INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
NPTEL ONLINE CERTIFICATIONS COURSE
Mathematical Modeling;
Analysis and Applications
Lecture-19
Continuous Time
Lotka-Volttera Competition Model
With Dr. Ameeya Kumar Nayak
Department of Mathematical
Indian Institute of technology Roorkee

Welcome to the lecture series on Mathematical Modeling and analysis in the last lecture we have discussed the phase Diagrams and Null-cline functions, for a non linear differential equations' and in the present lecture we will discussed about continuous time Lotka volterra competitions model, so especially in this model, we will just use like two phase which can just compute each other for the resources food or like the same species or like other phase can be present there over for like they can be just compute for like different resources or different sources of foods.

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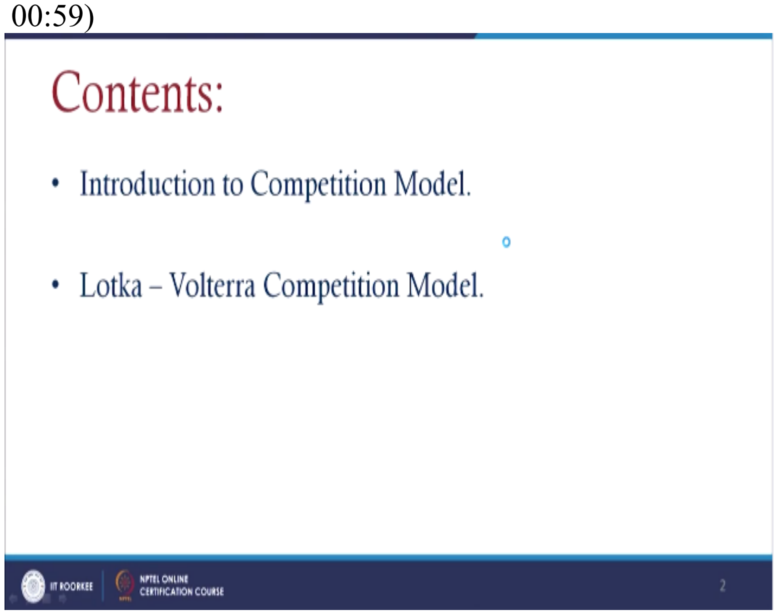

So second phase we will just go for this Lotka volterra competitions model since this scientist like Lotka and Volterra they have discussed this model for two phase's model when there is like two phase present and they can just compute for this same resources. So if you just go for this competitions model if instead of one phases in a population.

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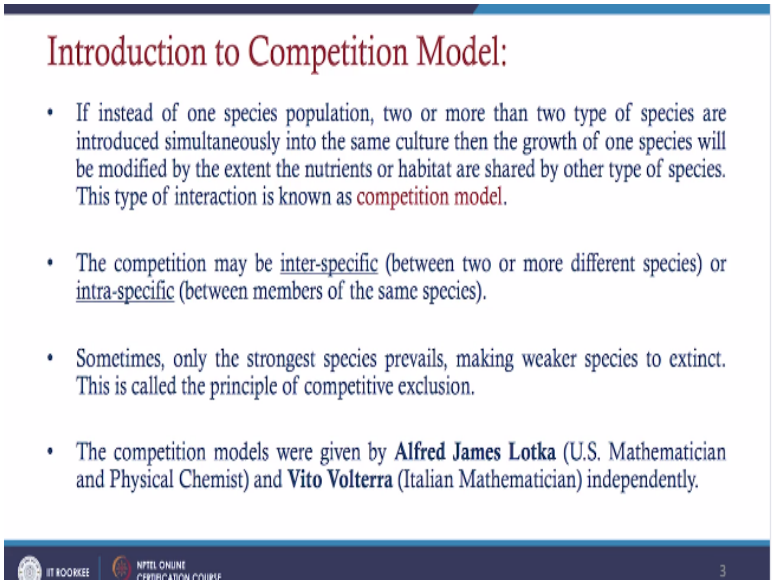

If they are like two or then two types phases are introduced simultaneously into the same culture then the growth of one special can be modified by the extent of this nutrients are habitat that are shared by other type of phase this type of interactions is known as like competiti0ons model here, the competitions may be like inter-specific between two or more different phases or inter-specific means like inside that population they can just like struggle easier to get the food.

So between member of the same phases especially it can be said to be, sometimes only the strongest phases prevails, making weaker species to decline and this is called principle of competitive exclusion, since somebody is stronger they can just to get the food and weaker they cannot survive in that situations the competition models first given by Alfred James Lotka he is a USA Mathematician and physical chemist and this Volterra who is a Italian Mathematician so independently they have just given their model.

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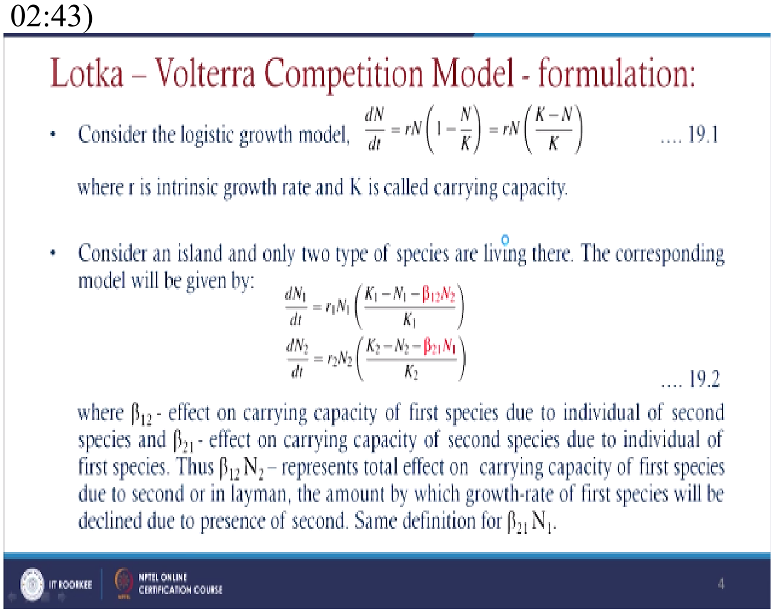

And if you just go for this Lotka-Volterra competitions Model, so a simple logistic model if you just consider that is in the form DN/DT this equal to rN into 1-N/K which can be especially written has rN into K-N/K where r represent the intrinsic growth rate and K is the maximum carrying capacity here so if we just consider an island where only two types species are living there corresponding model will be given by like dN1/dt, where intrinsic growth rate it will be present that is the corresponding to this like atmospheric conditions are like resources available this population level with respective to time will get change that is especially r1 and we have just tried and then this population level it will be controlled by different factors.

So first one it is some other species or certain other constant if it is or if it is present there then it will be decline by the other species if there are present over there, that is why we have just written here $-\beta_{12}N_2$ similarly if second species it is just present over there that rat of change of that population which can be written has r2 as the intrinsic growth rate N2 is the total population level and which can be affected by like the total population level N2 with the present of like first species whatever it is just present in that population level.

So that we have just explained it in a clear form here if you just see so β_{12} we have just written here that affects on carrying capacity of first species due to individual of second species present there and β2 on that affects the carrying capacity of second species due to the present of first species there, so that is why $\beta_{12}N_2$ that restricts the total effect the carrying capacity of first species due to the second or in layman we can just say that the amount by which growth rate of first was affected will be declined due to the presences of second, same it can be applied for like N2 population there and if you just consider her like constant $\beta_{12}N_{21} = 0$ here then it means that one of the species that will not be affected by other species there, but you can have a competitions model sine inside the populations there will be some weaker elements and some are stronger elements present there, they will just competitive each other for the food where resources there.

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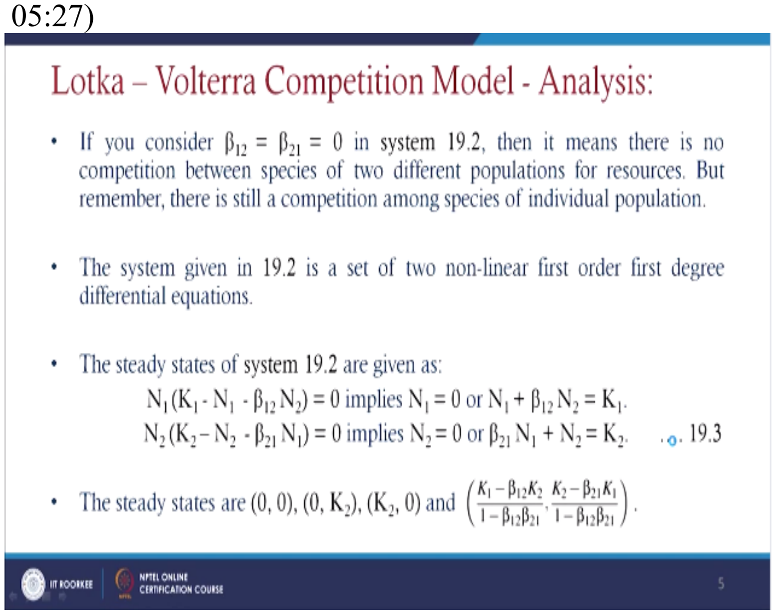

So that is why still a competitions will exist between this individual species there and this system if you just see here the equations represented in 19.2 here, so which represents two non linear first order or first degree differential equations and if we want to go for steady state here then we can just write this one as like N1 into K1 –N1 –β12 N2 this equal to 0 and also N2 into K2 -N2 –β21N1 is equal to 0 here, so this will just give you like N1 is equal to 0 or N1 +β12 +N2 this equals to K1 similar N2 is equal to 0 from the second equations and β21+N1+N2 =K2 here, so the steady state are like (0, 0) if you will just prove to N1 =0 we can just find K2 =0 also, so if you just prove to like K1=0 the we will have like K2 N2 =K2 here and if you just prove to like N1=K2 then we will have like K2=0 and another equilibrium point that is just occurring at K1-β12 by K2 by 1-β12β21K2-β21K1/1-β21 into β12.

So we will have like four equilibrium points now so this steady state (0, 0) this represent that no specie is present at that level or this represent the absences of both the specie, so if the both the species are absent so the populations cannot grow in that level, second one is like k10 this represent that the first species is present second specie is absent (0, K2) represent that first species is absent second species is present with maximum carrying capacity and the fourth one is which is represented as K1/β12K2/1-β12β21and K2-β21K1/1-β12β21 which represent both the species are co-existing at that level.

So if you just go for a Null-cline find this stability or on stability situations, so there are four possibilities if you just see here so foist possibilities is that if you just see here so K2-β21K1 if you just consider here so this means that K2/β21 either it is greater than K1 or it is less than K1 based on that we have just consider like four conditions here first conditions is that K2/β21 greater than K1 and K2 is greater than K1 β12 only this conditions we are just assuming here and second conditions if you like both these are like less here this means that K2/β21it is less than K1 and K1/β12 it is like greater than K2 similarly one by one this like sign changes we have just consider like different cases here so in this case we have just consider like the first case only it is the reverses process here K1 is greater than k2/β21 and K2 this same one we have just consider here, similarly in this four case we have just consider this one as like k2/β21 is greater than K1 and second one remain the same.

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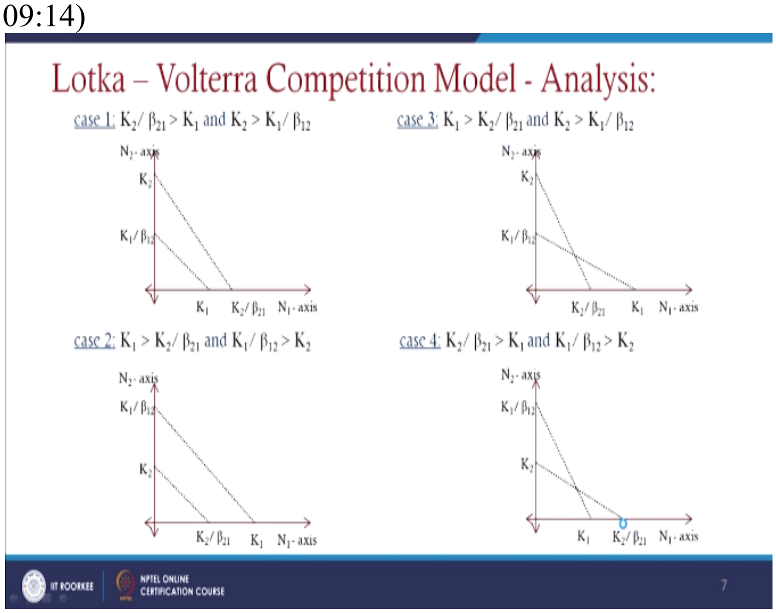

So based on this four conditions if you just plot the graph here for case one we can just find that we will have like K1 since K1 is less than K2/β21, so K2/β21 we have just consider it is a larger point here similarly if you just put here K2, K2 is greater than K1/β12, so that is why we have just joint all this points here to establish this relationship and we will have this exercise that is N1 axes and N2 axes similarly we have plot for like case 2, case 3 and case 4 here and for the like case 3 we are just getting that whenever K1 is like greater than K2/β21 here so we will have like k2/β21 it is less than K1 and K2 it is like larger than K1/β12 here if we are just connecting this points that is K1 and K1β2 so there is a interactions of this lines all this is in Null-clines here.
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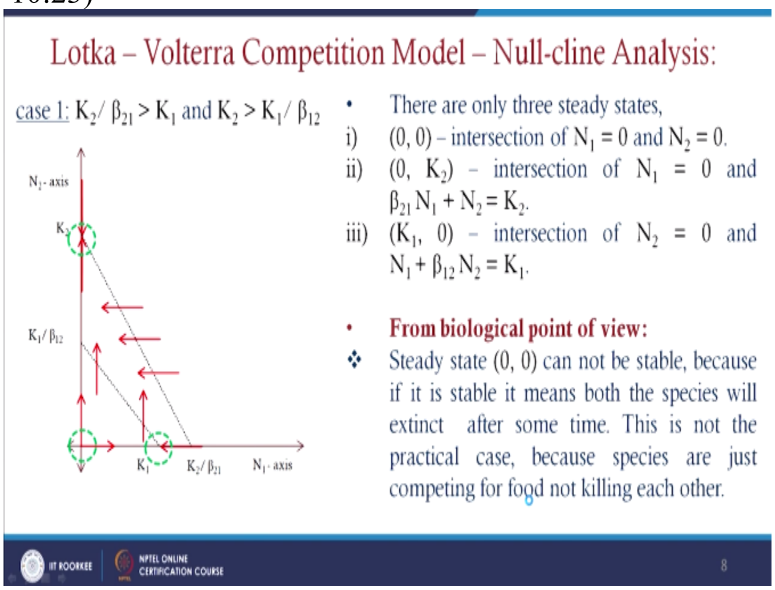

So if you just like for like further analysis of this points we can just find that whenever K2/β21 is greater than K1 and K2 is greater than K1/β12 here there are only three steady states here, first one is(0, 0) here second one is (0, K2) third one is like (1, 0) here and for the(0,0) if you just see that these Null-Cline of N1 and N2 there are just intersecting each other, since we have just consider that different Null-clines should inter set each other to get the steady state points and (0, K2) if you just see here that is nothing but intersection of N1 =0 and β21 N1 +N2=K2 line, similarly if you just see this K1 01 line here that is nothing intersection of N2 axes and N1+β12 N2=k1 axes here.
So from the biological point of view if you just see steady state(0, 0) cannot be stable because it is stable it means that both the special will extent after certain time so both will dies out in that situations that cannot be stable positions so this is not the practical case because species are just competitive each other for food not killing each other they will just survive each other whatever the resources it will be available, so some species they can eat and they can survive in stand or in type of situations so that is why we cannot say that this(0, 0) point is stable point here and if you just see the K2 point here if you just consider any like point above this K2 or below this K2 always we can just find that this level or whatever the point we can have chose this tools just move towards K2 and since this will just give you like increasing function there.

So towards the point K2 there and if you will find any like values which is above this K2 that will just coming towards the point K2 so that is why this K2 is stable point in that scenario and same thing we can just, we cannot observe that K1 point since if you put any value here K2 by β21 if you just see here so K2 /β21 it is always larger than K1 so this mean that whenever this population of K2 it will be larger compare to the fast populations so than they can just used this food and we cannot find a like stable conditions at K1 level also.

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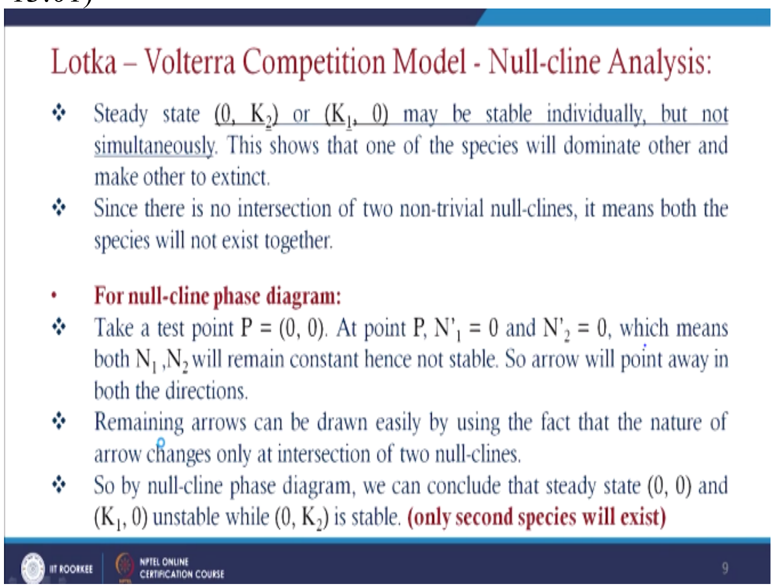

So you just go for this further analysis of this model here steady state (0, K2) and (K1, 0) may be stable individual if you just see but not simultaneously sine this is shows that one of the these species will domain at other and make other to extent there this means that one will survive and other will just dies out in that scenario.

So, if you just see this graphs here, whenever K2 is a stable point here this means that either we can just consider either K2 or K1 here, since, there is no intersection of two non trivial, null-cline for this case here it means both the spaces will not exist together. And if we will just analyze this test case for the point (0, 0) here at point P N1'= 0 to get this null-cline lines and N2' = 0 which means that both N1 and N2 remain constant. Hence, are not stable also.

So, that is why we are just pointing this arrow in both this away from the both these directions, So this means that, this is just going there and this is just going in outward directions. Since, N1' and N2' both are N1 is constant and N2 equals to constant there. And, remaining errors can be drawn easily by using defect that whenever nature of arrow changes only at the intersection of two null-clines and so while nun-cline phase diagram we can conclude that study state (0, 0) and (K1, 0) is unstable while (0, K2) is stable here only second species will exist.

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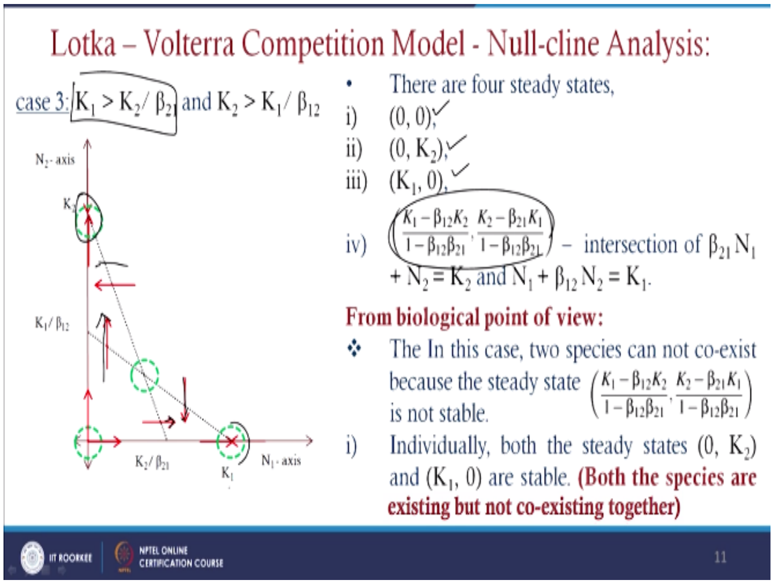

And if you just go for like, for the second case here, .so when K1 is larger than K2/β21 and K1/β12 is greater than K2 here we can just plot this lines so first if you just here (0, 0) it will just intersection of N1 and N2 axes and 0 K2 that is nothing but this line here so this intersection of N1=0 and the second line that is β21 N1+N2=K2 since if N1=0 obviously K2=N2 there and third point it is(K1, 0) which is intersection of this N2=0 if N2=0 we will N1=K1 and for this case we can just find that K1 represent the stable state here but K2 it is on stable and there is no intersection of Null-cline cell are here so for the test point if you just consider as a P=(0, 0) if you just proceed like our earlier case but we have just consider here

So we will have an N1'=0 N2'=0 this means that N1 and N2 will remain constant hence not stable so always if the food is are the resources it is utilize by the species they cannot be constant sine either food resources will just goes down and one species will complete with other and one will be decline and other will like survival over there over or further we can just say that one will like grow up other will just decline there and the Null-cline diagram shows that if you just see here at different levels we can just find that(K1, 0) if you just put in the equations we have just find that it is just present a stable state while (0,0) and (0, K2) are unstable so first case will exist In this case.

Third case if you just consider like K1 is greater than K2/β21 here and K2 is larger than K1/β12 we will have this like equilibrium points (0, 0) (0,K2) (K1,0) and we can just find that the fourth point that is existing as K1-β12 K2 /1/β12 β21 K2/-β21 K1/1-β12 β21 that is the intersection of this lines which is just drawn from a like k22 K2/β21 and K1/β12 2K1 here since the intersections of Null-cline it is just present there so it is just give you this steady state conditions and if you just analysis this will just give you also a like intersection point which is represented by this value here.

And in this case two species cannot co-exist because the steady state this one it is not stable if you just see here all the value there are just going downwards and for this axes we are just finding that all the value they are going outwards also, and in the above of this lines if you just see, this is just point towards this N2 axes and in the like, in the left ward directions and if you just see below this points they are just point towards this like away from this (0, 0) point, so that is why this will not give you a steady state individually if you just see here so we can just find that both these species are stable but not co-existing together here, since K2 level we are just finding a stable point here and like K1 level we have also obtained a stable point here, so individually both the steady state (0, K2) and (K1,0) are stable but no co-existing together.

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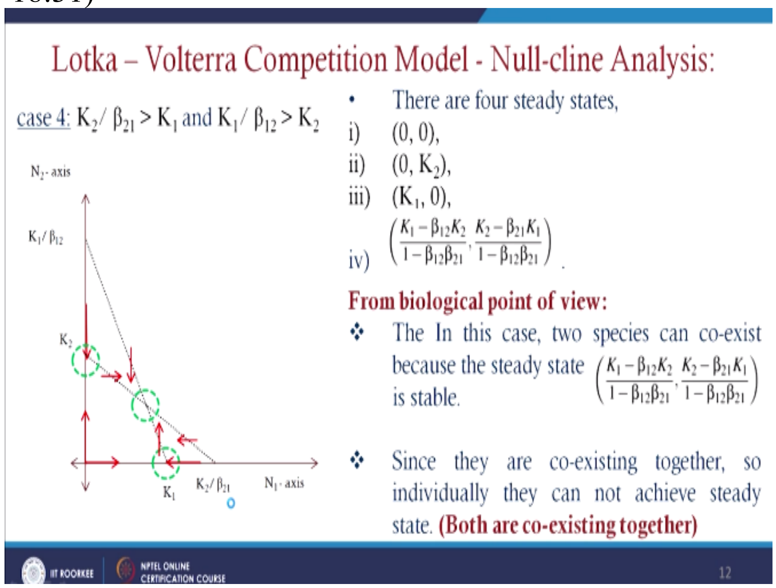

And if you just go for like case 4 here they are like four steady states especially if you just since there is a intersection of this two Null-clines this existing here, so we will have this points (0,0) (0,K2) (K1,0) and this co-existing conditions and in this case, two species can co-exist because they steady state this one is stable here sine if you just see the above this lines point are pointing towards like to the axes there N1 axes and if you just see here the points which is existing below this line there also pointing towards this stable point which is existing for like co-existing for this population level hence this represent this stable conditions since both are existing are co-existing together so individually they cannot archive steady state.

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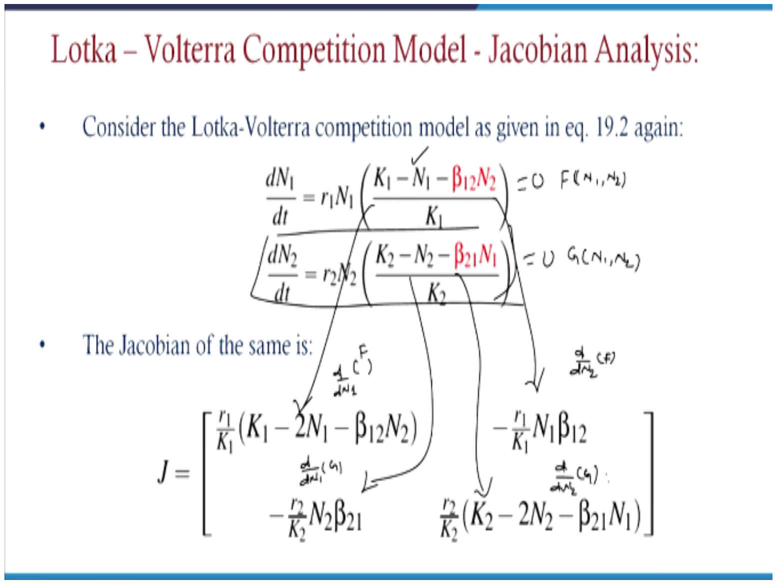

So further we will just go for like Lotka-Volterra competitions model that is, we will just do this like Jacobian matrix analysis here if you just consider this Lotka-Volterra competitions model which can be written, already written that one that is in the form of D1 DN1/dt this is written as r1N1 into K1-N1-β12N2/K1 dN2/ dt which is written as like r2N2 into K2-N2-β21N1/K2 here and if you just put this one is equal to 0 this=0 we can just find the steady state conditions but first order diffractions with respective to N1 if you just do that will just give you this first here if you just do this like diffraction with respect to N2 that will just to provide and if you just go for like first order diffraction of N1 of this equation here then we will just get it as like – r2K2 r2/k2 N2β21.

So similarly we can just to get after like diffraction with respect to N2 we will have like r2/K2(K2-2N2-β21 N1) here so this is nothing but the like diffractions with respect to D/DN1 for the first function we are just doing this one if I will just tried this as like FN1 N2 here this is like G of N1 N2 here, so then we are just diffracting this one F here this is D/DN2 of F here this is D/D N1 of G here this is D/D N2 of G here.

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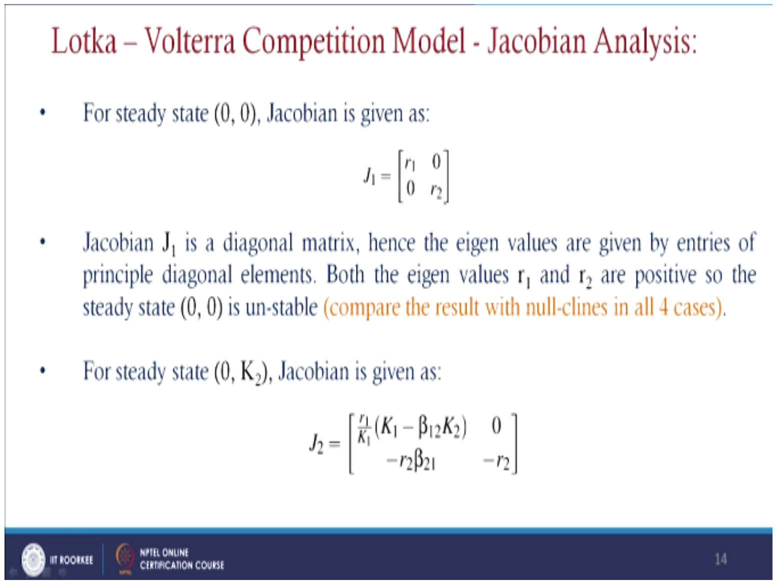

So then at the steady state (0,0) we will have this Jacobian values if you just see here N1 N2 if you just put 0 then can just find this values as a like first value as r1 here then 0, then 0 and r2, if you just see since K1, K1 it will just cancel it out here N1 N2 is 0 so that is why it is just coming as r1 here so N1 is 0 so N2 is 0 here also so same thing here that is r2/K2 into K2 here so this both the quantities are 0 so that is why it is just coming as a sample here.

And jacobian J1 is nothing but the diagonal matrix here so we will have this eigen values that is nothing but the diagonal entries here r1 and r2 both are positive, so the steady state is (0,0) unstable here and we can just compare this result with the Null-clines in the all the four cases you can just see so for a steady state (0,K2) if you just see the Jacobian matrix is just given this values if you just directly put N1=0 and N2 =K2 we can just obtain this definition here..

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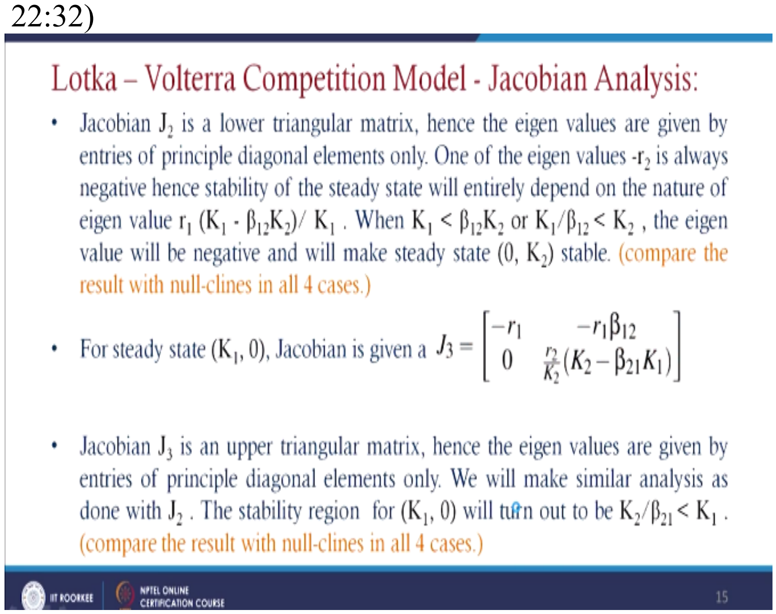

And if you just go for the analysis of this Lotka-volterra competitions model in this eigen values analysis here jacobian j2 if you just see here these are lower triangular matrix so this diagonal entries are nothing but the eigen values here and If you just see one of the eigen values –r2 is always negative since the stability of this steady state will be entirely depend on the nature of the eigen value r1 into K1–β12 k2 here so when if we can just assume here k1 is less than β12 K2 or K1/β12 less than K2 the eigen value will be negative and will make this steady state (0,K2) is stable.

This means that we just try to compare this both this methods for this Lotka-Volterra competitions model here and for steady state (K1, 0) we will have this jacobian that is in the form of this here and if you just see here J3 is a upper triangular matrix and the eigen values are nothing but we can just take principle diagonal elements here, so the stability region if you just see here we can just find (K1, 0) will turn out to be K2/β21 it is less than K1 we can just compare this one with the result of Null cline also.

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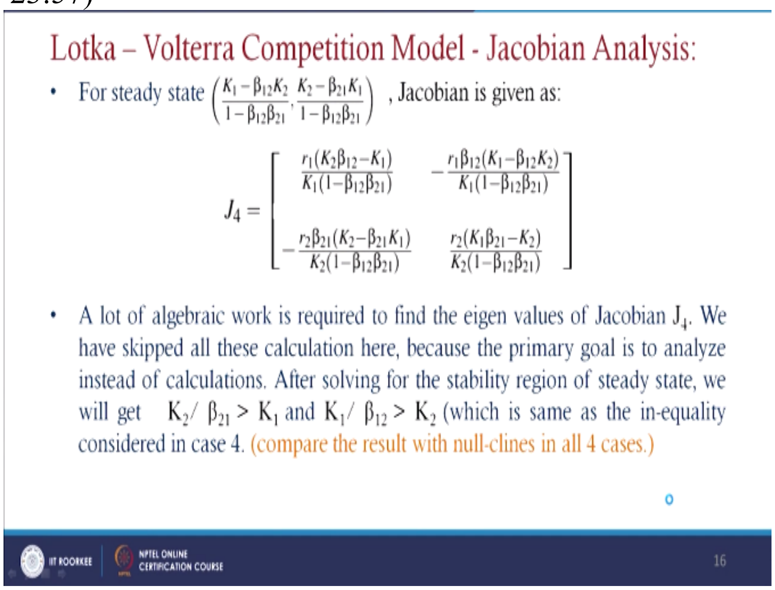

Similarly if you just go for like for steady state at this point jacobian will be give as this one here and for this we have to do like lot of algebraic works to find the eigen values for this jacbian here and we have to skipped all these calculations inside here is a task here the sine the primary goal is to analysis the calculations here after solving this stability region of steady state here will just to get the different region we can just define here K2/β21 is greater than k1 K1/β12 is greater than K2 which is same for the in equality it is consider in case 4 for this Null-clines we have just consider

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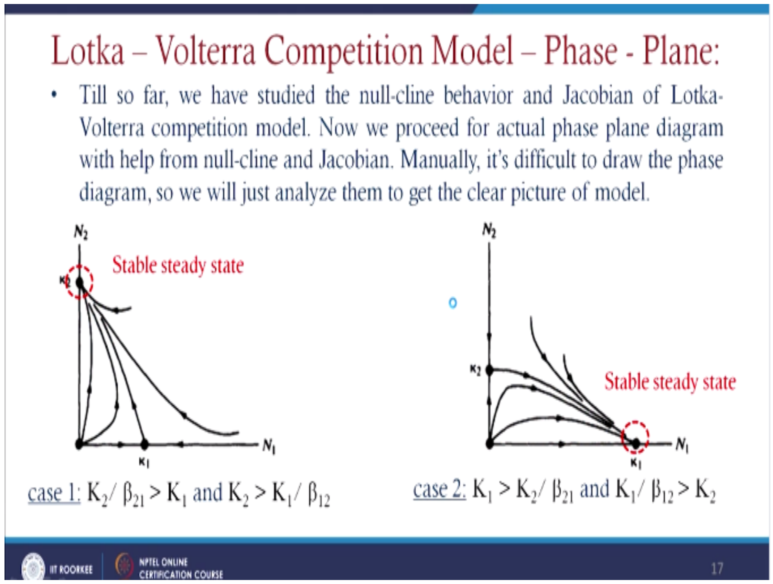

And till so far we have studied this Null-cline behavior and Jacobian of Lotka Volterra competitions model now we can just proceed for actual phase plane diagram with the help of Null-cline and jacobian so manually it is difficult to draw the phase diagram, so we will just analyzer them to get a clear over view of this model if you just see for the first model here we can just find that in Jacobian senses that if we are just finding here r1 and r2 are both are positive so the steady state (0 ,0) is on stable we can just say that (0, 0) is on unstable point here.

Second analysis if you just do here like for (0, K2) we have obtain that one of this eigen value that is (0, K2) is stable since we will have lie K1 is less than β12 K2 and k1/β12 is less than K2 the same thing we have just consider for this case also so that is why we have just find that K2 is a stable find for this case and point wise analysis we can just to do this arrows if you will visualize in case Null-clines here we can just to find that K2 is also a stable point if you just put all this arrows in line form there that lines are nothing but if you just see here these are the lines here.

Which is just point toward the point K2 here so for the second case if you just see here than we have just obtain in Jacobian model K1 as the stable point here (k1, 0) if you just see the stability region (k1, 0) will turn out to be K2/β21 is less than K1 the same thing we have obtain for this Jacobian as well as in null-clines for this case also.

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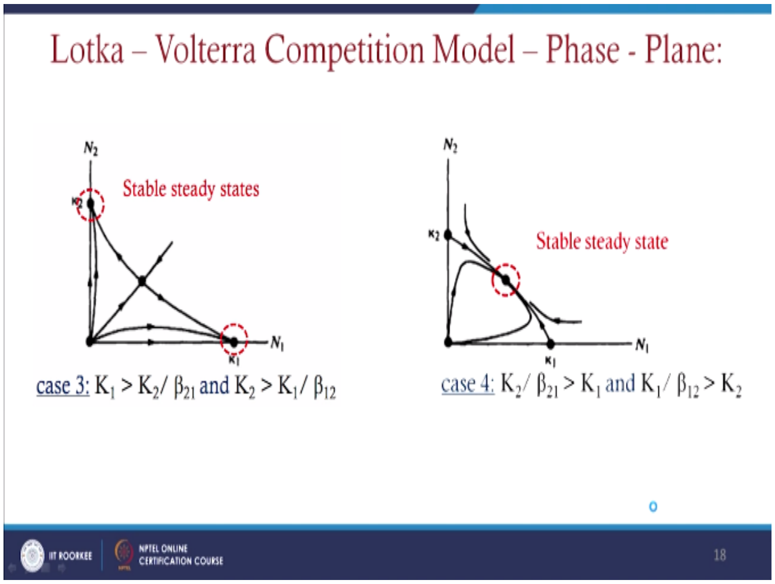

Similarly we have analysis the stable state for like third case that is K1 is greater than k2/β21 and K2 >K1/β12, so we want to visualize that both this like quality approaches and the Null-clines both should be same so that is why this analysis has been prove to it over there. So, that is why for case three we have obtained this line analysis whatever this arrows we have put it over there. So, always it is just pointing towards the point K2 and for this point if you just see these points, it is just moving towards this point and this is also moving towards the point, but at this level if you just see these points are just going outward from this point. In this level we cannot say that this is a stable point here. And for case four we have just find that this is the stable steady state point that is the co-existing point if you just see and at this level if you just see these points are pointing outwards here and in this level also you can just see the, this points are outwards moving. If you just see this one, but at this level if you just see all these points are pointing towards this point here. So, that is why it is represented service stead.

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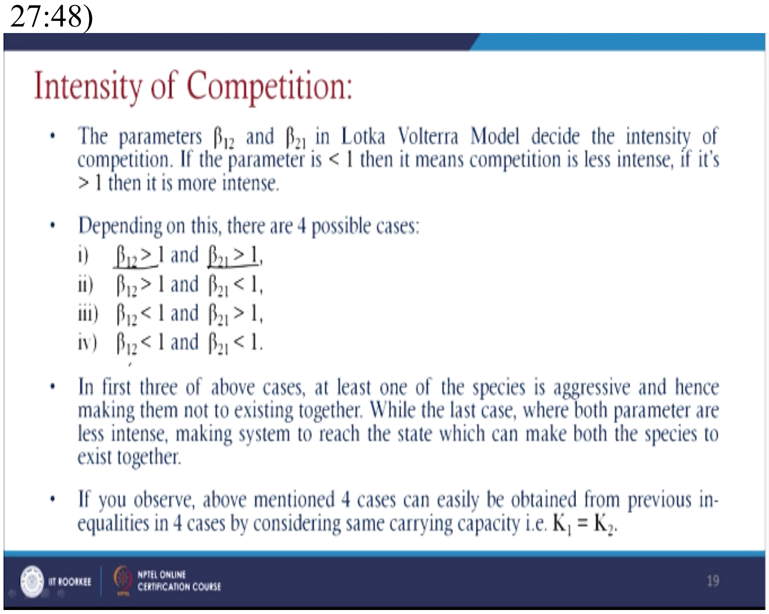

So, if you just go for this like intensity of this competition level if you just analyze, this means that β12 and β21 parameter have it is just performing for this Lotka volterra model. So, then if this parameter is less than 1 then the competition is less intense and if it is greater than 1, if you just see it is more intense. So, Depending on this behavior we have like four possible cases: first one it is β12 it should be greater than 1 and β 21 it should be greater than 1, second case we have just assumed as β12 is greater than 1 and β21 is less than 1, third case β12 less than 1 and β21 greater than 1 and β12 is less than 1 and β21 is less than 1.

So, all the possible cases we have just assumed for β12 and β21 in a coupled form, so in the first three of the above cases if you just see, at least one of the species aggressive and hence making them not to co-existing together. While in the last case, if you just see both these parameters are less intense making the system to reach the steady state which can make both the species to exist together. Since both this parameters are less than 1 here and if you observe, all these above four cases easily be obtained from previous in-equalities four cases by considering same carrying capacity that is K1=K2.

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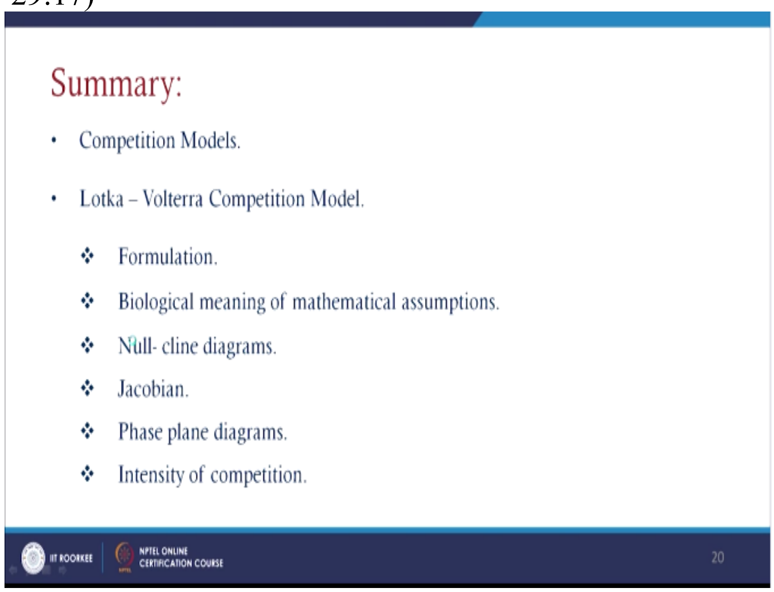

And in this model, we have just discussed like first the competition models where like more than 1 species they just exist. Then we have discussed about Lotka volterra competition models. So, first we have formulated the model and analyze the model then we have just tried to explain the, what is the biological existence of this model with the mathematical assumptions, then we have just tried to analyze this model using null-cline functions, then this Jacobian and tried to compare both this like null-cline functions and Jacobian with the phase plane diagrams, then final phase we have just discussed this intensity of this competition levels. Thank you for listen this lecture.

For Further Details Contact
Coordinator, Educational Technology Cell
Indian Institute of Technology Roorkee
Roorkee-247667
Email: etcell@iitr.ernet.in,etcell.iitrke@gmail.com
Website: www.nptel.iitm.ac.in

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Prof. Ajit Kumar Chaturvedi
Director, IIT Roorkee
NPTEL Coordinator
Prof. B.K.Gandhi
Subject Expert
Dr. Ameeya Kumar Nayak
Department of Mathematics
IIT Roorkee
Produced By
Sarath.K.V
Graphics
Binoy.V.P
Production Team
Mohanraj.S
Web Team
Dr. Nibedita Bisoyi
Neetesh Kumar
Jitender Kumar
Vivek Kumar
Dharamveer Singh
Gaurav Kumar
An Educational Technology cell
IIT Roorkee Production