INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NPTEL

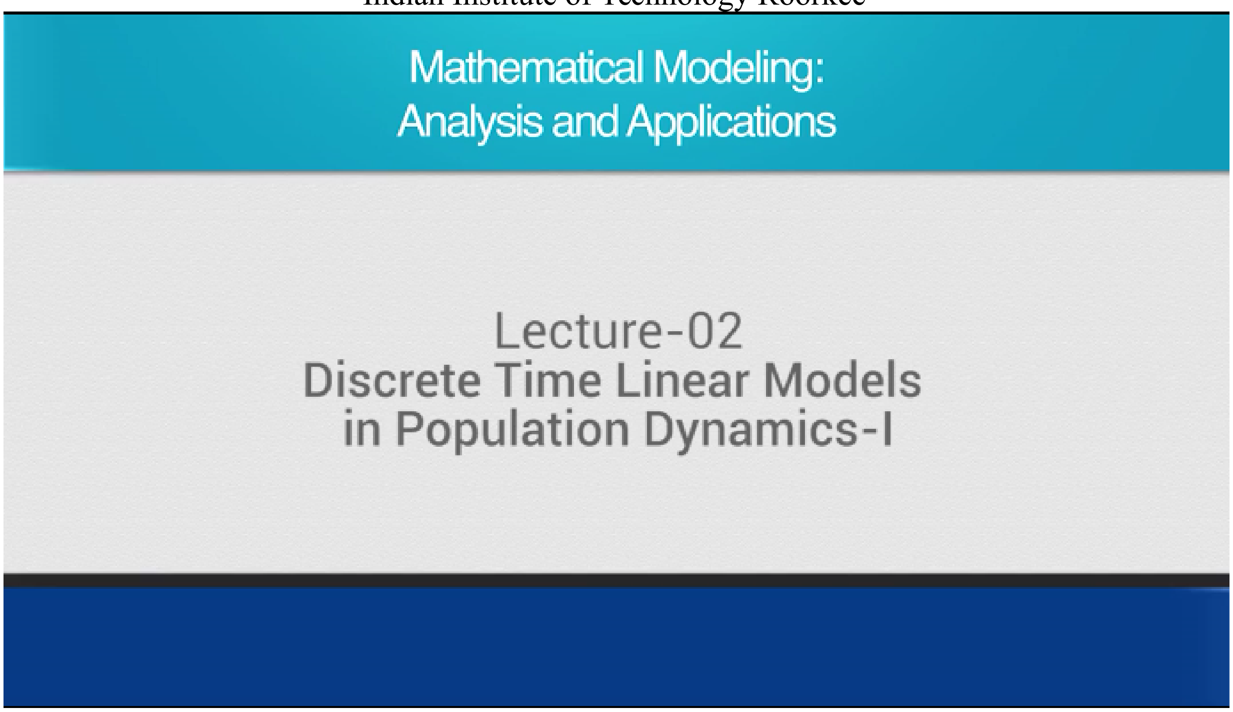
NPTEL ONLINE CERTIFICATION COURSE

Mathematical Modeling:
Analysis and Applications

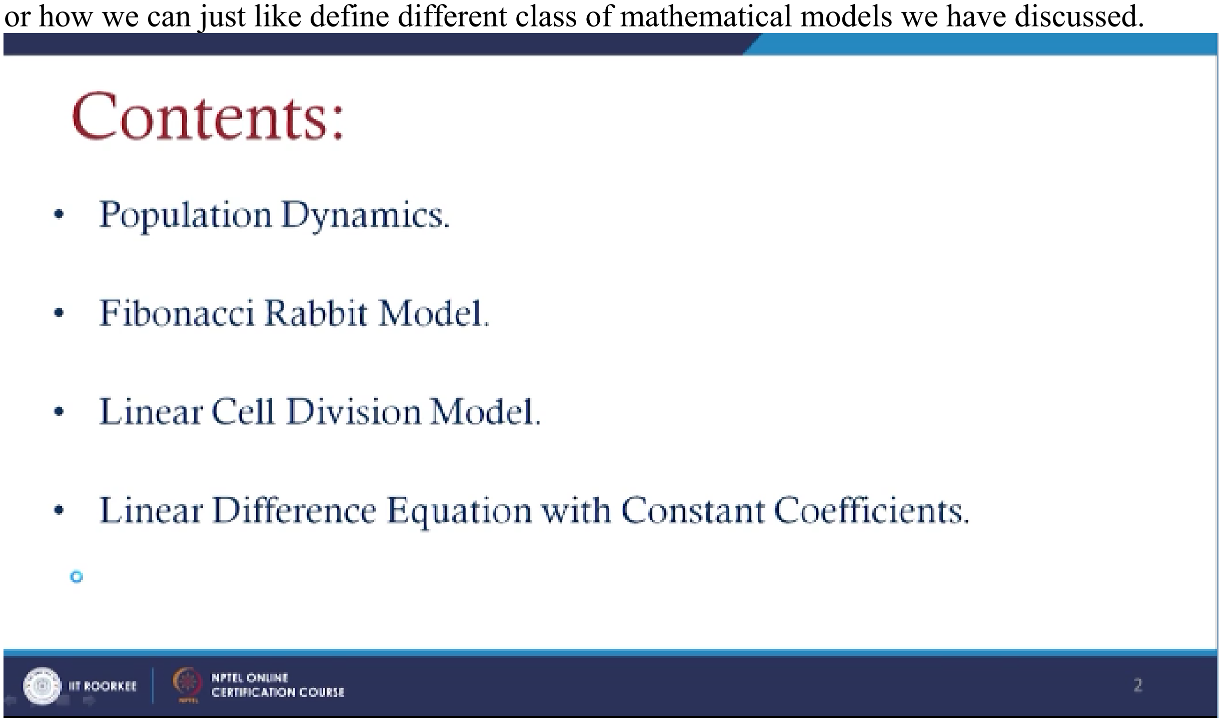
Lecture-02
Discrete Time Linear Models
In Population Dynamics - 1

With

Dr. Ameeya Kumar Nayak
Department of Mathematics
Indian Institute of Technology Roorkee

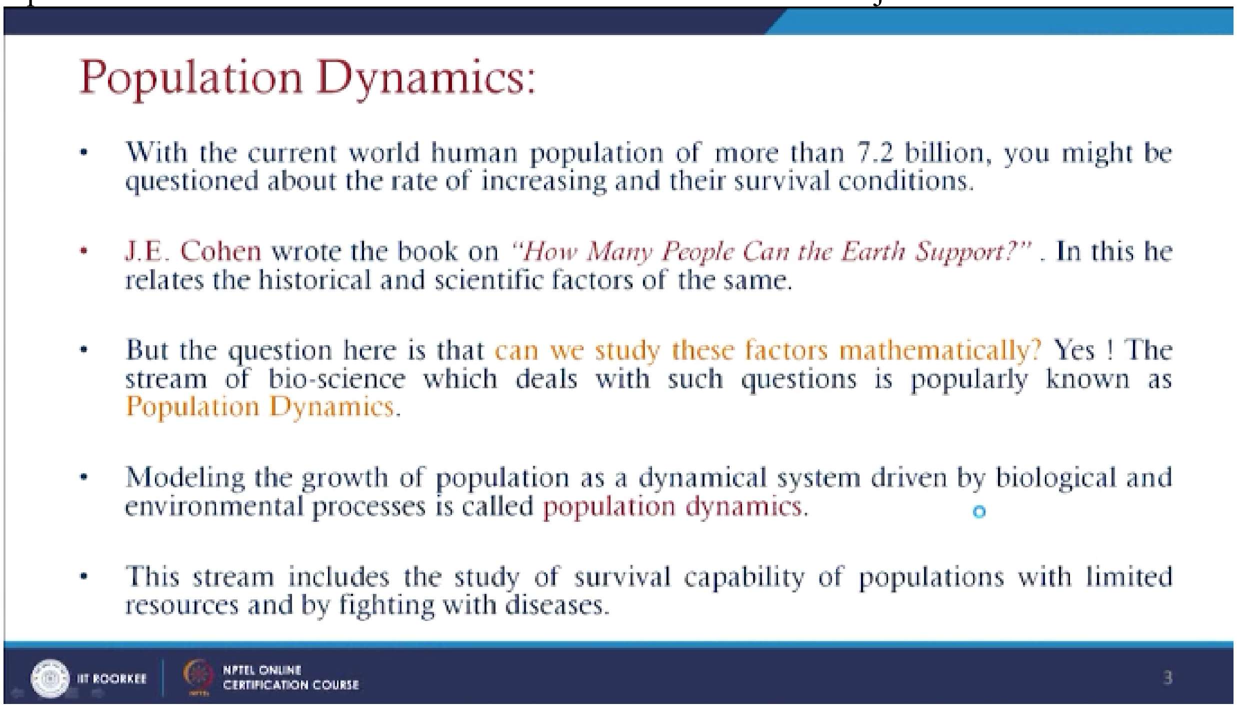


Welcome to the lecture series on Mathematical Modeling Analysis and Applications. In the last lecture we have discussed different class of mathematical models, and how we can just classify or how we can just like define different class of mathematical models we have discussed.



And in this lecture we will just go for the discussion of discrete time linear models in population dynamics.

So first we will discuss about what is Population Dynamics then we will just go for Fibonacci Rabbit Model, then Linear Cell Division Model, and how we can just use these difference equations with the constant coefficients in a linear firm that we will just discuss.



So with the current world for human population of more than like 7.2 billion, you might be questioned about the rate of increasing and their survival conditions, so like J.E Cohen wrote the book on How Many Population Can the Earth Support? Is this he relates the historical and scientific factors of the same, but the question here is that can we study this factors mathematically? Yes, the stream of bio-science which deals with such questions is popularly known as population dynamics.

So when we'll just go for modeling of the growth of population as a dynamical system, already in the last lecture I have discussed about the dynamical system, so if we'll just go for this growth of population as a dynamical system driven by biological and environmental process then it is called population dynamics.

This stream includes the study of survival capability of populations with limited resources and by fighting with diseases, sometimes you can just find this population level always depends on like birth rate and death rate, so if we'll just consider the total populations size at level, then we have to take like birth rate minus death rate, and how this time is affecting, relating to the other factors that we have to also consider. Specifically if we'll just consider all this factors at a time then we'll have a complete population dynamics model.

Fibonacci Rabbit Model:

- Consider the problem of rabbit generation which was originally postulated by Fibonacci.
- A young pair of rabbits, one of each sex is placed on an Iceland. A pair of rabbit does not breed until they are 2 months old. When, they are 2 months old, each pair will produce another pair in each month.
- Let's assume that f_n is the number of pairs of rabbits present after n months, for n = 1, 2, 3, …. so on.

  f_1 = f(1) = Number of pair of rabbits after 1 month = 1.

  f_2 = f(2) = Number of pair of rabbits after 2 months = 1.

  f_3 = f(3) = Number of pair of rabbits after 3 months = 1+1 = 2, and so on.

- Hence, the mathematical model (linear difference equation) will of the form:

$$f(n) = f(n-1) + f(n-2), \text{ for } n \geq 3. \qquad \text{…. 2.1}$$

So if we'll just go for like Fibonacci Rabbit model, consider the problem of rabbit generation which was originally postulated by Fibonacci, so that's why it is called Fibonacci rabbit model, so if you just consider a pair of rabbits one of each sex is placed on an Iceland, a pair of rabbit does not breed until they are 2 months old, when they are 2 months old each pair will produce another pair in each month.

Let us assume that F function F(n) is the number of pairs of rabbits present after n months, for n equals to suppose, since we are just starting like initial conditions 1, 2, 3, and so on we can just consider. For the first month if you just see F1, it is just given as F(1) here that is the number of pair of rabbits after 1 month, so this will be 1 here. So F2 means number of pair of rabbits after 2 months, this will also be 1, since after 2 months they can just give birth of like 1 pair. So F3, that is nothing but F(3) here, number of pair of rabbits after 3 months, if you just see first one pair was there they have given like one pair, so the total pair is 2 here, and after this it will just continue it like the next birth baby is, they can be, like given birth after 2 months again. Hence Fibonacci developed a mathematical model considering this linear difference equation which is in the firm like f(n) = f(n-1) + f(n-2), for n greater or equal to 3.

Since it is a linear difference equation with order 2 here, so we will have like at best 2 initial conditions it is required, so to deal with that equation or such type of equation especially we'll just go for this linear homogenous difference equation solution form.

Linear Homogeneous Difference Equation:

- The general form of linear homogenous difference equation of order 'k' with constant coefficients is:

$$A_n = c_1 A_{n-1} + c_2 A_{n-2} + c_3 A_{n-3} + \ldots\ldots\ldots + c_k A_{n-k} \qquad \ldots.\ 2.2$$

- Since it's of degree k, the same number of initial conditions are required. Let's say $A_0 = a_0; A_1 = a_1 \ldots\ldots A_{k-1} = a_{k-1}$.

- **Procedure to solve above type of equation:**

1. Assume that $A_n = m^n$ is the solution of the equation and substitute this in equation 2.2.

2. This will get simplified in a polynomial of degree k. This is also termed as auxiliary equation. The roots of auxiliary equation are called auxiliary roots and these roots help in getting explicit form of solution.

So the general form of this linear homogenous difference equation of degree K with constant coefficients is written as An = c1 An-1 + c2 An-2 + c3 An-3 + ck An-k, where c1, c2, c3 up to ck are constants here, since it is a k-th order difference equation, the same number of initial conditions it is required, let us say A0 = small a0 here, A1 = small a1 here, Ak-1 = small ak-1, so if you just go for this solution of this homogenous difference equation, first we will just substitute An = m to the power n here, and if you just substitute this in equation 1, this will just simplified to polynomial of degree K here, this is also termed as auxiliary equation. The roots of these auxiliary equations are called auxiliary roots and these roots helps in getting explicit form of the solution.

Linear Homogeneous Difference Equation:

3. If all the roots of characteristic polynomial are real and distinct then, solution will be in the form of

$$A_n = b_1\alpha_1^n + b_2\alpha_2^n + \ldots.. + b_k\alpha_k^n \qquad \ldots.\ 2.3$$

where α_i 's are roots of poly. and b_i 's are obtained after substituting those k initial conditions for i = 1, 2,…,k.

4. If any one of the roots, say α_1 has multiplicity m then solution will be in the form of:

$$A_n = b_1\alpha_1^n + nb_2\alpha_1^n + \ldots + n^{m-1} b_m\alpha_1^n + b_{m+1}\alpha_2^n + \ldots + b_{m+k-1}\alpha_{k-m-1}^n \qquad \ldots.\ 2.4$$

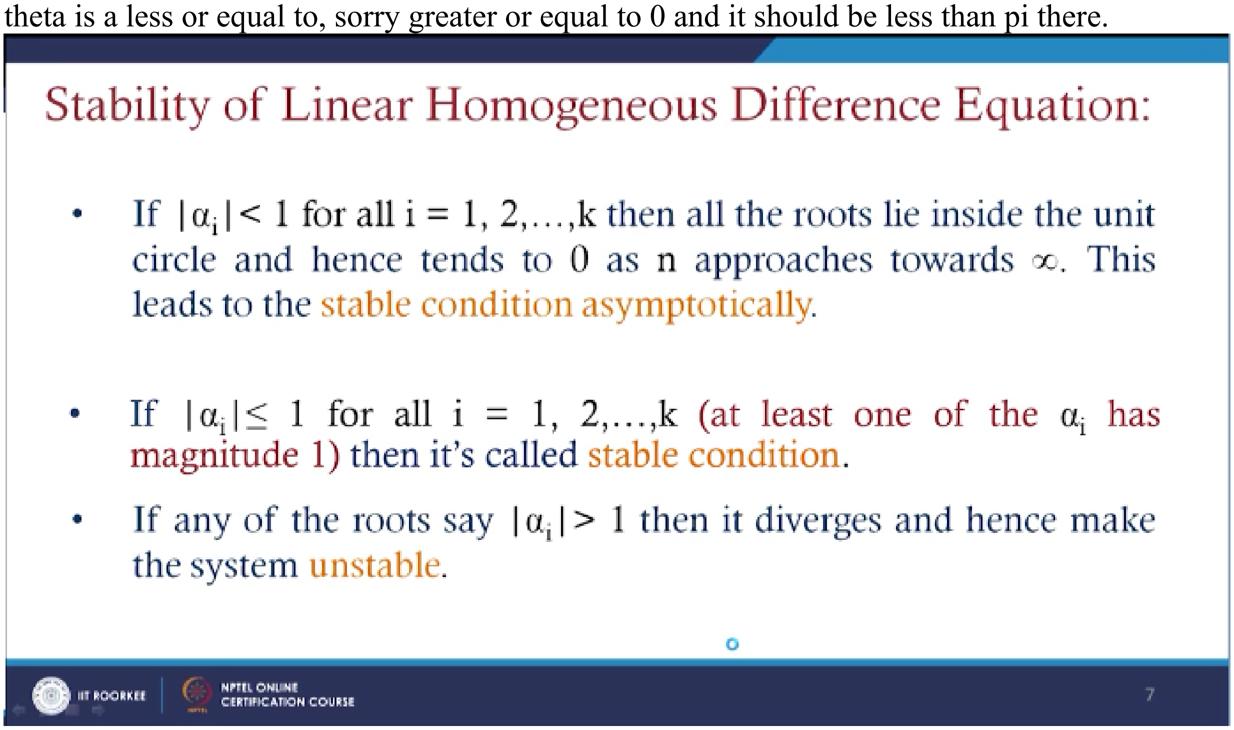
5. If the roots are in complex conjugate pairs say $\alpha = u \pm iv$ then this will lead to the solution in the form of (for order 2)

$$b_1 r^n \sin(\theta n) + b_2 r^n \cos(\theta n), \qquad \ldots.\ 2.5$$

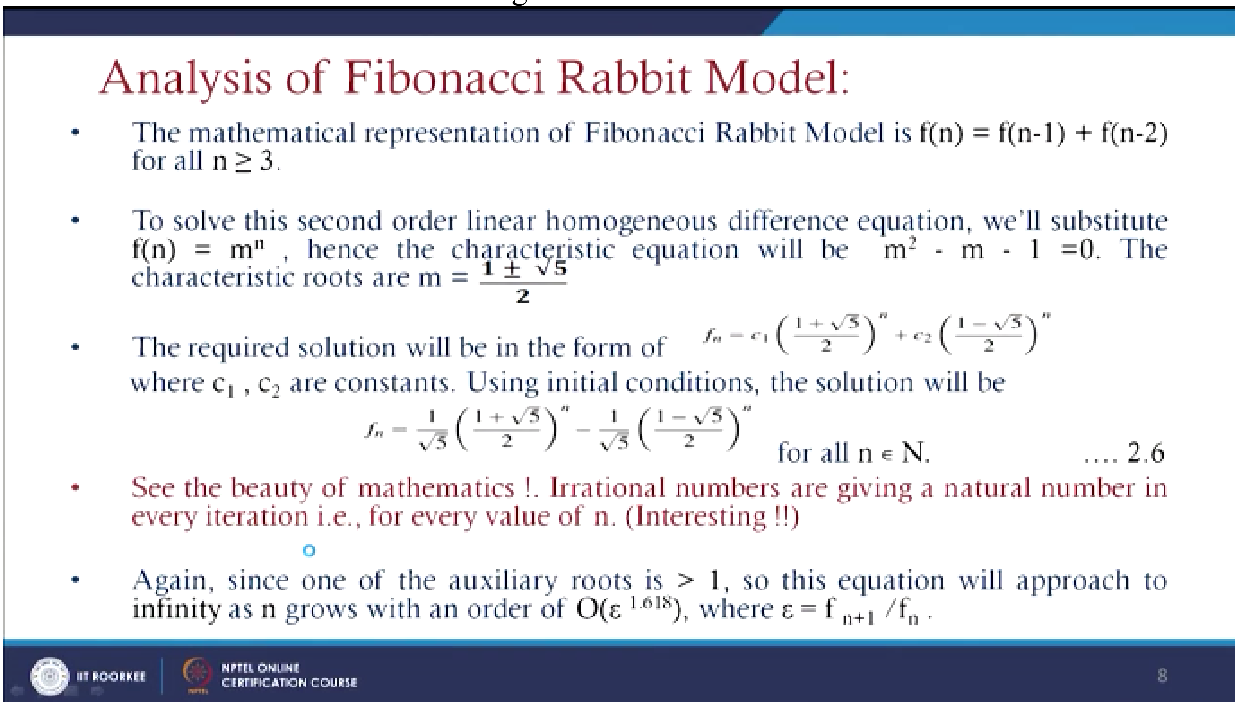
where $r^2 = u^2 + v^2$ and $\tan(\theta) = v/u$ for $0 \le \theta < \pi$.

So if then we will have this roots of this characteristics polynomial are real and distinct, then the solution will be in the form of An = B1 alpha 1 to the power n, B2 alpha 2 to the power n, up to Bk alpha K to the power n, since it is solution it is assumed in the form of like M to the power N, since we will have this distinct roots like alpha 1, alpha 2 up to alpha k so that's why in place of M we can just put alpha 1, alpha 2 up to alpha K, we will have this with the solution, where aI's are the roots of this polynomial and Bi's are obtained after substituting those k initial conditions, since they are like constant coefficients so it can be obtained if we'll just put the initial conditions for I = 1, 2 up to k there.

If any one of the roots say alpha 1 has multiplicity m, sometimes if double root is existing both the roots are same, then in that case the complete solution or this auxiliary solution of this equation will be in the form of An = B1, alpha 1 to the power n, so simply we can just multiply n here, nb2 alpha 1 to the power n up to, since the number of roots if you'll just consider here multiplicity m here then we will have n to the power m-1, bm alpha 1 to the power n + bm+1 alpha 2 to the power n it is the next root, immediate root, since we will have like alpha 1 has the multiplicity m then other roots are also existing there, so that's why other roots it will be written in the form that we have just explained in the earlier solution. If the roots are in complex conjugate pairs, say alpha = U plus or minus IV then this will lead to the solution in the form of b1 r to the power n sine n theta + b2 r to the power n cos n theta, where r square can be written as u square + v square and tan theta can be written as v/u for theta lies between 0 and pi here, so theta is a less or equal to, sorry greater or equal to 0 and it should be less than pi there.



So if we'll have the system then we will just go for this stability condition of this linear homogeneous difference equation, in the first lecture I have just told that if you will have a system then the systems would be stable and the system would be consistent, so once we are just getting this solutions whether this solution representing a stable solution or not that we will just go to test. If alpha I is less than 1, for all I = 1, 2 up to K, then all the roots lie inside the unit circle and hence tends to 0, as n approaches towards infinity, this leads to a stable condition asymptotically, if you just see here suppose if you'll have a circle here then this condition is just coming like this form here towards the origin, so that's why we are just saying it is a asymptotically stable. And if alpha I less or equal to 1 at least one of the alpha is 1 there for all I = 1, 2 up to K then it is called stable condition, this means that you will have a like constant value there and we will have like a straight line in that sense.



If anyone of the roots say alpha I greater than 1 then it diverges and hence make the system unstable here, and if we will just go to, go for the analysis of this Fibonacci rabbit model here then this mathematical representation of this Fibonacci rabbit model is f(n) = f(n-1) + f(n-2) for all n greater or equal to 3, and to solve this suppose if you'll just put here f(n) = m to the power n, hence we will have this characteristic equation that is in the form of m square – m – 1 here, this equals to 0, since directly if you'll just put here f(n) = m to the power n in this equation here then we will have this equation there. And the root for this quadratic equation will be in the form of m = 1 + or – root 5/2, and the required solution whatever we have just explained in the previous slide that can be represented as f(n) = c1 since constant it will be there 1 + root 5/2 whole power n + c2 1 – root 5/2 whole power n here, and if you just use the initial conditions, the solution will be like f(n) = 1/ root 5, 1 + root 5/2 whole to the power n – 1/root 5 1-root 5/2 whole to the power n here.

And if you'll just see here that for all m belongs to n we will have a natural number here, since in this equation if you'll just see all the irrational numbers are involved and in analysis always we can just find that when the irrational numbers are involving, any expression then always they are producing a irrational number, but the beauty of the equation is that you can have a natural number when you can just assume any natural number which is in the form of n here, so again since one of this auxiliary roots is greater than 1, so this equation will approach to infinity as n grows with order of like epsilon to the power 1.618 which is nothing but 1 + root 5/2 value here that is especially called golden ratio, where epsilon is defined as Fn+1/Fn here.

Linear Cell Division Model:

- Assume that a cell division takes place concurrently and at each time, cell is producing α numbers of daughter cells.
- Number or density of cells at the beginning of cell division process is C_0.
- After n generations, number of cells shall be C_n , given as

$$C_n = \alpha C_{n-1} \text{ where } n \in N \qquad .... 2.7$$

n = generation number.

- This model is in the form of simple linear difference equation and also known as Malthus model named after famous English Economist Thomas Robert Malthus.

Then we will just go for linear cell division model, assume that a cell division takes place concurrently and at each time cell is producing suppose alpha number of daughter cells, so in that sense first we will have like big cell then this division takes place, so then we will have like different cell size or different distribution of cells, and the number or density of cells had beginning of cell division processes suppose C0 here, after n generations number of cells can be CN suppose given as CN = alpha Cn-1 where n belongs to N, n = generation number here.

Linear Cell Division Model:

- Now assume that, cells are producing themselves at a rate of b and simultaneously cells are extincting at a rate of d.
- If cell population is changing only due to births and deaths of cells i.e. without any migration effect, then $C_n - C_{n-1}$ gives the difference of number of births and deaths over the interval t_{n-1} to t_n . If b and d are constants then, $C_n - C_{n-1} = (b - d) C_{n-1}$, or

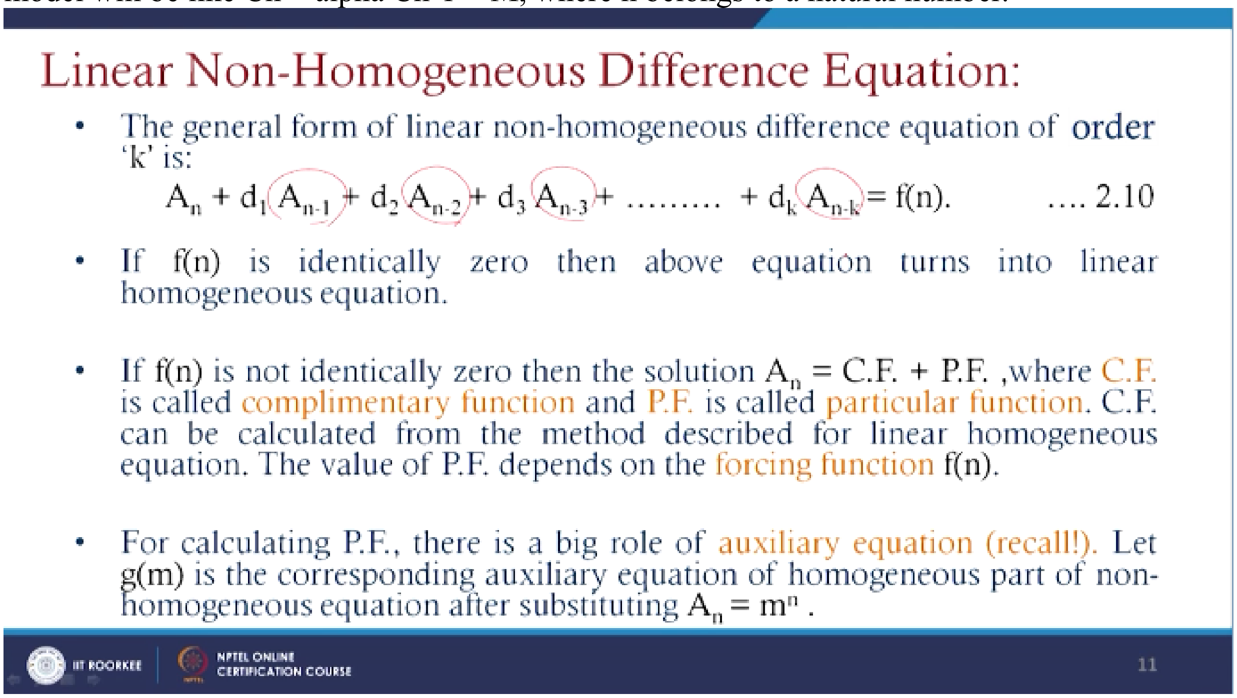
$$C_n = (1 + b - d) C_{n-1} = \alpha^n C_0 \text{ where } n \in N, \text{ (for } \alpha = 1 + b - d \text{)}. \qquad .... 2.8$$

- If further migration of cells is allowed with a constant rate of migration M (M>0 for immigration and M<0 for emigration), the evolved model will be:

$$C_n = \alpha C_{n-1} + M, \text{ where } n \in N. \qquad .... 2.9$$

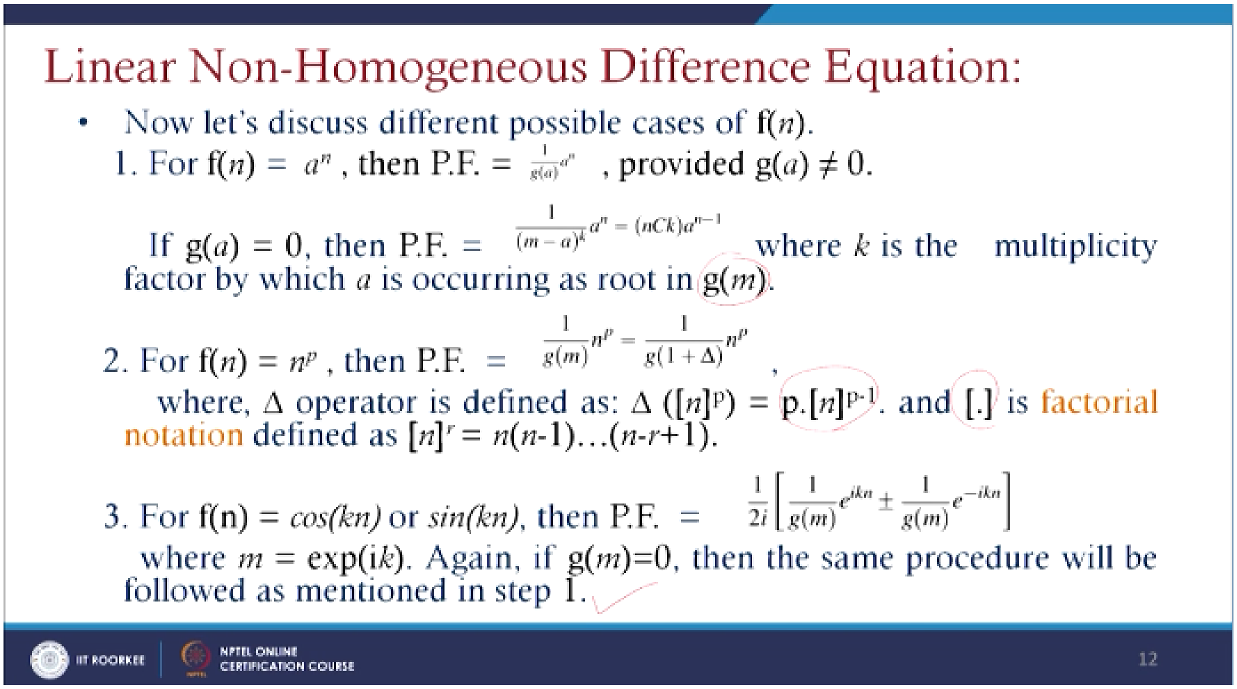
- This equation is in the form of linear non-homogeneous difference equation. **Stability** of the same can be analyzed from the previous analysis.

This model is in the form of simple linear difference equation and also known as Malthus model here, and this model is named after famous English economist Thomas Robert Malthus developed this model, so now in this model if you'll assume that cells are producing themselves at rate of B and simultaneously cells are extincting at a rate of D suppose, the cell population is changing only due to births and deaths of cells, that is without any migration effect then we can just write Cn – Cn-1 gives the difference of numbers of births and deaths over the interval tn-1 to tn there, if b and d are constants then Cn - Cn-1 it can be represented as b-d Cn-1 here or Cn can be written as 1 + b-d whole into Cn-1 which can be written as alpha to the power n C0 where n belongs to N, if alpha equals to 1 + b – d here, if further migration of cells is allowed with a constant rate of migration suppose M here, M greater than 0, so especially we are just considering M greater than 0 for migration and M less than 0 for immigration here, then evolve model will be like Cn = alpha Cn-1 + M, where n belongs to a natural number.



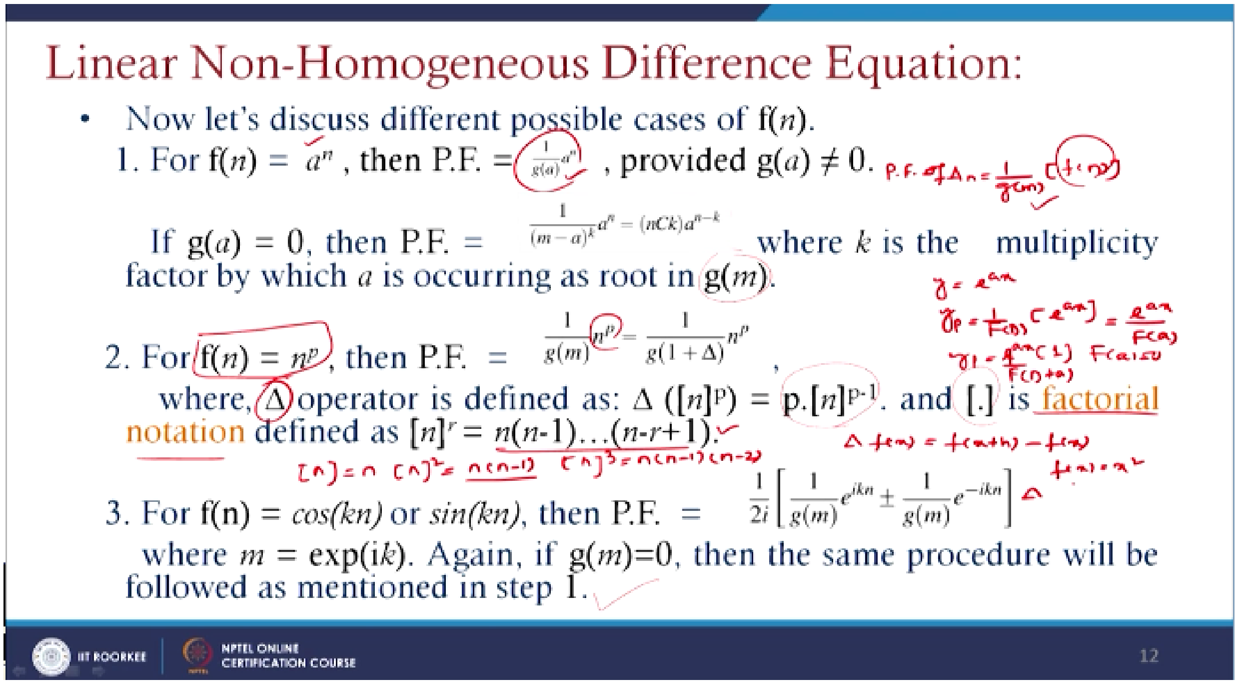
This equation is in the form of linear non-homogenous difference equation, and stability of the same can be analyzed from the previous analysis, so if you'll just go for this linear non-homogenous difference equations then the general form of this linear non-homogeneous difference equation of degree k can be written as An + d1 An-1, d2 An-2, d3 An-3, dk An-k this equals to Fn suppose, if Fn is identically 0 then above equation turns into a linear homogeneous equation.

If Fn is not identically 0 then the solution of An can be defined as complementary function + particular function here, so where the CF represents here complimentary function and PF is particular function here, and this complementary function can be calculated from the method described for linear homogeneous equation, but for the particular function it can the value of this function depends on the forcing function Fn there, since a Fn represents the right hand side of this equation here, for calculating particular function there is a big role of auxiliary equation, so auxiliary equation already you have known that one, let g(m) is the corresponding auxiliary equation of homogeneous part of non-homogeneous equation after substituting An = m to the power n, especially whenever we are just discussing this linear non-homogeneous difference



equations that defers from this like continuous difference equations, that is continuous difference equations means we can just say that it is a differential equations and in case of this difference equations if you'll just here that we will have two types of solutions, that is one is called your complementary function, another one it is called particular function here, and for the complementary function especially we are just putting that variable, it is in the form of like M to the power N, then we are just determining these roots if you just see this slides here that if we will have like a polynomial that is for order n here, that is An + d1 An-1, D2 An-2, D3 An-3 up to Dk An-K this equals to Fn.

Then when we will just go for the solution of this difference equation we will have two parts here, one it is called complementary function part, another one it is called particular function part here, so if you'll just find this complementary function then we have to put An = M to the power N here, and if you just put An = M to the power N then we can just reduce this polynomial that is in the form of like a M to the power N + D1 then we will have M to the power n-1 d2 M to the power n-2 + up to dk m to the power n-k, this equals to 0 here, and if you just take common here that is m to the power n-k which is a lowest order term of m here then we can just write this one as a m to the power k + d1 m to the power k-1 up to dk this equals to 0 here.



And if you'll just see here then m to the power n-k which is not equal to 0 then we can just consider this polynomial of degree k, that is a specially we are just writing it as here as a g(m) this equals to 0 here, and if you'll just find the root of this equation g(m) which will provide the characteristics roots for this difference equation, and if we'll just evaluate this characteristic roots then we can just write this characteristic roots as in the form of like a complementary function which takes the values are in the form of like, if you'll just write here then our solution it is written in the form of An = m to the power n, so that's why whatever this order of this equation it will just involve, the same number of arbitrary constants it will involve for this complementary function also, so that's why this solutions can be written in the form of like complementary function which can be written as if you'll just see here that is alpha 1, you're variable that is in the form of m1 here, so m1 to the power m alpha 2 m2 to the power n so likewise we can just write since it is a polynomial of a degree k here so alpha k, mk to the power n is the complementary function.

And if you'll just go for this particular function here, then this particular function can taken as like, since our earlier value if you just see this difference equation that has been reduced in the form of g(m) and if you'll just write this particular function for this complete difference equation that is in the form of this one here, then this variable An can be written as 1/g(m) operated on this right hand part that as Fn here.

And if you'll just operate this one this is nothing but the particular function of An here, so especially whenever we are just operating this 1/g(m) function with the Fn we will have like different conditions here, for the first case if you'll just choose here Fn = A to the power n then like our continuous case whenever we are just using this differential equations then especially we are just assuming this generalized solution that is in the form of Y = E to the power X, then we are just writing this particular function, if the function is in the form of like YP then especially if 1/F(d) if it is operated on E to the power X we have just replaced this one by the form of E to the power X/F(a).

Similarly in case of discrete values that is in case of ordinary difference equation if you'll just apply here F(n) as A to the power M suppose then we can just write this particular function as 1/g(a) A to the power N, but the question arise is that if G(a) = 0 here, if G(a) is not equal to 0 then directly we can just write 1/g(a) A to the power N, since it is a geometric series here, but if g(a) = 0 then this particular function can be written as 1/m-a whole to the power k, a to the power n here, especially if you'll just go for this like our continuous case that is in a different ordinary differential equation case there also if we are just writing this F(a) = 0 there then we have just replaced YP as in the form of 1/F(D + A) and which is operated on E to the power X, so E to the power AX operated on 1 here, that is your XB of X1 so.

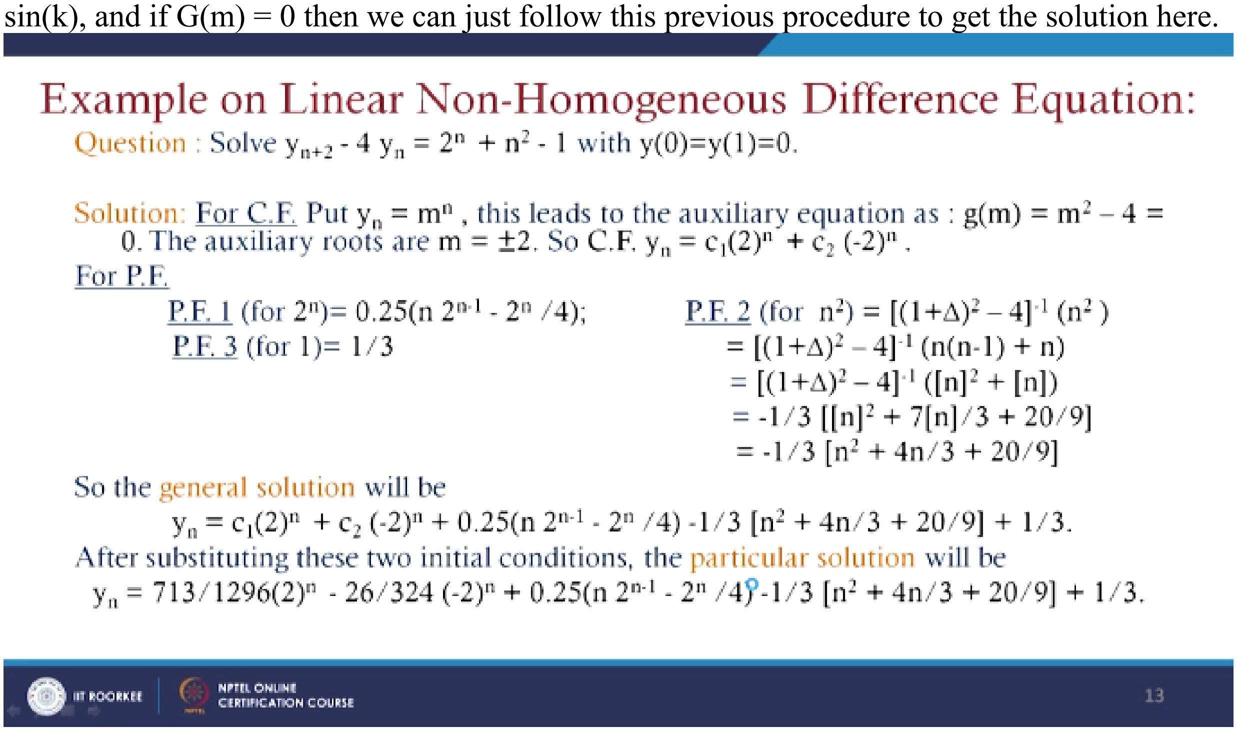
Similarly if you just go for this geometric expression here then we can just write this particular function as 1/m-a whole to the power K since it is generated as polynomial of degree K that is G(m) and which operated on this function A to the power N, and if you just see here this represents a, like a combination series that is in the form of NCK A to the power n-1, since this is a geometric series up to n-1 terms we can just consider that is in the form of K here, that's why it has been written in the form of first term if you just see it can just take as the constant term, then simultaneously it will just increase, likewise if you'll just see like sine X suppose, sine X means it has Taylor's first term it just takes a constant value there 1, then especially it can just incremented after wards, so if you'll just take the geometry expansion of this function which involves this power function here and which has this denominator parts takes the 0 value then we can just write this particular function as NCK A to the power n-1.

And similarly the next case is that whenever we'll have this function that is in the form of like F(n) = n to the power p here, and if you'll just put this F(n) = n to the power p here then this particular function can be written as 1/G(m) so whatever it is, I just written there, then it is operated on F(n) function that is as n to the power p here, so p is like an arbitrary constant here, and if we'll just write in this form here then we have to use certain difference operators to get this particular function here, the difference operator that is used for this class of problem is that forward difference operator here, so forward difference operator if it is operated on this function that is a bracketed function we have just define, this bracketed function is defined in the form of like bracketed of n, it can just take the value as n here, and if we are just taking this bracketed function that is n square here which can just take the value as n(n-1), if you'll just take this bracketed function of NQ then it can be written as n(n-1) (n-2). Especially if you'll just see this bracketed function here then whatever this power or order it is just written here that takes the number of terms before that, and it is just tagged as a product form there, so that's why it is just represented as a factorial notation.

Officially if you'll just take this power of this bracketed N as R here, then especially it will involve R terms so that's why we have just written as the product of n(n-1) to n-R + 1 terms here. And we have known that whenever this difference operator is operated that is like forward difference operator, so if we are just assuming a function F(x) which is written in the form of F(x+h) – F(x) this is the forward difference operator formula, and if we just assume suppose F(x) is a polynomial suppose like X square, then if we just use this difference operator here that is with space size suppose A is here, then we can just write del(x square) which can be written as x+h whole square – x square which can be written as X square + 2xh + h square – x square here. And if you just see here so x square, x square it will just cancel it out and we can just write here that is in the form of 2xh + hx square here.

And next time if you just apply this del operator here, so if del will operated on a constant function that will just provide you like another if you just apply del square of x square here, then this will just provide you del(2xh) + del(h square), so h square since h is a constant here, so h square – h square this will just to give you 0 value and especially if you just use for this function here then we can just write here as 2(x), so that's why x+h –h, since only here if I'll just see here only x it is there, so if I'll just use this function F(x) = x only, so first value it can be written as F(x+h) so that's why x+h minus this is F(x) as x here, so x, x cancel it out so it will just provide you 2h square, so if you'll just keep this space size x =1 suppose, then we can just write del square of x square, this equals to 2.

Similarly like derivative here, so if same way we can just use this del operator for this function here, n to the power p we can just write this one as p n to the power p-1, since we have just seen that if the del operator, if it is used for x square here so then if you just consider this space size is equal to 1 here especially, this can be written in the form of like 2x+1 here, and the first term if you just see that is nothing but 2x it is there, so that's why this difference operator if you just operated here that can be written as p n to the power p-1 here, and if you just consider suppose this F(n) as like a trigonometry function here, trigonometry function means if this roots are coming in the imaginary form then we can just write this functions are in the form of cos(x+I) sin(x), so that's why this function if we can just write as cos(x) or sin(x) here that is in terms of n if you just write, cos(kn) and sin(kn) then this particular function can be written as 1/2I into 1/g(m) E to the power IKN + or – 1/g(m) E to the power –IKN, where m = especially it can be written as exponential (ik), since exponential (ik) means we can just write this as cos(k + I) sin(k), and if G(m) = 0 then we can just follow this previous procedure to get the solution here.



Based on different cases of like linear homogeneous difference equations we have consider one example where we have consider this equations as yn+2 – 4 yn this equals to 2 to the power n +

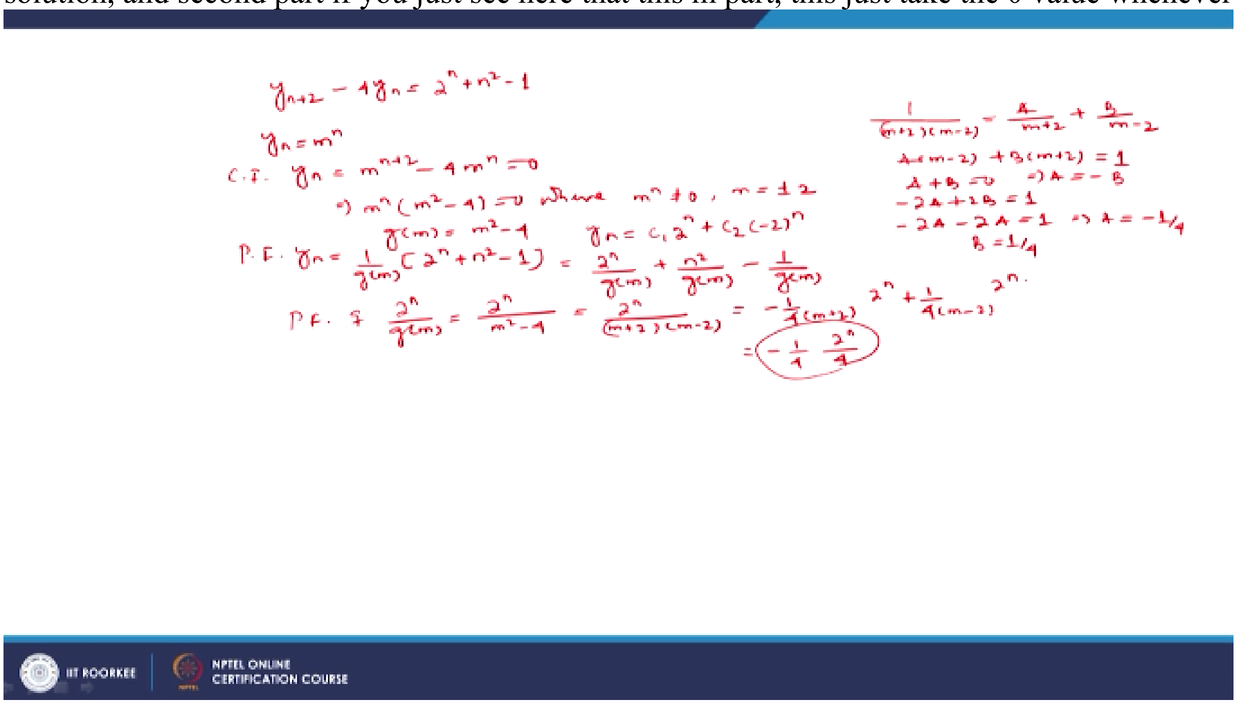
n square -1 with the boundary conditions as y(0) = y(1) = 0 here, especially if we are just going for this solution of this linear homogenous difference equations we have to consider one complementary function and one particular function, and the total solution it will be like some of this complementary function and particular function. In case of like ordinary differential equations that is in a continuous case, here we are just considering this discrete cases, so in continuous case we have just assume this function as y= e to the power mx to find this complementary functions, and here we are just using one geometric expansion that is in the form of like yn = m to the power n to get this complementary function here.

So if you just put suppose yn = m to the power n in this equation, so we have this equation in the form of yn+2, that is -4yn these equals to 2 to the power n + n square – 1 here, and directly if you just put here that is yn = m to the power n as a geometric expansion here then for the complementary function we have to keep it as like complementary function of yn this can be written as like m to the power n+2 – 4 m to the power n this equals to 0 here, this implies that m to the power n into m to the power 2 - 4 this equals to 0, where if you just see m to the power n this is not equals to 0, so that's why we can have like m = + or – 2 here, and if we just consider m = + or – 2 then we will have two real and distinct roots here, and if you just see here the polynomial that is expressed in the form of G(m) that x as m square – 4 here, since G(m) is a polynomial for degree 2, the solution for this complementary function that will involve 2 arbitrary constants, hence the solution for this complementary function yn can be written as like c1 m to the power n so that's why we can just write 2 to the power n + C2 (-2 to the power n) is the complementary function here.

So if you'll just see here that we will have this auxiliary roots that is m = + or – 2 here, so the complementary function for yn it is expression in the form of c1(2) to the power n + c2 (-2) to the power n here, then we will just go for this particular function, so if we will just go for this particular function here then we have to consider like the particular function of yn which can be expressed in the form of like 1/g(m) 2 to the power n + n square -1, and this can be written in the form of like 2 to the power n/g(m) + n square/g(m) – 1/g(m) here, and already we have discussed that if any function that is in the form of A to the power n/g(m) is existing then directly if G(a) is not equals to 0 there, then we can just write that function as A to the power N/G(a) if you'll just see this previous slide here that you can just find that particular function of A(n) which can be expressed as 1/G(a) A to the power n if G(a) is not equals to 0 here, so that's why if you'll just see here the particular function for 2 to the power n here then we can just find that, so 2 to the power n directly if you'll just write here particular function of 2 to the power n/g(m) this can be written as 2 to the power n/g(m), so g(m) is nothing but m square - 4 here, so this can be written as 2 to the power n/m+2 into m-2 here.

So first we will just do this partial fraction to get the independent form of this m+2 and m-2 in the, like denominator side here, so that's why if you'll just use the partial fraction to get this division form of this m+2 and m-2 we can just write this one as 1/m+2 into m-2 this equals to A/m+2 + B/m-2 here, and if you'll just try to find the solution here then we can just write here that is A into m-2 + B into m+2 this can be written as 1 here, if you just take this coefficients of m which can be written as A+B = 0, then we will have like -2A +2B this equals to 1 here.

From the first equation if you'll just see this implies that A = -B here, so if I'll just replace here directly -2A, -2A this equals to 1 this implies that A = -1/4 here, and since we have written here A = -B so that's why B can be written as 1/4 here, and finally we can just write this coefficient as in the form of like -1/4 since A/m+2, 2 to the power N + 1/4 this one is your m-2, 2 to the power n here, and already we have seen that in case of like 1/m-A, sorry this is G(a) is not equals to 0 we can just write that one as 1/G(a), A to the power n, so that if you just see from this slide here that 2 to the power n/m+2 here so directly if you just replace here m/2 then directly we can just write this one as -1/4 2 to the power n/4 here, this is the first part of the solution, and second part if you just see here that this m part, this just take the 0 value whenever



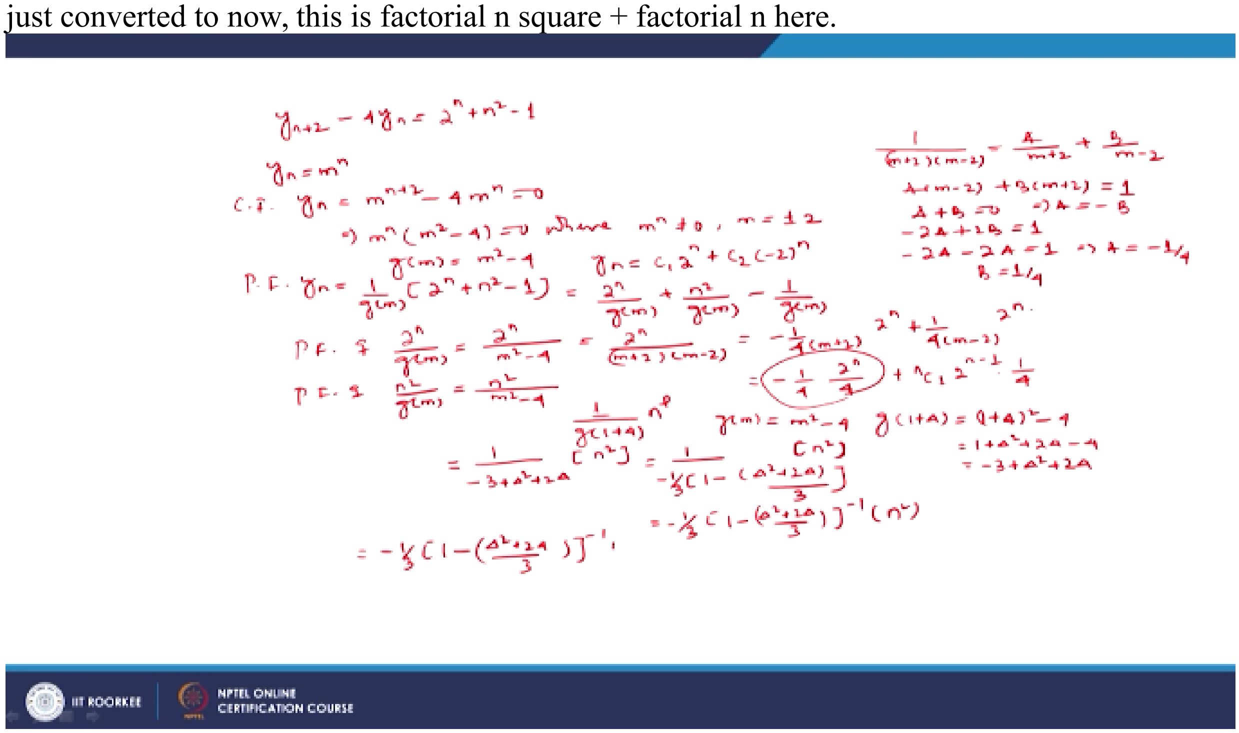
we are just putting directly m = 2 here, so for this we will just use the other formula that is in the form of like if you just see here that is 1/m-a whole to the power k, a to the power n this can be written as NCK A to the power n-1, so that's why if you just see here that K is 1 that is the first root since we are just writing their way m-2 only, so K is 1 so we can just write that one as NC1 A to the power n-1 is the particular function for that part, so that's why we can just write this one as plus, we'll have NC1 and then second is 2 to the power n-1 into 1/4 whatever it is just present there.

So finally if you just see this particular function for 2 to the power n/g(m) so that can be written as like here that is 0.25 into n 2 to the power n-1 -2 to the power n/4 here, so that we have just derived there.

Next part is that if you just see here, first part it is over, so next part particular function of, we'll have m square/g(m), so if you just see here this is written in the form of m square/g(m) is m square – 4, and which is nothing but if you just see our previous slide here, then this is just taking in the form if F(n) = n to the power p then the particular function of 1/g(m) n to the power p which can be written as 1/g(1 + delta) into n to the power p, so directly we can just write there as 1/g(1+delta) n to the power p, so if you just see here we will have, g is written as in the form of m square – 4, so that's why G(1+delta) it can be written as 1 + delta whole square – 4.

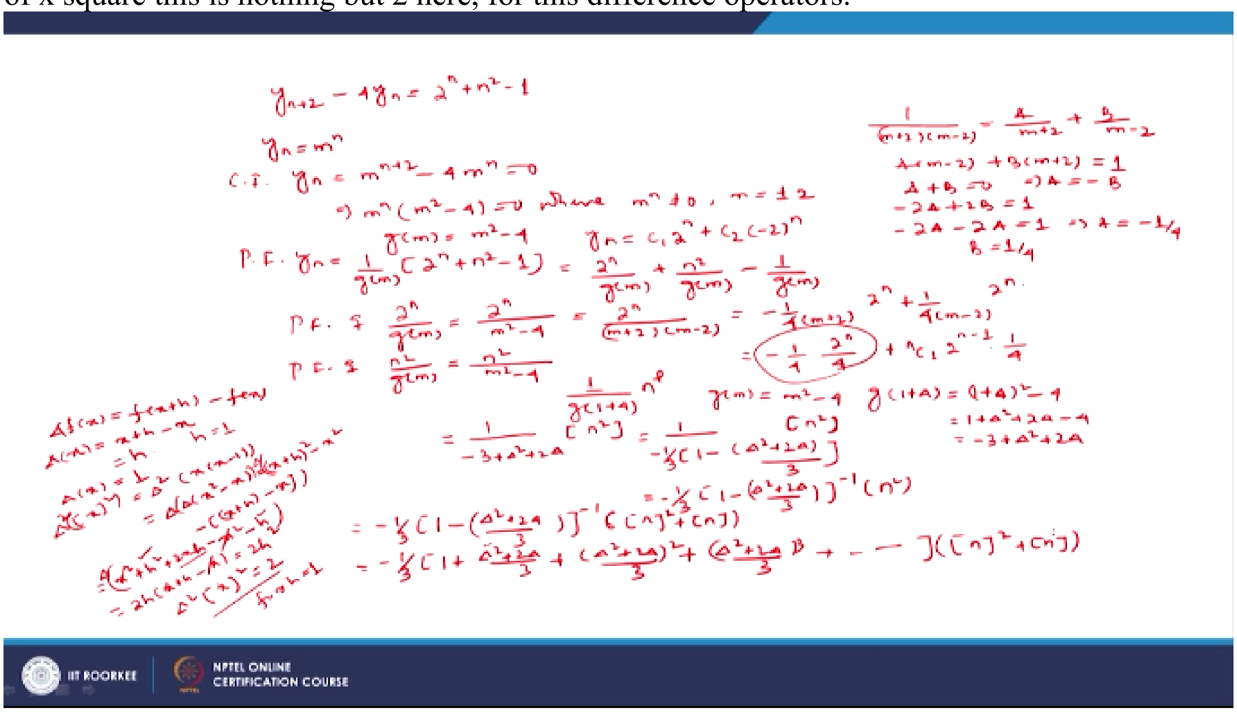
So which can be written in the form of like 1 + delta square + 2 delta – 4 here, and which can be written as -3 + delta square + 2 delta here, and directly if you'll just write here the particular function of this one which can be written as, so you'll have like denominator side that is 1/-3 + delta square + 2 delta operated on n square here, so since we cannot operate this operator in, like if you'll just keep this one in a denominator part we cannot operate that one, so that's why we have just kept it into the numerator part here, so that's why to get it in numerator part we can just write this one as like if you just take common here -1/3, so -1/3 if you just take common so then we can just write this one 1- del square + 2 delta/3 here, operated on n square, and which can be written as, so -1/3 so this is 1-delta square + 2 delta/3 whole inverse operated on n square here, so already we have known also that if we are just using this operators, delta operators on this factorial form then it is easy to handle, so that's why our first aim is that we have to convert this n square into the factorial part here, so if you just see here first we have just tried to convert this n square into the factorial part, so that's why we are just writing this n square as in the form of n(n-1) + n here, so if you'll just see this part here then it can be written as n square – n + n which can be written as nothing but n square here.

And if we want to write this n square in factorial form then this factorial m square which can be written as n into n-1, and factorial of m it can be written as n only, so that's why we can just write or shift this n square value as in the form of n into n-1 + n here, so that's why we can just write this one as bracketed n square + bracketed n here, and if you just see here we have already taken common if you just see here that is -1/3 1 – delta square + 2 delta/3 whole inverse, so it is just converted to now, this is factorial n square + factorial n here.



And now if you just take this expansion here so then you can just find that this expansion can be written as -1/3 so take this expansion 1 + delta square + 2 delta/3 + delta square + 2 delta/3 square + delta square + 2 delta whole cube, so likewise we can just do this expansion operated on bracketed n square + bracketed n here, and already we have known that this delta operator that is nothing but our forward difference operator only, so delta of F(x) which can be written as F(x+h) – F(x), so if delta is operated on a polynomial form here, so suppose if you'll just operated delta on x only which can be written as x+h – x so which can be written as only x here, h here, so if h = 1 here then we can just write delta(x) which is nothing but 1 here.

And similarly if since we are just operating this delta function on this factorial forms here, so if suppose we are just operating delta square on suppose this bracketed x square here, so bracketed x square means it can be written as delta square x into x-1 here, which can be written as delta square, x square – x here, and which can especially written as x+h whole square – x square – x+h – x here, and if you'll just expand this x+h square here which can be written as x square + h square + 2xh – x square, so x, x square it will be cancel it out, minus this will be like x – x it will just cancel it out, so –h it will be there. So we will have like, first delta we are just operating so that's why we can just write delta of this one, delta of this one only, so that's why first delta it has been operated, if you just operate the second delta then we can just find that h square means it is a constant, so they will take the 0 value, so if you do operate on h it will also take the 0 value, so next if you'll just operate on this value here, so it will just to take 2h into x+h – x here, so it will just provide you 2h square, and if you just put here h = 1, so delta square of x square this is nothing but 2 here, for this difference operators.

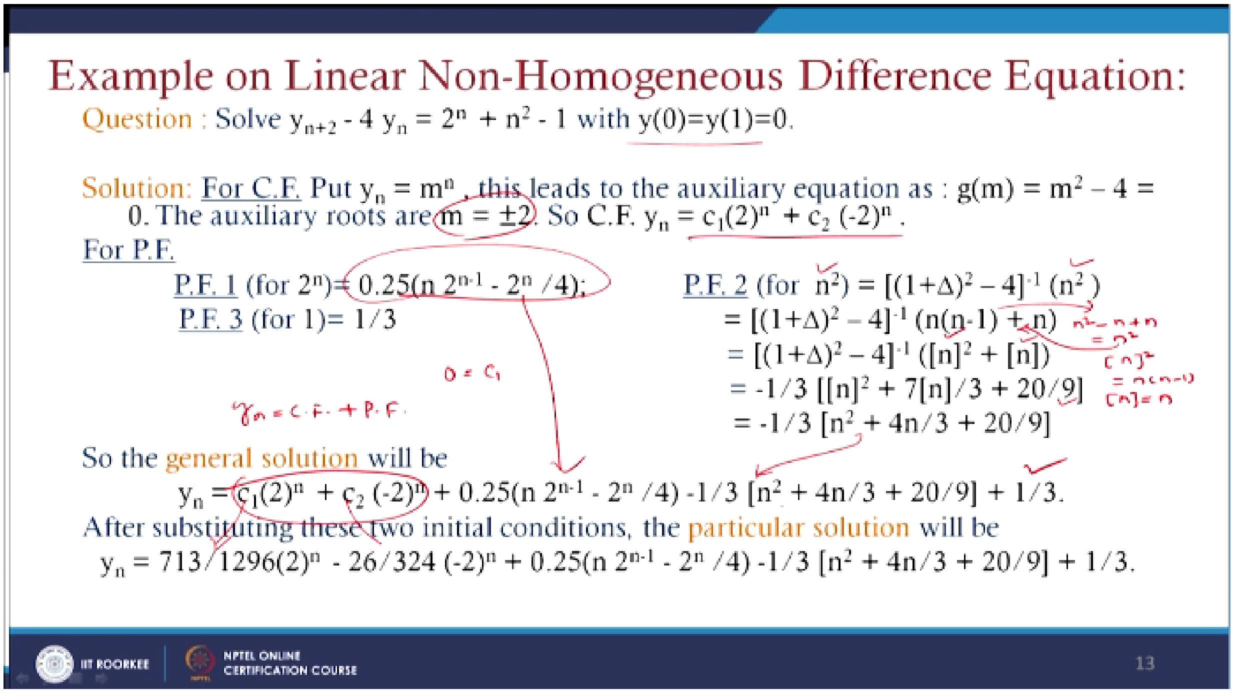


So similarly if you'll just see here delta square if it is operated on n square this will just giving you a constant here, and similarly if 2 delta it will be operated on n here so it will just give you a constant value, but if delta square it will operated on n here so that will just give you a 0 value here, so if you'll just eliminate all this operators so then you can just find this operated value as bracketed n square + 7n/3 + 20/9 here.

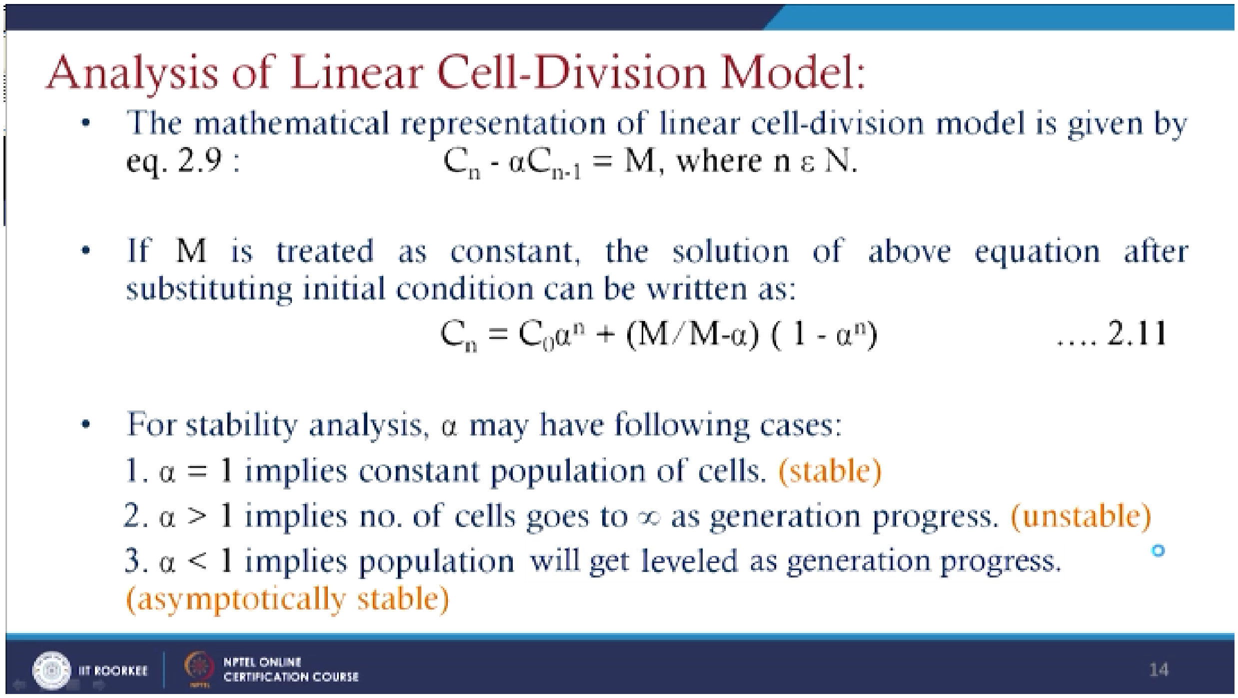
And finally if you just multiply this -1/3 into bracketed n square this one, so bracketed n square means we can just write this one is n(n-1) + 7 bracketed n means this is 7n/3 + 20/9, so it will just take the value as -1/3 into n square + 4n/3 + 20/9, and this one usually you can just visualize from here also, that is first value if you'll just write here that is -1/3 bracketed n square + bracketed n, second value if you'll just operate this one delta square n square here, I have explained also this one in the previous slide if you just see here, that is delta (n to the power p) it can be written as pn to the power p-1 here, so same way we can just express also there that is delta of pn which can be written as n(p) n-1, sorry np, this is p(n)p-1, so if you'll just see here delta of n to the power p, directly we can also visualize here that delta square if it is operated on n square here so directly if you just put this one, so it will just give you 2(n), first one, first operator it will just give you 2(n), then the second operator if you'll just do then it will just give you 2, only 2 here.

Similarly if you'll just operate 2 delta on n square or bracketed n square, so it will just give you 2(n) here, similarly if you'll just operate delta square on bracketed n this will just give you 0 and 2/3 delta if you'll operate on n this will just give you 2/3 here, so that's why if you'll just write here so first value it will just give you 2/3, so then 2 delta if you'll just operate so it will just give you 2/3 x 2(n) and immediate next value if you just use so it will just give you 2/3, then the next, immediate next value if you'll just apply here so we will have like del to the power 4 + we will have like operated, if you'll just see here del to the power 4 + we will have 4 del square, then we will have like 4 del q/3 operated on bracketed n square + bracketed n, since after this you can just find all other values are 0, since added or differences it will just give you 0 value there, so from this only you can just find this particular values that n square + 7(n)/3 + 20/9 here, so hence if you just see here 3 particular functions we have just obtained, then this 3 particular functions it can be added together to get this complete particular function for yn here, and the general solution it will be like yn equals to first the complementary function + particular function, so that's why this complementary function it can be written as like c1(2) to the power n + c2(-2) to the power n this part it is just written.

And second part is that is particular function which can be written as like 0.25 n 2 to the power n-1 – 2 to the power n/4, second particular function if you'll just see that is nothing but -1/3 n square + 4n/3 + 20/9. Last one if you just see here that is remained is -1/g(m) here, -1/m square -4 here, and which can be written as like only 1/3, since directly if you'll just put this value that is -1/-1 whole square -4 here, so it can be written as -1, so it will be just taken as, if you'll just see here -3 here so particularly you just take 1/3 value, so 1/3.

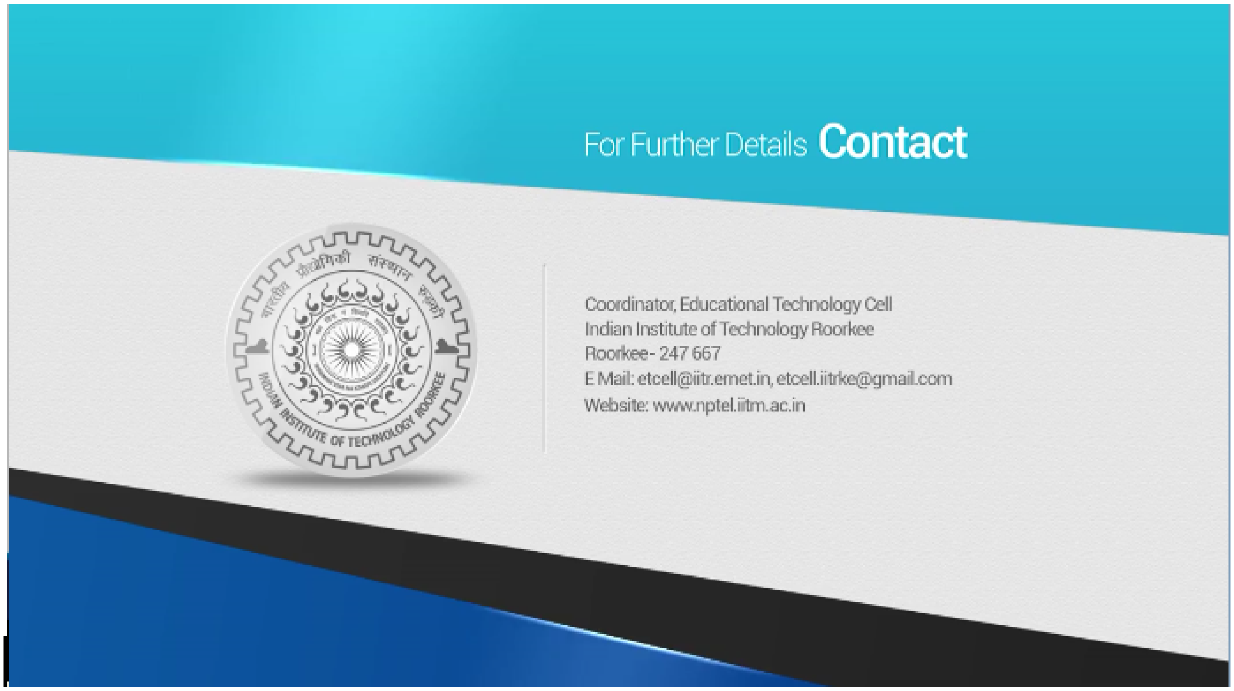


And if you'll just find this complete solution based on this initial conditions, the particular solution will be in the form of, since we have to put this boundary conditions also that is in the form of y0 = y1 = 0, then the total solution it will be in the form of 713/1296 (2) to the power n – 26/324 (-2) to the power n + 0.25(n 2 to the power n-1 – 2 to the power n /4) – 1/3 n square + 4n/3 + 20/9 + 1/3, it is easily obtained since directly if you'll just put this values like yn is given here, so if you'll just put here that y0 = 0 here, so 0 equals to you can just put c1, so first value this is 2 to the power 0 + c2 0 also, so all remaining values if you'll just see here that will just 0.25, so n is 0, so -2 to the power n/4 here, so it can just take the value then -1/3 n = 0, n = 0, then 20/9 + 1/3, so from this equation we can just separate c1 and c2, we will have two equations since next equation we can just put this one as y(1) = 0, so this means that n = 1 will just put and yn total value is 0 here, and we will have like two set of equations and from there itself we can just find c1 and c2 value, so once you are just obtaining this c1 and c2 value directly you can just put this c1 value as this one and c2 value is this one here, so this is the final solution for this problem here.



So if you just go for the analysis of linear cell division model the mathematical model representation of linear cell division model is given as equation 2.9, if you'll just see the previous slide here, that is Cn = alpha Cn-1 + m here. So Cn – alpha Cn-1 this is m for n belongs to n here, if M is treated as constant here, the solution of above equation, after substituting initial condition can be written as Cn = C0 alpha to the power n + M/M-alpha into 1 – alpha to the power n here. And next step is the stability analysis, for stability analysis alpha has the possibilities like alpha = 1, so if we will just consider alpha = 1 this implies that constant population of cells, then we will have a stable condition, if you'll just consider alpha greater than 1 then we will have like no cell, number of cells goes to infinity as generation progress, then we will have a unstable condition, if alpha less than 1 then we will have like the population will extinct at generation progress will have asymptotically stable condition.

So now we will just conclude this session, so first we have discussed the introduction of this population dynamics, then we have just discussed about this Fibonacci rabbit model, and how this formulation has been for trend and how we are just obtaining the solutions, then we have discussed about different class of linear difference equations, so with the constant coefficients and we have also discussed nonlinear class of models, then we have discussed like different linear cell models and their formulations. Thank you for listen this lecture.



For Further Details Contact

Coordinator Educational Technology Cell
Indian Institute of Technology Roorkee
Roorkee – 247 667
E Mail:- etcell@iitr.ernet.in, iitrke@gmail.com
Website: www.nptel.iitm.ac.in

Acknowledgement
Prof. Ajit Kumar Chaturvedi
Director, IIT Roorkee

NPTEL Coordinator
IIT Roorkee
Prof. B. K Gandhi

Subject Expert
Dr. Ameeya Kumar Nayak
Department of Mathematics
IIT Roorkee

Produced by
Sarath K.V

Graphics
Binoy V.P

Production Team
Mohanraj.S
Pankaj Saini

Web Team
Dr. Nibedita Bisoyi
Neetesh Kumar
Jitender Kumar
Vivek Kumar
Dharamveer Singh
Gaurav Kumar

An educational Technology cell
IIT Roorkee Production

WANT TO SEE MORE LIKE THIS
SUBSCRIBE