measure based on this like real part of this eigen values and then if this non-hyperbolic in nature and we will have include this linear function of the analysis of the stability.

And then, finally we have discuss about null clines that is nothing but the phase diagram analysis based on null clines and with this we conclude our lecture thank you for listen this lecture series

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