INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
NPTEL
NPTEL ONLINE CERTIFICATION COURSE
Mathematical Modeling: Analysis and Applications
Lecture – 18
Qualitative Solution of Differential Equations Phase Diagrams-11
With
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Welcome to the lecture series on the mathematical modeling analysis and applications, in the last lecture we have discussed this phase diagrams for 1d model and then we have discussed about 2d model for like couple system and uncoupled system. So in that sense we have just found that for this like 1 dimensional system we have visualized start.

Whenever we are just going for phase diagrams, we are just finding first the steady states and once this steady state is received then that point where this system is unstable or stable so that we can just take from this phase diagrams. And whenever we are just analyzing we are just finding that, if suppose we will have like positive slope their then the point at which it is occurring that represent unstable conditions there but if it is represent negative slope then it is just provide some stable conditions.

And in 2d uncoupled system it is easy to get the solutions and especially we have just tried to find like the steady state points and at that point we have just determined this is Jacobean to find this like Eigen values and depending on Eigen values we have predicted that whether it has like stability or un-stability is preserved. But if we are just going for like a couple system, then especially we have reduced again to a uncoupled system or two a matrix where this diagonal entries are there and depending on this Eigen factors plotting we have just find it out that whether this point is stable or unstable.

And in this lecture we will just go for like qualitative approximation of this differential equations, that is phase diagram two here. So how we can just say that this weather we are just approximating this functions with like a linearized system and weather this linear system which is represented as Jacobean matrix form, is it providing a exact solution whatever we are just expecting from this analytical overview of this equations, so that we will just analyze with this lecture.

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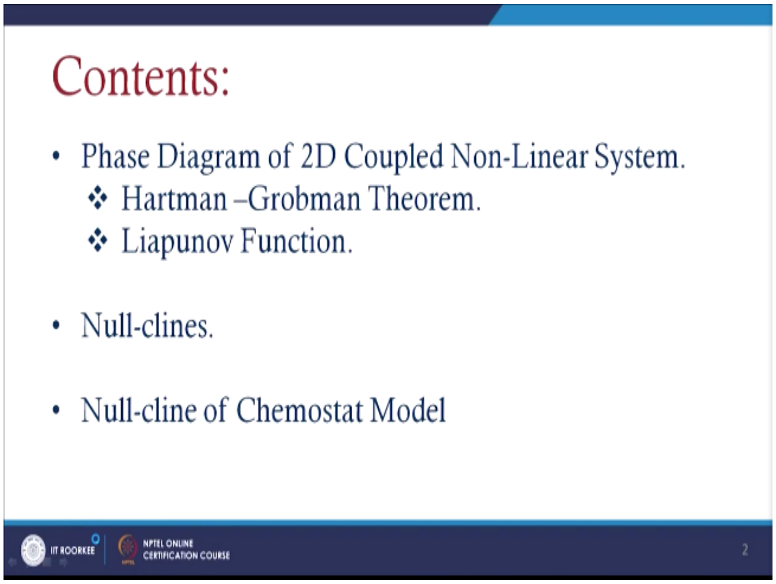

So first we will just go for like phase diagram of 2d coupled non linear system. Since this I told in the last lecture we have discussed for 2d coupled linear system and this nonlinear system can deal with that two methods that is first one is, Hartman, Grobman theorem. And the second one is like Liapunov function and this can be analyzed through null clines also. So then in the final phase of this lecture we will just go for this non cline of Chemostat model.

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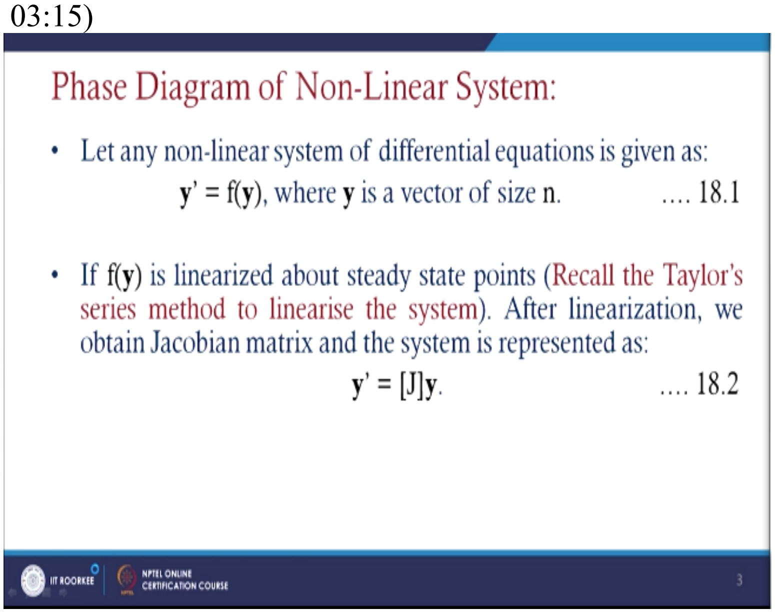

So any non linear system it can represent as we have discussed in the last lecture that the generalized representation of system can be retain as like y dash is equals to f of y, where y is a vector of size n. this means that we will have like system of an equations here, and if it is non linear system of differential equation first what we are doing is, we are just linearizing that system and if f of y is linearized about the steady state points.

Then especially we are just using this Taylors rigid function parties, in the last few lectures we have discussed that Taylors method how we can just analyze this things, this means that we are just assuming this two functions that is in the form of f of x plus hy plus k, and capital G of x plush and y plus k and we are just assuming this functional values which is represented it has x dash is equal to f of xy, and y dash is equal to g of xy that assume to zero. Then we will have like first derivative terms present in the Taylors rigid function and from there itself we are just getting this Jacobean matrix.

This is nothing but the linearization technique we are just using to get this nonlinear differential equation to linear form here. And once we are just obtaining this,

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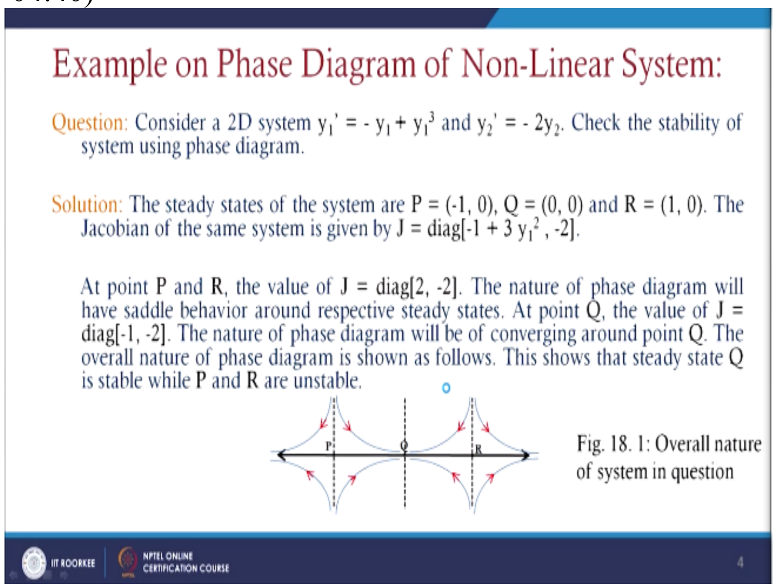

Linearized system, so then we are just using this phase diagram to analyze whether we will have a like stable condition where it is just appears there, if we will have like steady state points. For this analysis we will just consider to the system like y1 dash equals to or y1 prime is equals to minus y1 plus y and q and y2 prime is as minus 2 y2 here.

So first if we will just find the steady state of the system we will just keep this y1 dash equals to zero and y2 dash is equal to zero here. If you'll just put these two values zero here and then from the first equation we can just get it has like y1 if you just come on here, then y1 square minus 1 this equals to zero and second one we can just get it as y2 equals to zero here. And from the first system we can just get either, y1 equals to zero or y1 equals to plus one or y1 equals to minus 1. Hence the steady state of this system are like first point it is minus 1, 0 and the second one is 0, 0 and third one is 1, 0 here.

And if we will just go far this Jacobean of this system, so Jacobean especially we can just write it as like matrix of first differences with the respective y1 here, so that will just provide you a minus 1 plus 3 y1 square, and second no y2 is present there so that's why this is like zero value. And the third one if you just differentiating with respective y1 here and there are no factor associated with y1 here.

So that will just give zero, and the last value if you'll just differentiate with respective y2 here that will just give you minus 2 here. Hence the diagonal entries for a Jacobean matrix it is just written as diagonal as minus 1 plus 3y square minus 2 here, and at the difference point like since we have just defined p as minus 1,0 here and q as 0,0 here, and r as 1,0 here. So then at this points like p and r since this is the square function it is present here, so that's why this minus 1 and plus 1 both it will just provide the same value so that side point p and r if you'll just see the value of Jacobean it is just represented as a diagonal of 2-2 here, so this means that along the x axis it has like a positive value along the y code in it will have an negative value here, and if you will just go for the nature of phase diagram here.

We will have several behaviors around the respective steady states here. Since if you will just see here p as the points like minus 1,0 here, then q as the point as 0,0 here and r is the point that takes as 1,0 here. And if you just analyze this behavior of this phase diagram here at the point q you will have this like value as minus 1q it is 0,0 here.

So if you just put 0, 0 values for this Jacobean here we will have this value as Jacobean equals to like minus 1, minus 2 here. So both this values are negative here, this means that diagonal entries negative means you will have this Eigen as negative here, so the nature of this phase diagram we will converse towards the point q here, since the both diagonal values are negative here, and if you just see this like the overall nature of this phase diagram that is at the point q if you see at the point p suppose if you all just see here, so if you'll just put minus 1 zero here.

So minus 1 means, if I will just put y1 equals to minus 1 here so that will just give you three then three minus 1 this will be 2 and minus 2 we can just get, so 2 minus 2 means, you will have a positive value 2 and negative value minus 2 here, and along the x axis you just see, so that Eigen value is 2 that's why this will just deviate from that point. And if you just see along the y axis here the Eigen value is minus 2 so that's why this points it will just move closer towards 0,0 here. This means that sorry, this is minus 1, 0 here.

Since whenever we will have a minus 2 means, it will just push towards that point and whenever we will have like positive value it will just retired from that point. So this whenever it is occurring both this points p and r if you just see that is minus 1,0 and 1,0, over ally if you'll just see p and r this will just give the unstable measure here and q will become a point which will just provide a stable measure for this equations.

And if you just go for like a further analysis of a phase diagram of a non linear system, so in the previous example we have studied the qualitative measure of the system after linearizing the same about the steady states and as you know that always whenever we are just linearizing the system it is just a approximation to the system not the exact system we are just considering for this solution process. So whenever we are just approximating the system whether this has represent the same,

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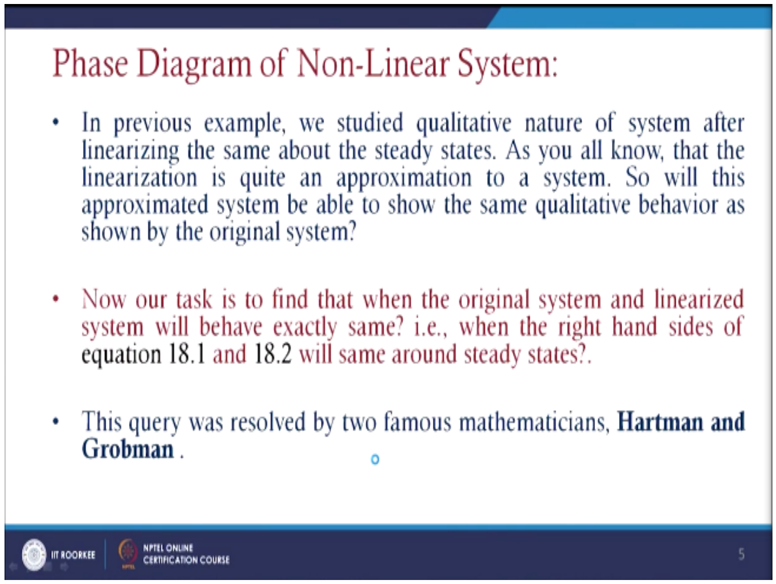

Qualitative behaviors as shown by the horizontal system or not, that we have to take also, so that we can just take, so this means that whether this original system and linearized system will be behave exactly the same or not and if you will just go for this analysis so we can just consider this equation like 18.1 and 18.2,

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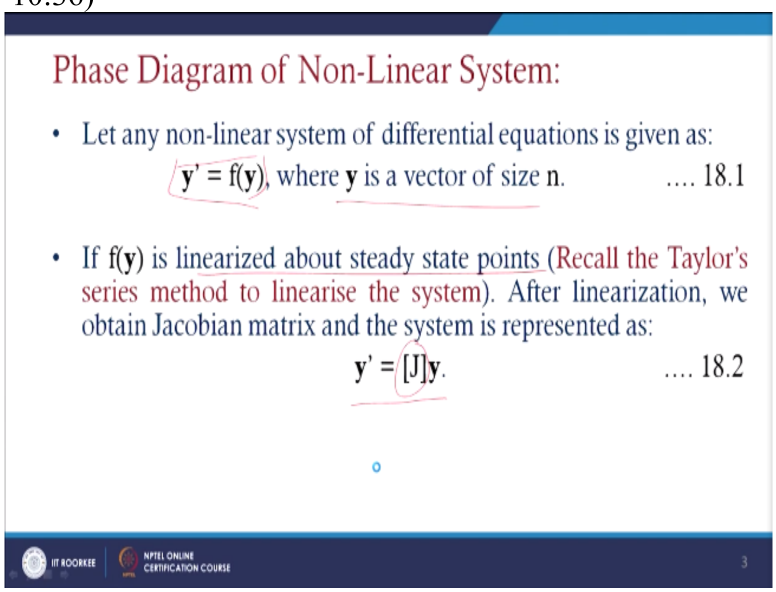

If you just see here, so why does its equal to f of y here and y dash that's equal to jy. So this is the approximated or the linearized system we have just considered and this is the exact system we have just considered here. So whether this system both the sides in the right hand side, they are just behaving like a steady states at the same level or not that will just discuss here. And this query was resolve by two famous mathematicians one is Hartman and another one is Grobman.

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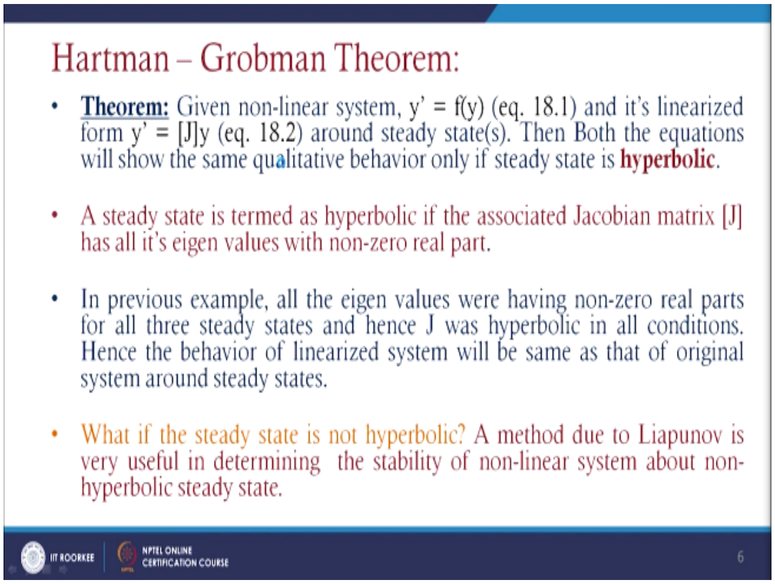

So if we will just go for this theorem that is developed by Hartman, Grobman. So the statement of the theorem states that, suppose we will have a nonlinear system y dash is equals to f of y, and its linearized form y dash is represented as like jy here, since we are just using this Taylor series expansion tool in the linearized system. Obviously many methods are available but we have just picked one method that is Taylor series method here around the steady state.

Then both these equations will show the same qualitative behavior only if the steady state is hyperbolic. Then the question arises that what does it mean by hyperbolic here, so the statement for hyperbolic is that, a steady state is termed as hyperbolic if the associated Jacobean matrix has all its eigen values with the non zero real part. So at least we have like real part which as certain values except zero to provide that the system is in or the state is a hyperbolic phase.

So in the previous example if you just see, all the eigen values where having non zero real parts, this means that if you just see that we have just discussed here, that is the diagonal elements are coming as 2- 2, -1,-2 here, so that's why all real parts are existing, hence we can say that the system is hyperbolic in nature. And that's why this statement states here that the three study states already we have discussed in previous slide that as like non zero or real parts.

Hence j was hyperbolic in all conditions and hence we can just say that the behavior of linearized system will be same as that of original system around the study state by the condition of the hyperbolic system. Since the theorem states that if you will have a system like y dash is equal to f of y and its linear is form is expressed as y dash is equal to jy that we had made it and around the steady states then if you will have a existent of a real part then we can just say that this qualitative approach of this approximated solutions is exactly equals to the real system solutions.

So the question arises that, if the steady state is not hyperbolic, then what we can say?

So for that a method due to Liapunov is very useful in determining the stability of non linear system about non hyperbolic steady state.

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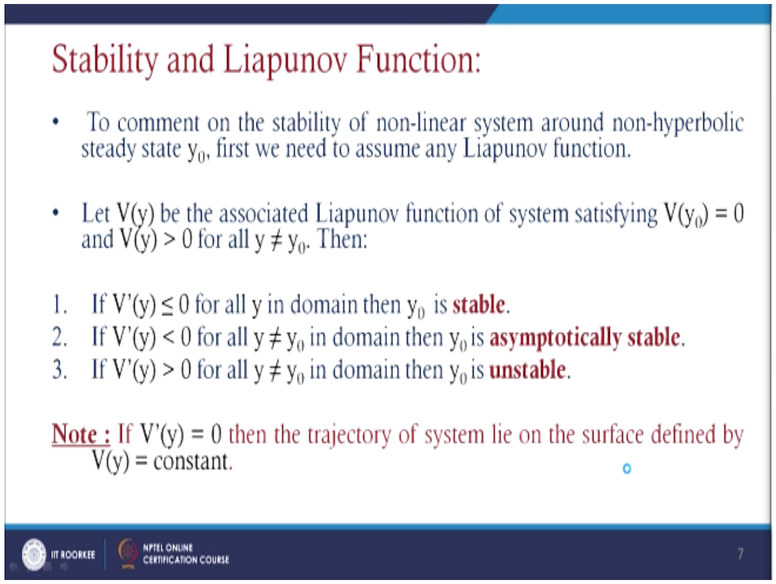
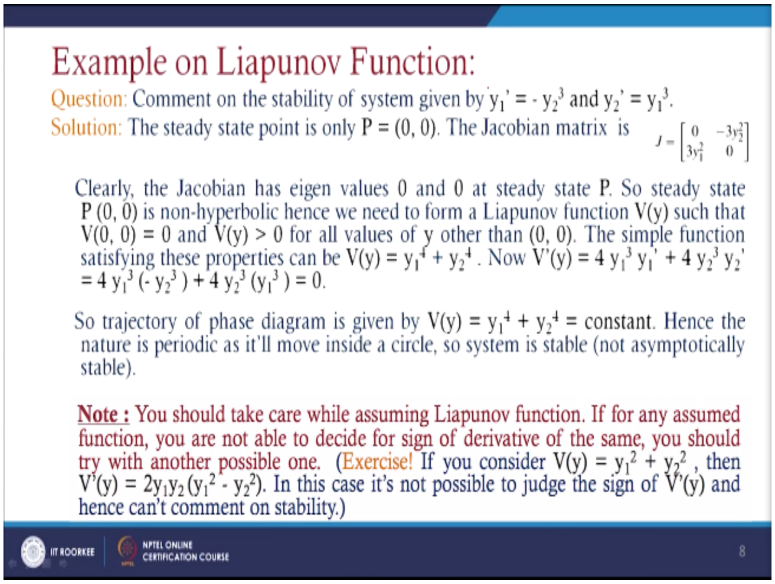


So if you just go for this Liapunov function, then we have to comment on the stability of a non linear system around non-hyperbolic steady state y0 here, so first we need to assume any Liapunov function how we can construct this Liapunov function that we have to concentrate. So for the Liapunov function, first we will determine a function as suppose v of y satisfying v of y0 equals to zero and v of y greater than zero for all y not equals to zero.

Then if we are just assuming this conditions based on this differentiation we can assume that whether this system is stable or unstable or asymptotical stable. So if we are just assuming this function v of y as a Liapunov function here. So the first condition we are just assuming v of y0 is equals to 0, v of y greater than 0 for all y is not equal to y0.

Then if v dash y is lesser equal to zero, conditionally we are just assuming both this v dash y equals to zero and it is less than zero. For all y in the domain then y0 is stable, and if v dash y strictly less than zero for all y are not equals to zero in the domain, then y0 is asymptotically stable. And if v dash y is greater than zero for all y not equals to y0, inside the domain y0 is completely unstable point or the system will be unstable at that point y0.

And if the condition remains is that v dash y equals to zero only, then the trajectory of the system lie on the surface by v of y equals to the constant. Since this will roam around that point level only. So for this we have assumed way function that is example we have considered here. So suppose the system is represented as,



Suppose y1 dash equals to minus y2 cube here and y2 dash equals to y1 cube here. And for this systems, if you will just see the steady points or the steady state points are like first one y1 dash equals to zero then you will have y2 equals to zero and if you just consider y2 dash equals to zero, this means that y1 equals to zero here.

So the steady state point is 0, 0 here. And if you just consider this is Jacobean here then this Jacobean will be represented as a first differentiation with respective y1 here so there is no amount of associated with y1 so that's why it take the value as zero here and if you just take differentiation with respective y2 here, that is minus 3 y2 square. Second function is just a differentiative with respective y1 this is 3y1 square here, second there is no operation of y2 term so that's why this take zero there.

And if you just see this Jacobean clearly we can just find that this Eigen values are 0 and 0, since the diagonal entries at the steady state we here. So the steady state p is 0, 0 is since we do not have any like real part like non zero real part, so that's why this 0, 0 is non hyperbolic. Since by basic definition of hyperbolic we will have like non zero real part. So if it is like a non hyperbolic then we can just use this Liapunov function.

So if we will just formulate this function v of y, in such equation that v of 0, 0 equals to zero and v of y greater than zero for all values of y except 0, 0, then the simple functions satisfying this properties can be written as v of y equals to y1 to the power 4 plus y2 to the power 4, this is the guest corrected methods it is exactly we do not know how this behavior of this function will perform there.

So now if you just take this differentiation here, since the next condition is that we have to find that v dash y it should have either less than equals to zero or greater than zero or less than zero or equals to zero that we have to testify. So if you just see here, v dash y that is nothing but the differentiation if you just take for this function here, we will have like 4y1 cube y1 dash plus 4y2 cube and y2c dash here already y1 dash it is known as minus y2 cube and y2 dash is y1 cube. If you just put all this values we are just getting this value as zero, since this to hold these quantities are equal here.

So the trajectory of this phase diagram it is just given by v of y which is written as y1 to the power 4 plus y2 to the power 4 this equals to constant here. That we have just defined in the previous slide here, v dash y equals to zero then this will be defined as v of y equals to constant here. And if it is constant definitely you have known that this nature series periodic and it will move inside a circle since it can be written as y1 square plus y2 square this equals to constant. So it will just move inside this circle.

Hence the system is stable here so not asymptotically stable. And whenever we are just discussing about this Liapunov function so we should have to take care in seriously since this is the guest corrected method by assumption we are just considering this function. If for any assuming function you are not able to decide for sign of derivative of the same you should try with another possibilities.

So suppose if you will just consider function like vy as y1 square plus y2 square, then v dash y it can be written as, if you will just find the derivative here so 2y1 and 2y1 dash plus 2y2 into y2 dash here and y1 dash is defined as minus y2 cube and y2 dash this y and q. if you just put all this values then you can just find minus 2y1y2 into y1 square plus y2 square.

And in this case you cannot say that whether v dash y it will just give you this like positive or it is negative or it is equals to zero that is doubt full you cannot comment on the stability of this conditions. So you should have to be careful whenever you are just defining this Liapunov function. So then we will just go for non clines, a systematic way to draw a phase diagrams. So till so far we have,

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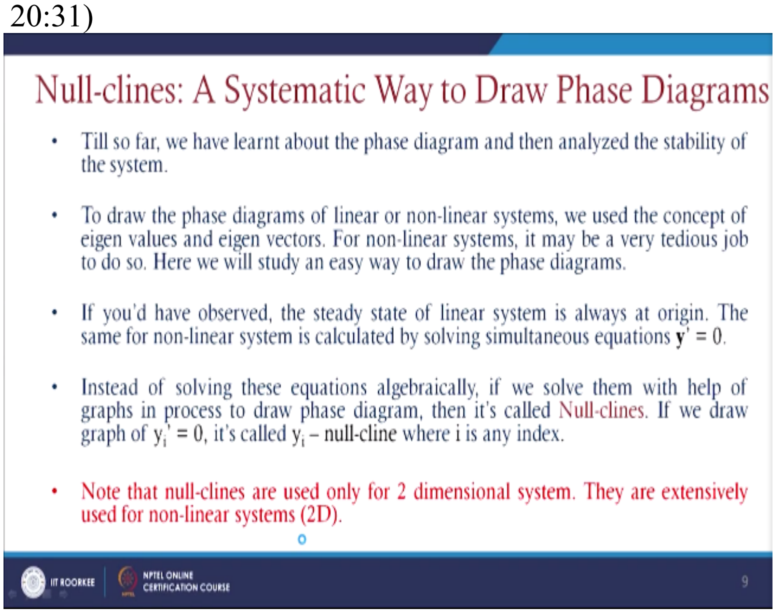

Learned about the phase diagram and then analyzed the stability of the system. Whenever we are just drawing this phase diagrams of linear on nonlinear systems, we used the concept of Eigen values and Eigen vectors. Already we have discussed that whenever we will have like uncouple system we have used the Eigen values, whenever we have used couple systems then we have used this Eigen factors.

And for non linear systems we have used like our Liapunov function for non hyperbolic functions and hyperbolic function we have used also the same Eigen values there. And here we will just study an easy way to draw the phase diagrams for non linear functions so if you would have observed here the steady state of linear system is always occurring at the origins. And similarly the non linear system it can be calculated by solving this first order differences is equals to zero also, and instead of solving this equation algebraically, if we can just solve them with the help of graphs then this is called null clines.

And if we will just suppose draw this graph of like yi dash equals to zero then it is called yi null cline where I is any index. So the basic idea is that directly we are just putting this graph of yi dash equals to zero here and from the different graph intersection points we can just find the steady state points and after that we can just visualize that whether this system it is just performing at different point levels how it is just performing.

So if we have to do this like stability analysis using this null clines here, we have to put some points that is when null clines are used for two dimensional systems it can be used and they are exclusively used for non linear systems. Since null clines cannot be used to ever where, so there is some certain restrictions that's why we just put this statement as a, note that null clines are used only for two dimensional systems and they are extensively used for non linear systems that is for a 2 dimensional non linear systems. And if you just go for one example on null clines here.

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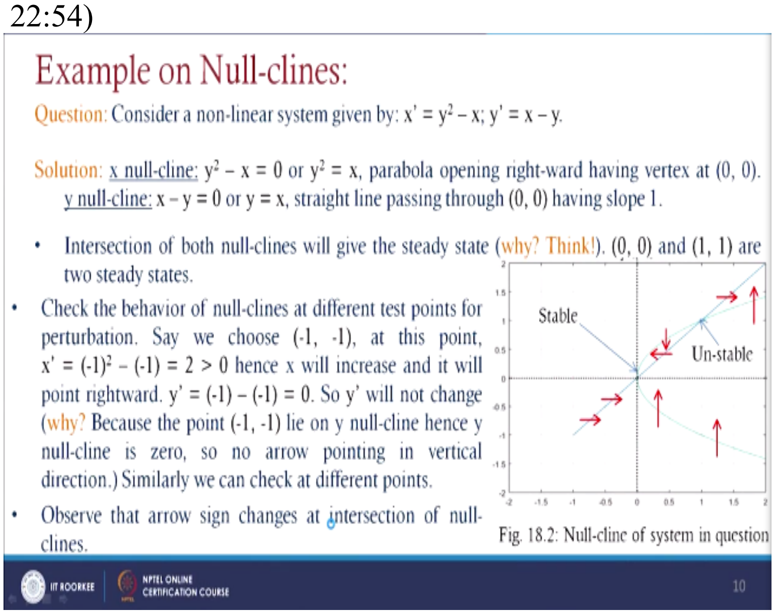

So suppose consider a non linear system given by x dash equals y square minus x and y dash equals x minus y here. So first way to draw this null clines we have to put x dash equals to zero and y dash equals to zero. So if you just put x dash equals to zero and y dash equals zero we will have this equation that is in the form of y square minus x is equals to zero or y square is equals to x. which represent open formula or a formula opening right ward having what I said is 0, 0, that we have just represented here.

And second null clines if you just see here that is y dash equals to zero that is nothing but x minus y equals zero, which just provides y equals to x this is a line, that is a line passing through 0, 0 having slope 1 at origin and whenever we will have this intersection of this two null clines that will just provide you in this steady state condition, if you see here this intersection points occurring 0, 0 and 1, 1 here. And at this points if you'll just see this just provides us x dash equals to zero and y dash equals to zero, so that's why they are steady state points. And if you just go for this like analysis of their behavior of null clines at different test points. So we will just take a small part close to the steady state points and then we can just visualize that how this like part of points behaving in the closer the steady states depending on that we can just say that whether it has a like stable point or unstable point.

So if you just choose the point like minus 1, 1 here, so then we can just find that x dash is taking the value as y square minus x so that's why its just providing 2 here, so any point if you just choosing here suppose minus 1,1 here. Then it is just providing a like positive value here, so this means that the points x will increase it will point forward so that's why we have just putting this symbols here, this means that just giving a monotonically increasing function so always it will try to increase along the like positive x axis.

So that's why it is just the arrows are putting towards this right ward direction. And if you just consider this function for y dash here, then we can just find that this is minus 1 this is equal to zero here so y dash will not changes here, because the point minus 1 lie on the y null cline here since this is just lying on this line here and y null cline is zero here, so arrow no arrow that's why we have kept any arrow finding in the vertical directions.

Similarly if you just check all other points like suppose 1,1 we have kept it over here, then we can just put like 2,1 so if you just put 2,1 here so then we will have like positive value, so that's why it will just a point out wards there. So similarly at all other points we can just test we have just a point this arrows in different directions, if it is an negative then it will approach towards this steady state points and if it is like positive then it will just point outwards from the steady state points that we can say from this graph. And if we will just go for this,

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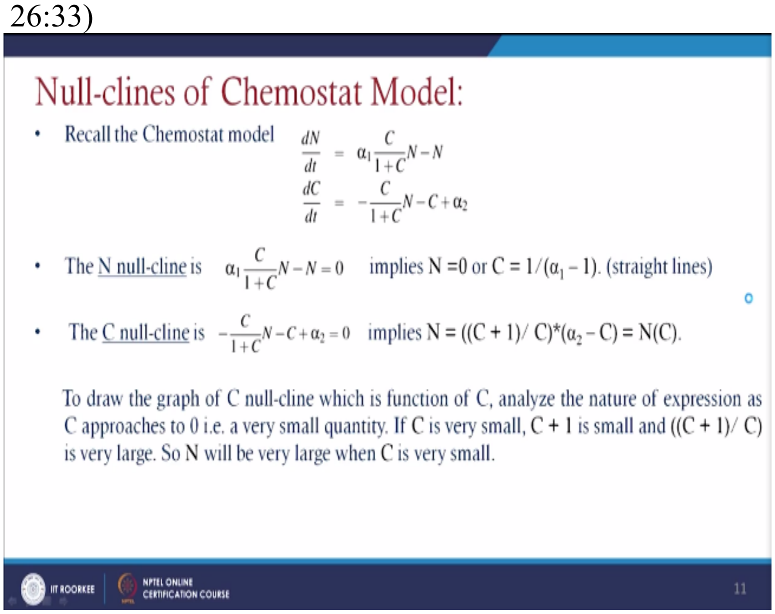

Analysis of null clines function for Chemostat model here, so this Chemostat model especially it is represented in dn by dt equals to alpha 1 c by 1 plus cn minus n here and this dc by dt which is represented as minus c by 1 plus cn – c plus alpha 2, already we have defined this Chemostat model in the earlier lectures and if you just go for this null cline function here or the null cline values we can just put dn by dt equals to zero and dc by dt equals to zero here.

And if you will just put this n dash equals to zero and c dash equals to zero you can just find that alpha1 into cy1n plus cn minus n equals to zero which will provide one value as n equals to zero here and another value that is c equals to 1y alpha 1-1, so both are straight lines.

And if you just see this c null cline so then we can just find that this cannot be separated it out here so that's why we you will just find n value from this equation then it can be written in the form of c plus 1 by c into alpha2 minus c that is nothing but n is a function of c here. And if you just plot a graph of c null cline which is a function of c to analyze the nature of expression as c approach equals to zero here, so if c approaches zero this means that c is very small value here then c plus 1 it is also like 1 plus some values there.

So then n will be very large and so we can just say that when c is very small and it is very large for this case here.

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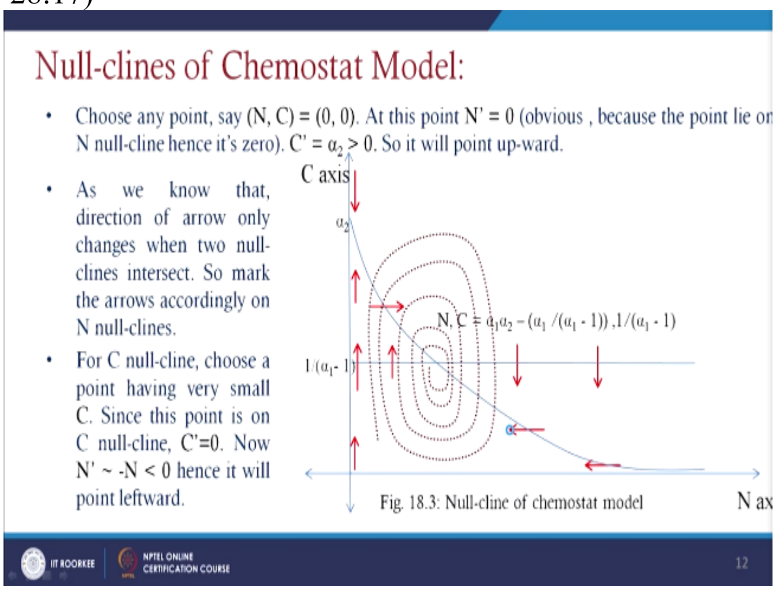

And if you will just plot this points and if you just go for this null cline here, so at n, c as 0, 0 whenever we will have like resources is zero and population is zero, so obviously this point will lies at n null cline so both will be zero, so at that point we can just see that whenever n is equals to zero and c is equals to zero the resources alpha 2 here.

So directly we can just say that c dash equals to alpha 2 greater than zero, so it will point upward from this 0, 0 point. And as we know that the direction of arrow only changes when two null clines intersects, since we will have this null clines that is in the form of like n, c if you will just see here the null clines are n c then we will have another one that is n c equals to alpha 1, alpha2, minus by alpha1 by minus1 and 1 by alpha 1 minus 1 here, these are the steady state points especially I can just say.

And whenever we will just go for like c null cline functions choose a point having a very small c then we can just find that c dash equals to zero and n dash will have approach to minus n here, so this is less than zero hence it will just point left wards if you will just see. And from this graph we can just find that whenever we are just starting with a like initial population levels certain population it is just present here.

Then resources plenty available then after certain time we can just find that this resources will be like optimized with this population growth level and finally we can just find that this resources will goes down when this population will get on increase and in the physical scenario we can just find that whenever this population level it is just a in a optimal level it will just reach then this resources will be reduced variant and then this will decline this population level again it will just increase this will just follow like cyclic process.

And if you will just go for this like physical observation of this null cline for this Chemostat model, so when there are,

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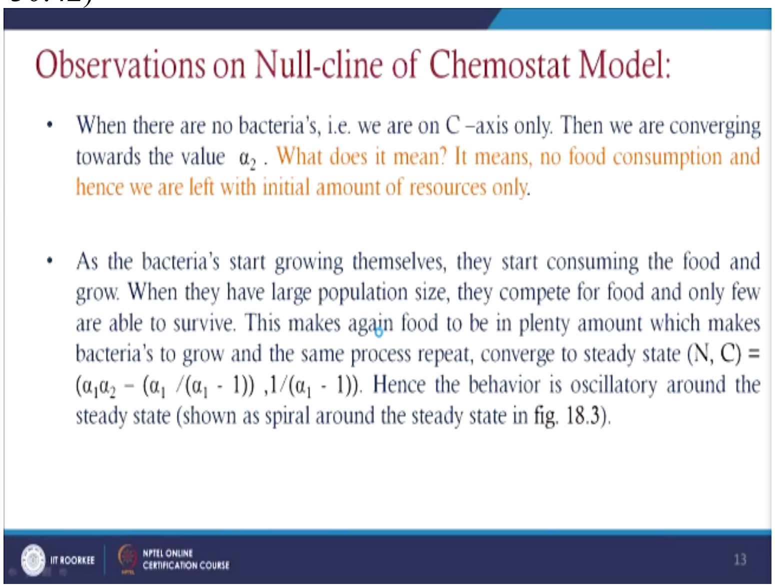

No bacteria we are in this c axis only. And then we are converging towards the value alpha. So this means that if there is no consumption of food then we can just find that we are left with only initial amount of resources, and as the bacteria's start growing themselves, they start consuming the food and grow. When they have like the large population size, they compute for food only few are able to survive.

Then again this makes the food to be in plenty amount which makes the bacteria again to grow and the same process repeat and converge to other steady state that is alpha1, alpha2 minus alpha 1 by alpha1 minus 1 and I by alpha 1 by minus 1. Hence the behavior is oscillatory around the steady state as we shown here that is a spiral state it is just showing over in this range here, and finally we have obtained this steady state received at like alpha1, alpha2 minus alpha1 minus alpha1 by alpha minus n and 1 by I alpha minus 1.

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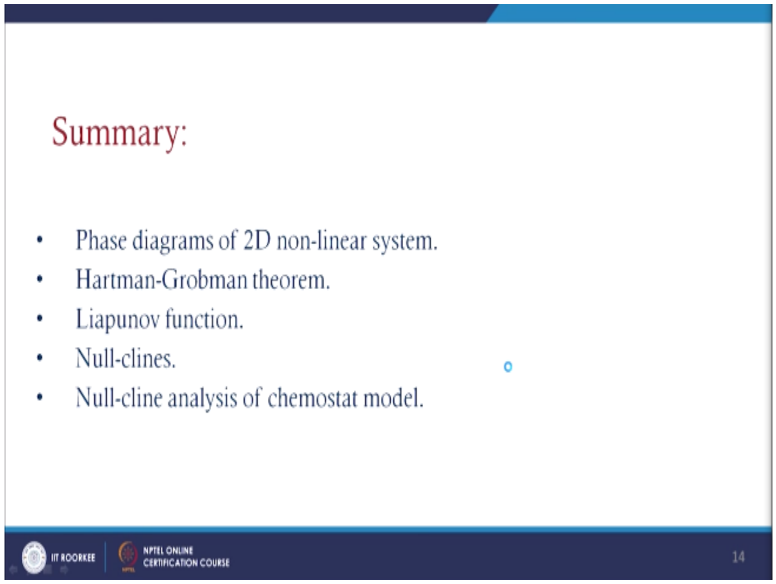

So in this lecture we have discussed like phase diagram of 2d non linear system. Then Hartman Grobman theorem, then Liapunov function, and null clines, so null cline means it is an analysis of like different stable points whenever we will have 2d non linear system. And then finally we have discussed about this null clines for Chemostat model. Thank you for listen this lecture.

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