INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
NPTEL
NPTEL ONLINE CERTIFICATION COURSE
Mathematical Modeling:
Analysis and Applications
Lecture- 17
Qualitative Solution of
Differential Equations-Phase Diagrams
Part 1
With

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Welcome, to the lecture series and mathematical modeling analysis and applications. In the last lecture we have discussed this single spaces population model and based on that we have just discussed this inter gin growth rate of the population and in the last page we have discussed this allee effect, allee effect especially this depends on special class of population modeling and that convergence analysis we have also discussed there if and in the present lecture we will just go for this discussion of like phase diagram here.

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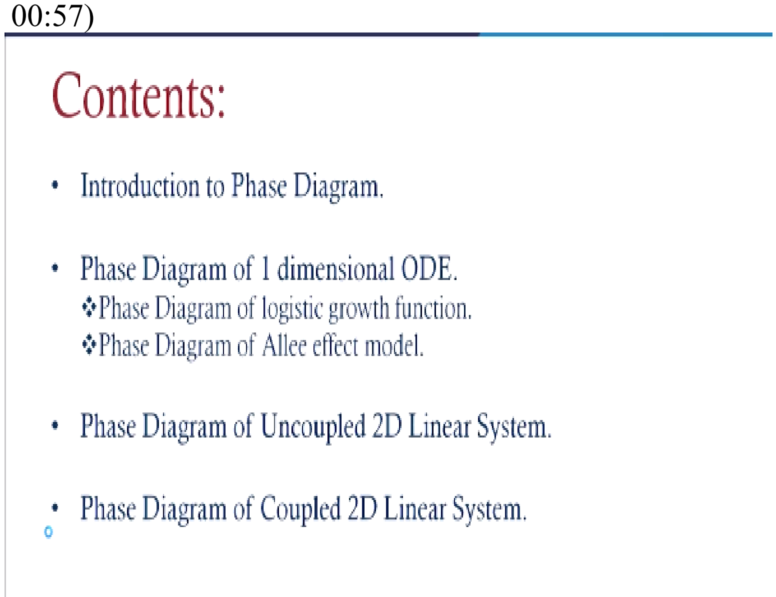

So first we will just discuss here like what is phase diagram is? Then the different class of phase diagrams this means that in one dimensional ordinary differential equation how we can just do this phase diagram analysis? That we will just discuss then we will just go for this phase diagram analysis for 2d uncoupled system, then we will just go for coupled system here and especially if you will just see here for uncoupled system it is very easy to do this analysis in a graphical sense. But in case of coupled analysis we can just find like lots of troubles and difficulties which we can handle by using different techniques so first if you will just go for this introduction to phase diagram here.

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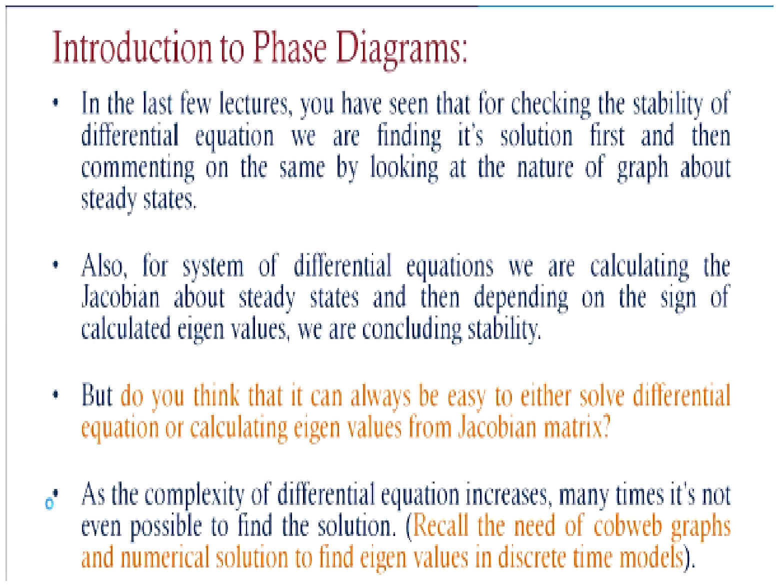

That especially why we are just going of this phase diagram is that? When we are just seeing that this stability of differential equation we are just finding and it is very difficult to find the stability analysis in case of first and our differential equations, are seconded differential equation. Then we cannot comment also anything by looking here the necessary of the graph about the steady states.

So especially whenever we are analyzing any population dynamics model first we are just finding this steady state population level and the steady state population level difference on the fasted on the differences on it should equals to 0 there, and especially if we are just getting some points and that point whether it may just provide a steady state or whether it is just providing a like stable solution or unstable solution that we cannot predict it out.

Also for the system of differential equation we are calculating the jacobians about steady states and then depending on the sign of calculated Eigen values, we are concluding the remark of the stability but especially it is difficult to find this jacobian matrix or when we are just calculating the Eigen values and we cannot say that what type of we have especially it can just shown whenever we are just going for like stability condition and un stability conditions.

But do you think that it can be always be easy to either solve differential equation or calculating Eigen values from jacobian matrix. Especially now it is not easy always, so here is the complexity of differential equation increases many times it is not even possible to find this solution so especially if you can just remember this cobweb graphs and the numerical solution to find this Eigen values for descriptive models.

So it is very difficult to find this Eigen values also to find this solution in cobweb of graphs there. So if we cannot handle like complex differential equation in EG form to find this like stable conditions or unstable conditions at these steady states, then it is an easy or is difficult to go for some solution process where we can just find these solutions in complete form. So for that we are just going for this phase diagrams here.

And for the phase diagrams especially, we can proceed with two methods, so whenever we are just finding these difficulties first is the numerically solution for the differential equations. Then the graphical solution of a differential equations and if we are discussing only the graphical solutions the graphical method which just we are going to discuss is known as phase diagrams here.

Especially if you will just see that the solution process we have discussed also this is graphical solution in cobweb graphs also there but that is for steady state solutions not to take like stability or unstability conditions but here if we are just proceeding towards the numerical solution of any differential equations which is just representing this your population balance modeling. Then we have to first start a stress analysis of the solution in a graphical form.

And this is nothing but the like phase diagrams especially it is said to be. The graphical method which we are going to discuss is known as phase diagrams also called phase portraits here. In layman terms, I have explained that phase diagrams are similar to be painting or cartoon sketch of an image. Obviously if we can just see this like when some thefts has been occurred so if somebody has seen his figure then this case at least has been called and asked to what is it is like fundamental sketch of his figure.

So then this like eye witness he is just providing the idea that you can just put his hairstyle this one fair style there and eyes are there so based on that this sketch at least put a like graphical abstract that how he looks to be. The same analysis we will just do for this phase diagram here. So appending her case gives us only the love some idea about the characters or the objects in it but not give us the complete clear vision of this same.

Similarly phase diagram of differential equation provides us certain information to steady the stability but not gives the complete solution here, this means that obviously if we are just plotting this abstract solution of any differential equation in a graphical sense, so completely it can just provide this physical behavior how it is just performing in the system not the complete scenario point to point where it has changes occurring where this like very as soon as coming that we cannot predict it out.

Hence instead of finding the exact solutions if you suppose it is difficult to find the exact solution in case of the numerical solution or in analytical approaches we can just visualize the things from the graphical sense and we can analyze the differential equations qualitatively and that is why this phase diagrams are qualitative approach to solve the differential equation. So obviously it is not providing the exact solution of the differential equation and we are just visualizing this system in a like graphical sense here.

So that is why we can just say that it is just a approximated solution of the system and approximate solution especially it can be just provide your qualitative idea that how the solution behaves or performance there. So that is why this phase diagrams are qualitative approach to solve the differential equation we are just estimating that in that form.

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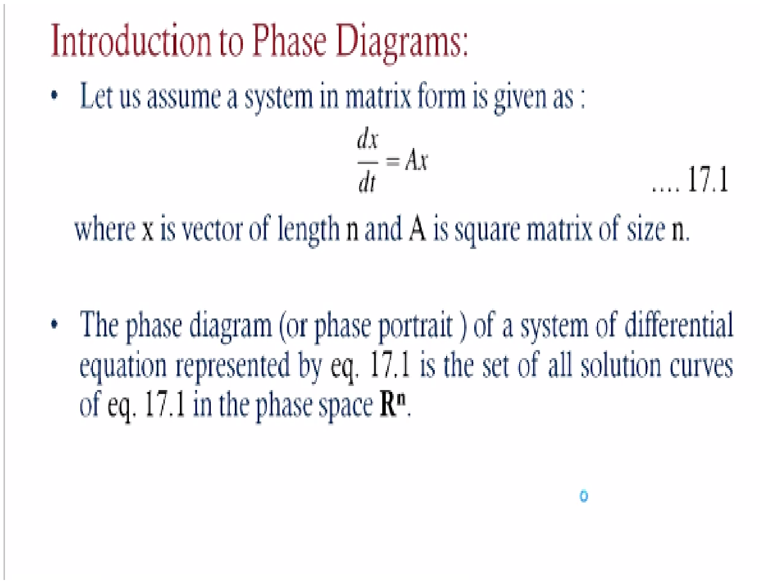

Let us assume a system of matrix suppose if we are just going of this phase diagram here, first we are just considering a firsted or linear equation here that is in the form of DX by DT equal to AX and if it is a like system of equations then x is a vector of length n here and a is a square matrix of size n and then this phase diagram or the phase portrait of this system of differential equation represented by DX by DT equal to AX is the set of all solut6ion curves in the phase space R to the power N.

Since we will have like N variable are involved with this equation or we will have like N set of equations then it can just provide this graph in A N dimensional space. Especially we are just considering the real space so that is why we are just writing here r to the power n.

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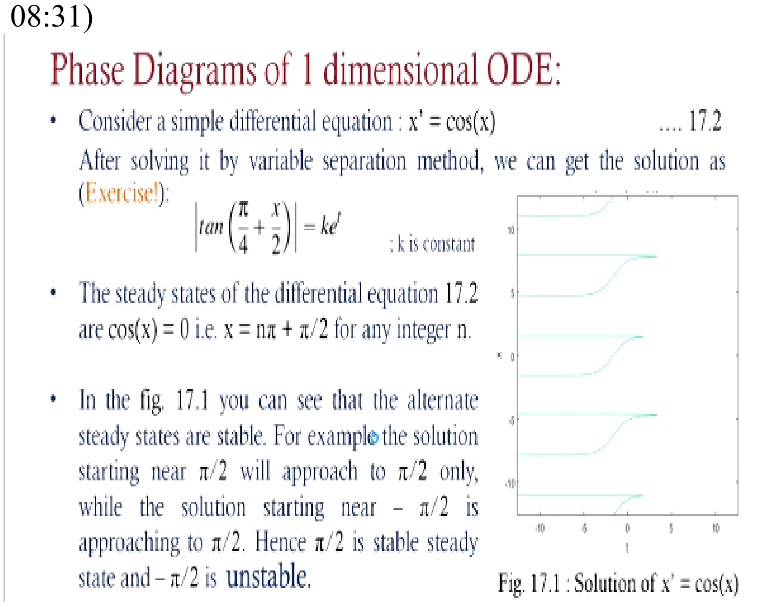

So if you will just go for a simple analysis of 1 dimensional ordinary differential equation, consider differential equation that is in the form of a, x appose equals to cos x. I can just write this one as DX by DT is equals to cos x appose and if we will just find the solution of this differential equation using variable separation method we can just write this one as DX by cos x that is equal to DT and if you will just integrate both the sides there we can just write LN of sac x+ 10 x.

This equals to P + LN of C her suppose or LN of K I can just write and it will just try to find the solution for the system here so LN of it can be written as 1 by cos x here + sin x by cos x, which can be written as t+ LN K here and it will just go for like further manipulation of things then we can just write this one as 1+ sin x by cos x which can be written as p + LN K here and then it will just go for further analysis of this things then we can just write this one as LN of it can be written as sin x by 2+ cos x by 2.

Square by cos x so cos x can be written as like I can just write cos square x by 2- sin square x by 2 here and it will just cancel it all both these terms in the numerator and denominator then we can just find this as a like LN of 10 of 5 F, 4 + XY 2 which can be written as T= LN K there and the final solu6tion is can just be obtained as like 10(5) by 4 XY 2 this equals to KE to the bar t here.

Since I have not done the complete calculation here since it is faces less here but obviously you can just do this one. Since it will cancel it what how it one it or here sin x by 2 = cos x by 2 by cos x by 2 – sin x by 2 it will be there. So it will just taken one of cos x by 2 then it will just give you this 10 from there and obviously it can be written as like 10(5) by 4 + x by 2. Where K is a constant here and the steady states of this differential equation of this equation it will just see.

Since steady states are assumed whenever we are writing x 10 is equal to 0. So this means that cos x is equals to 0 we are just putting her so if cos x is equals to 0 that is nothing but X equals to n π +π by 2 here. For any integer it is a n here and it will just analyze this plot in a graphical sense here you can just see that alternate steady states are stable here for example this solution starting near π by 2, π by 2 means this is just coming here.

We will approach to π by 2 only it will just plot or close to its never end value then obviously it will just come to that values and obviously it will just see here when we are just starting this values near – π by 2 also here, then suddenly this graph is approaching towards π by 2 here also. Hence we can just say that π by 2 is stable, steady state which is just representing a stable steady state here and –π by 2 is unstable.

Since whenever we are just portorabing any L is close to π by 2it is not going to –π by 2 here. It is just tending towards π by 2 here, so that is why we are just considering here –π by 2 is a unstable state here and this can be achieved it will just go further also like 3π by 2 then this value which is also going towards the next immediate cyclic value there. so similar way we can just find like negative values and then we are just find that this periodically just providing a stability with respect to other values here and this is especially we are just analyzing in a one dimensional sense here.

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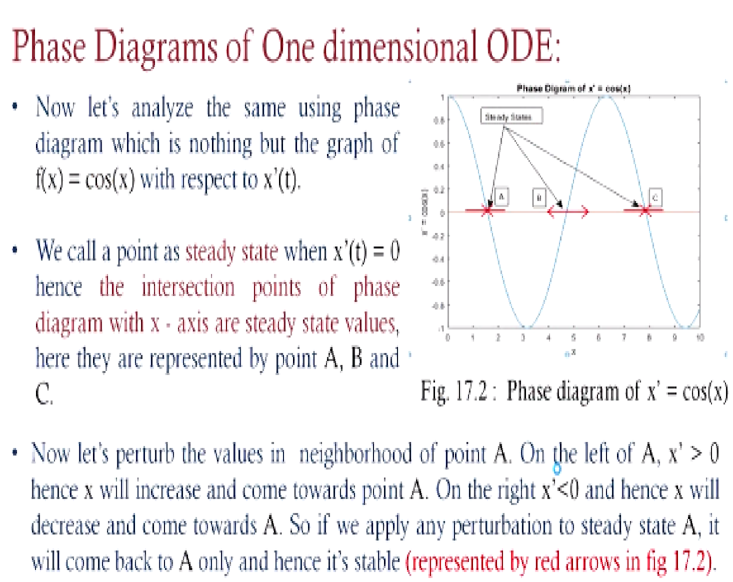

And it will just go for like further analysis of this phase diagrams in one dimensional sense here, even if you do know what is what it will just behave inside the system and from the like function itself it will just upload this graph here, suppose cos x graph just plot here, then we can just see that X des T is nothing but it is just represented in the form of F of X which is just written as cos x here and this is nothing but the graph of cos x.

And we can call a point as steady state since at that point we are just assuming that x des is equal to 0. Hence the intersection of this points of his diagram with the x, x axis it is just giving you the steady states here. Since especially we are just finding here x des equals to f of x this is equals to cos x and where ever cos x is 0 that is nothing but he point of steady states and especially from this graph you can just find that A and B and C are three points where this cos x intersects with the x, x's which is providing this steady state matrix.

Now, it will just pottered the values in the never hood of point A suppose, so any close never hood points it will just consider. On the left of point A we can just find that X des greater than 0 here. Since x will increase and come towards the point a here and it will just start form their height here we can just find that since x des is less than 0 here if you just see and at these levels you are just finding this is positive values.

Hence x will decrease and come towards the point A here it will apply any perturbation to steady state A here it will come back to point A only hence it is stable. So that is why we have written here it will just pottered any where use in the never hood of A since X des is greater than 0 on the left part of A here since it is positive here and it will just increase and come to our since it is increasing in this direction so it will just come towards the point A here.

And it will just see this right hand side here then you can just find that this is monotonical decreasing since this represents here negative values and this will just decrease afterwards so that is why it will just approach towards the point A here. So which is a represented by red arrows for this figure and it will just go for like further analysis of all other points here you can just find that for the point p here you can just find that this left hand side it is monotonically increasing.

Increasing means it will just move further outwards and it will just see here this right side this is negative values here it will just move further to the right words so that is why it will just move in this direction this will just move in this direction here. So that is why both this points will move far away from that point P. So that is why b is unstable point here and it will, just point C here then the left part it will just see here.

Left part means this is like monotonically increasing function here. Monotonically increasing function means afterwards it will also increase then it will just move towards the point C here and it will just see the right force on here then right force on it is monotonically decreasing so it will just decrease again it will just come towards the point C. So that is why we are just saying C is a stable point here and A is a stable point here but B is an unstable point.

Since the point C if you are just pointing here then this is like monotonically increasing means it is going far away and monotonically increases means it is also going far away. So this is especially called phase diagram of X des equals to cos X here. So that it is written here with similar arguments we can show that steady state point is unstable while point C is stable.

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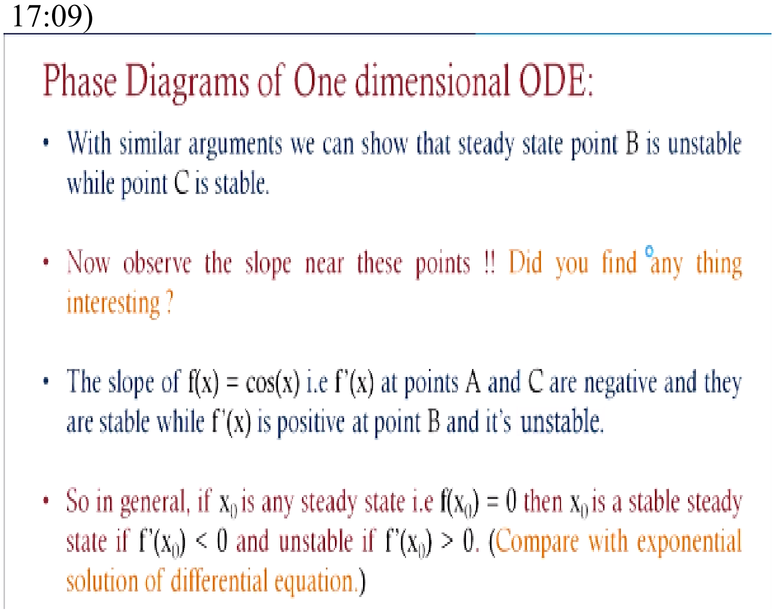

Now it will just observe the slope near this point, so have you find anything interesting? It will just see the slopes here especially the positive slopes especially this just moves like this one. It will just to see here or it will just plot the slopes here and it will just plot this slopes here and it will just plot this slopes here, so especially this slopes are like especially this called negative slopes and especially this is called positive slopes.

Since you know that the function whenever we are just putting these slopes like Y equals to X so then it is just representing the positive slope it will just find lit rain we are just saying here. So it will just see here the slope of FX equals to cos X I have des X point A and C are negative especially it will just say this is negative slopes but at the point B, you can just find that this represents a positive slope there. \

So hence it represents like if it is in steady state and especially we can just say that when the slopes are positive then we are just finding this points are unstable and if this slopes are like negative then the point represents the stable state there. So that we have just retain it for here in complete sense this lope of f of x and f des of x at points A and C are negative and they are stable while f des X is positive at point B and it is like unstable here.

So in general if X0 is any steady state that is f of x0 equals to 0 then x0 is a stable steady state if f des x0 is less than 0 and unstabele if f des x 0 is greater than 0. So we can just compare these solutions with exponential solution of differential equation, especially if you will just find like any differential equation also so the solutions are represented in the form of like 0 equals to e to be 4 like λ value g if you will just write, so it will be written as e to power –λ t.

So if λ is negative you are just saying it is like representing a stable solution and whenever λ is positive then you are just saying it is like unstable solution. This means that it will just take this difference like differential of this equations, suppose if you can just represent this solution in the form of like x as e to the power 2 t suppose and another solutions suppose x equals to -3 t then it will just take the differences her like DX by DT you can just find that this is true it will be for 2 t and DX by DT this is like -3 e to the power -3.

So you can just find that this represent the negative slope here this represent then the positive slope in case of positive slope we were just finding the system is unstable and in case negative slopes you are just finding that the system is stable there, so that we want to say that is why we have just written that compare with exponential solution of differential equations. So it will just go for a phase diagram analysis of logistic growth function here.

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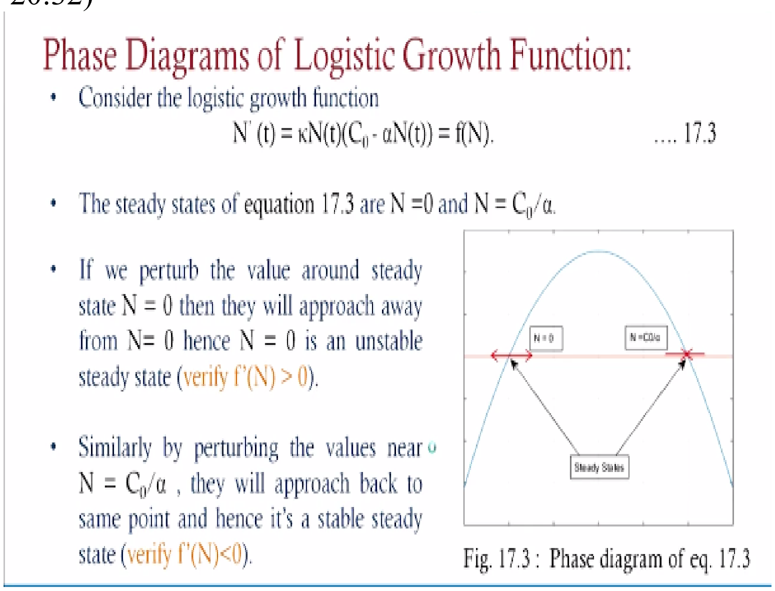

Consider the logistic growth functions suppose especially we have discussed this equations in the earlier lectures that is in the form of n des t equals to k n of t c0-α n of t, where c is nothing but this like concentration or this food or like the supply of the concentration it should be provided there and this n is represents the population growth level they are population size n of t and k is the rate of this population growth level or decline it can just be controlled.

Then this steady states for this equation that is can be represented in the form of a industry it goes to 0, so steady state means especially fasted or difference equals to 0 we are just considering it will just consider and thus says t equals to 0 here then we can just consider either n equals to 0 or n can be taken n as like c0yα from this equation. And it will just see here this means to find like the nature of the stability for this function here.

So we have to like potter the value around the steady state values so if you will just plot in a graph here we can just find that n equals to 0 suppose it is placed here and n equals to c0yα at this point, then at any cost to 0 it will just potter an value, then we can just find that in the right hand side it is monotonically increasing function, so this will just go and increase in and it will not tend towards the point n equals to 0.

And it will just see the right hand side here this is a monotonic decreasing function so that is why it will just go further outwards, so that is why we can just say that n equals to 0 is unstable point here. And it will just go towards the point c0yα here and it will just see the right hand side here, then this function is monotonically decreasing then again further it will just decrease towards the point c0yα here hence we can just say that this is a stable point here, and this is unstable point.

And that we have just written also it will just put the slopes also here then you can just find that this represents a positive slope here and this represents a negative slope here and especially from the last slide we have just concluded that when the slope is negative then we are just saying the point is stable and when this slope is positive then we will have a unstable point here, so that you can just verify yourself.

So then we have just written that one in a like phrase hereby pottering the values near n equals to c0yα, they will approach back to the same point and hence it is a stable steady state, so you can just verify while considering like this slopes would been negative there.

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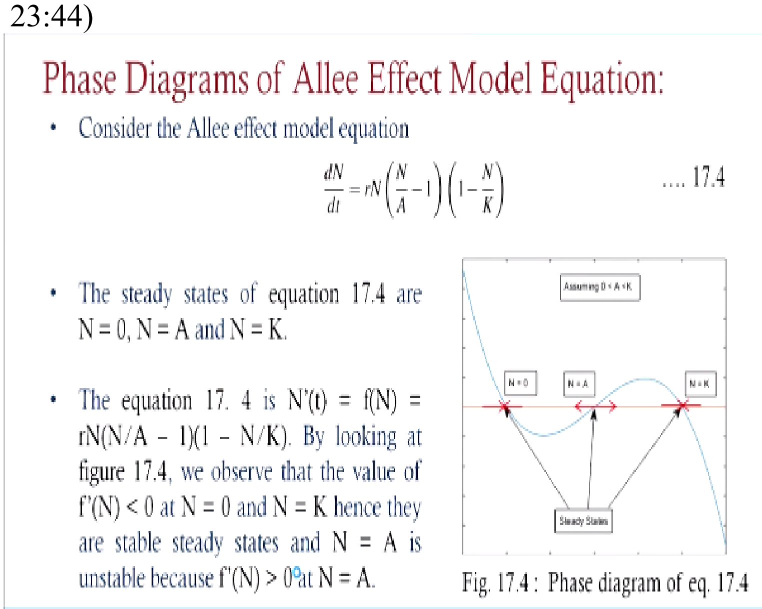

And this phase diagram analysis it will just consider for allee effect model, then for the allee effect model we can just find that this equation is represented in the form of dn by dt this equals to rn in to n by a-1 into 1-n by k and if we want to find this steady state here we can just put here dn by dt equals to 0, this means that you will have like n by n equals to 0 is a first point and second one is like n by a-1 this equals to 0 this implies that n equals to a .

And second term it will just see third term it will just see then 1-n by k this equals to 0 this implies that n equals to k. so three different points we are just getting for this steady states n equals to 0, n equals to a and n equals to k. and from this equation we can just find that n des t can be written as like f of n here which can be written.

And it will just plot this graph then we can just find that this three poi8nts n equals to 0 n equals to a and n equals to k. we can just find that f des n is less than 0 this means that at this point it will just put the slope here and at this point it will just put the slope here and it point it will just put the slope here then you can just find that this, at this two points we will have negative slopes and at this point we will have positive slope here.

And especially we can just find that this function behaving like it is just going at outwards from this point when we are just approaching towards this point A here. Since the left hand side if you just see here the point is further monotonical decreasing and at this level in the right hand side it will just see the function as further increasing outwards. So that is why it will just see here that n equals to A represents the unstable point and N equals to 0 and n equals to k represents stable steady state here.

And it will just go for like further analysis since both that we have just discussed till now are like one dimensional differential equations, now it will just go for two dimensional differential equations here, so two dimensional linear difference system if you will just consider here.

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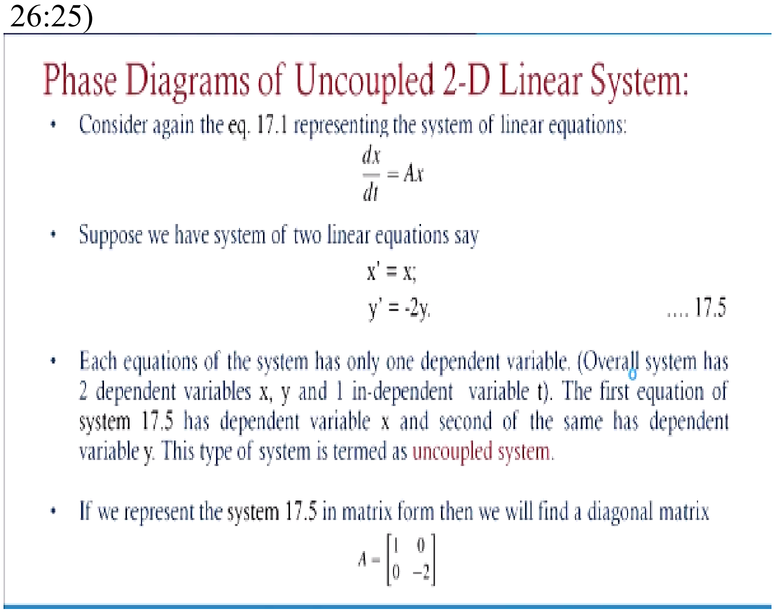

In the first phase it will just consider a uncoupled system here consider the same equation that is dx by dt equals to x whatever we have just defined for the equation 17.1 and if will just consider to linear set of equations in that form that is suppose x des equals to x and y des equals to -2 y here. Each of this system has only one independent variable it will just see here, sorry one dependent variable.

Since it is written as dx by DT equals to x here and Dy by DT equals to – 2y here. So x and y are dependent variables and t is the independent variable. So that is why we are just written each equation of this system has only one dependent variable x it is present here overall the system has two dependent variables x and y and one independent variable. The past equation of this system it will just see here has dependent variable x and the second has like dependent variable y here and this system since we have not coupled this system means x is not involved in the second equation and y is not involved in the first equation.

The this type of system is termed as uncoupled system, if we represent the system in a matrix form here since we have like diagonal matrix we have to form it out, so that is why it will just write this one in a jacobian form here. Then we can write the first derivative here that is with respect to x as 1 here and with respect to y there is no variable so you have just consider 0 here. Second one it will just see there is no variable x symbol in this equation. So that is why 0 it is here, it will just take the firsted or differences with respect to y, this is just giving in -2, so this matrix represents as it such giving 100-2 here.
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Phase Diagrams of Uncoupled 2-D Linear System:

- **Note:** The eigen values of a diagonal matrix are same as that of the entries at the diagonal
- So the eigen values of matrix [A] = 1 and -2. Hence the solution is $x(t) = c_1 e^t$ and $y(t) = c_2 e^{-2t}$. So you can observe that how easy is to find the solution of system when it's uncoupled!!.
- To draw the phase diagram of the same, first we will find a relation between two in-dependent variables x and y. By using simple algebraic method, we can find:

$$x^2(t)y(t) = k \text{ where } k = c_1^2c_2. \qquad \text{.... 17.6}$$

And if you will just see the Eigen values of this diagonal matrix are this same is that of enrich of the diagonal since already we have that if it is upper triangle matrix or if it is only diagonal entries are present here then the diagonal elements are nothing but the Eigen values. So the Eigen values of this matrix day is 1 and -2, that is nothing but the diagonal entries here and it will just write the solutions x of t that is nothing but c 1 into the power t y 2 equals to c2 e to the power -2t.
You can observe that this is very easy to get the solutions when the system is uncoupled especially it will just directly finds the solution you can also find that is nothing but dx by dt equals to x here, so dx by x it can be written as dt here, so just integrate this one and you can often all this same solution and if we want to plot this function in a phase diagram or if we want to find the phase diagram of this same.
First we will find a relation between this two dependent variables x and y by simple algebraic method we can just find that the relationship as x square in to y this is suppose k, since we will have a uncoupled system here x equals to c1 it e to the power t and OI as c2 e to the power -2t and since if you just see that x is dependent on t here and y is also depending on t here so we can just establish this relationship in the form of x and y here.
And if you just take like square of this function here so then it can be written as like c1 it will be for 2t here and it will just see here like y equals to c2 e to the power -2t this means that we can just establish this relationship x square in to y so it can 2t and – 2t so it can be like, since c1 is

involved as the constant her and c2 is involved the constant her then obviously if we will just take this constant as c1 square and c2 here then we can just write this is capital k here.

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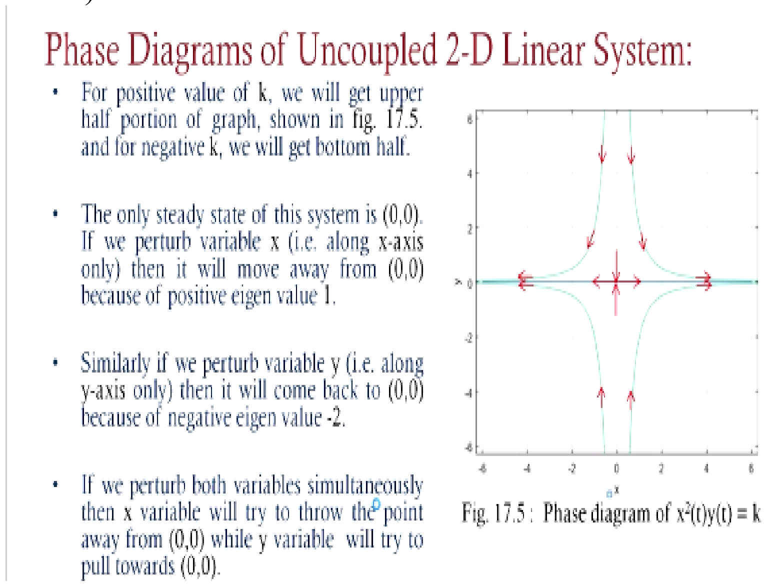

And if you 3will just go for the analysis of this equation here for positive value of k we can find the upper half position of this graph, this is shown as in fig. 17.5 here and for negative k we can just find this lower half of this graph and the only steady state is achieved at 00 level here and if you will just potter these values then we can just find that along the x axis if you will just see the values are moving away.

But if you will just see the y direction the value are coming towards the origin here, so x axis means this is just going outwards and y axis means this is just coming towards the origin here. So if you will just analyze this one that we can just find 5that if we pottered variable x along x axis this is nothing but x axis here, then it will move away from 00 because of positive Eigen values since two Eigen values we are just obtaining.

And for like one we are just finding that it is just going away from this origin and for -2 we are just finding that this is coming towards the origin there and that we have just written similarly if we plot out the variable y along y axis then it will just come back to the point 00 because the negative Eigen values here and if we will potter both these values at a time then when this -2 values like Eigen value for -2 if it is just forcing towards origin the this x value which is positive it is just taking outwards.

So then compile it can just represent an unstable condition or the point 00. So this is for the uncoupled system here.

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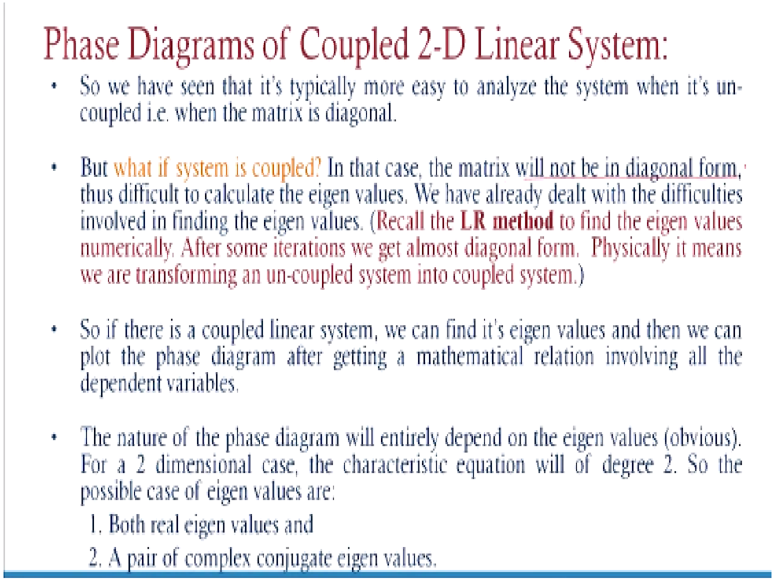

And if you will just go for like coupled 2d linear system then we can just fins that it is very difficult to handle since in an uncoupled form we are just obtaining this horizon in an independent form and we are just trying to establish these lessons within these two variables. And the matrix is diagonal also it is just coming so that is why easily we can also find this Eigen values in a compact form. But if the system is coupled in that case the matrix will not be in diagonal form.

Thus it is difficult to calculate the Eigen values we have already did with the difficult is in involved in finding the Eigen values, since we have discussed like power method and LR method to compute this Eigen values in the previous lectures and in that cases especially like K power method or LR method since we are considering here only LR method. Since after few, iterations we tried to get it in a diagonal form of mat ricers and once it is in a diagonal form then we can just predict that diagonal entries can be the Eigen values.

Physically it means that we are just transforming uncoupled system into a coupled system there. This means that the final form if certain iterations we are just doing or we are just trying to formulate like coupled matrix which has like all the entries are there in to a diagonal transformation form then we are just trying to find this solution that is in uncoupled system to a coupled system there or sorry if it is like coupled system then we are just trying to set this equations in a uncoupled form.

Then we can just represent that one in diagonal of form of this entries, so if there is a coupled in your system we can find its Eigen values and then we can plot the phase diagram after getting mathematical relation between them and the nature of the phase diagram will be entirely different in on the Eigen values obviously if you will just see in two dimensional case this characteristic equation will be of degree two there.

And so the possible Eigen values arte like both this Eigen values will be real or a pair of complex Eigen values we can just get over there and if we will just go for like theoretical analysis of this 2d linear system for the stability then we have to find first the Eigen values so let the Eigen values of 2d linear system are suppose a and b, so we are just considering first in the case of like when the Eigen values are both real here.

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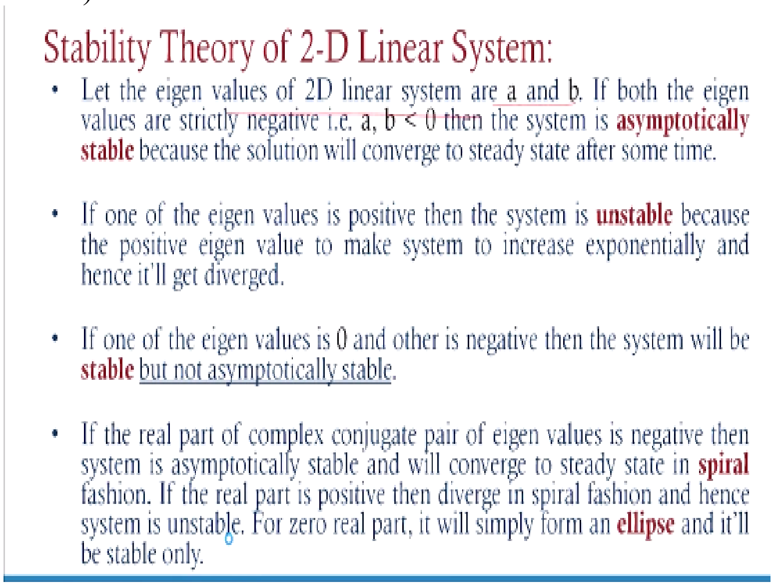

So if Eigen values are both real in the first case if you are just considering here 2 Eigen values are, a and b and if both the Eigen values are strictly negative suppose then we can just say that we will have like asymptotically stable condition because this solution will converge to a steady state after certain time or it will just take some time to reach to the steady state and if one of the Eigen values is a positive than the system is unstable.

Because the positive Eigen value to just make this system to increase exponentially and hence it will just get divorced and if one of the Eigen value is 0 and one of the Eigen value is negative then we can just say that this system is completely stable and it is not asymptotically stable since in asymptotically stable sense it just takes some time or periodically it will just come to what is the point of stability there so that is why it will just take some time so but particularly if one of the Eigen value is s0 and another one is negative, directly we can just say that it is stable.

And in the complex case if the real par of complex conjugate pair of Eigen values is negatives then the system is asymptotically stable so it will just form the spirals and will converge to the steady state in spiral fashion and if the real part is positive then it will obviously it will just diverge in spiral fashion and hence the system is unstable.

And if we will have like 0 real part it will simple form analysis and it will be stable only, since a completely it will just form a ellipse there and we can just say that the system is just lying in that phase only so that is why this system will represent a stable condition. So if you will just go for like role of Eigen vector in phase diagram.

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Role of Eigen Vector in Phase Diagram :

- For general case, it may be difficult to find the relation between dependent variables especially when both the eigen values are complex numbers. To avoid this, we will use the concept of eigen vector to draw the phase diagram.
- The relation between eigen values and eigen vectors is given as $Av = \lambda v$ where λ is eigen value and v is corresponding eigen vector.
- Consider again the uncoupled linear system represented by eq. 17.5. The matrix of the same is A = diag[1, -2]. The eigen vector for $\lambda = 1$ is $v = [1\ \ 0]^T$ and eigen vector for $\lambda = -2$ is $v = [0\ \ 1]^T$. (verify)

Then in general it may be difficult to find the relation between dependent variables especially when both these Eigen values are complex numbers to avoid this we will try to use the concept of Eigen vector to draw the phase diagram, so when this Eigen values are represented in a complete form then we can just say that just go for like analysis of the steady state points where there it is just represents a like stable solution or unstable solution.
But if it is complex pairs it is difficult to handle, so first we will just establish this relationship with the Eigen values with the Eigen vectors and this vectors representation it will just provide yours the idea in the phase diagram that whether this system is stable or not. So if we want to find this Eigen vectors for the system especially we are just writing this system as in the form of a b equals to λ b where these vectors here and each of these Eigen values will have a corresponding Eigen vector.
And since Eigen vectors are represented corresponding to each of the Eigen values, so Eigen vectors are not unique so in the first sense we can just consider the uncoupled linear system so if you will just this uncoupled linear system that we have just considered in example like equation 17.5 here, we will have this diagonal entries this one an -2 if you can just remember, yes! 1 and -2 and especially we are just writing here this, diagonal entries are 1 and -2.
The Eigen vector for λ equals to 1 if you will just consider since A is the complete system we have just written and we can just write this one as like A – λ in to v this equals to 0 here, so if you will just abstract this one from this diagonal elements like A-λ especially we can just since A is a matrix here, then obviously we are just writing here A is system that is in the form of like 1 0, 0 -2.
1 0 and 0 -2 it is just given corresponding to λ equals to 1 you can just find that we will have a vector that is in the form of 1 0 and corresponding to Eigen value λ equals to -2 you will have a vector 0 1 there. So this Eigen vectors it will be written in the form of like here there 1 0 and second one is 0 1 here and if you just draw this Eigen vectors so especially, so in the first case you can just find that this is nothing but the x axis here that is x equals to 1 and y equals to 0. In

the second case if you will just see this is nothing but x is equals to 0 and y equals to 1 here that is nothing but y axis here.
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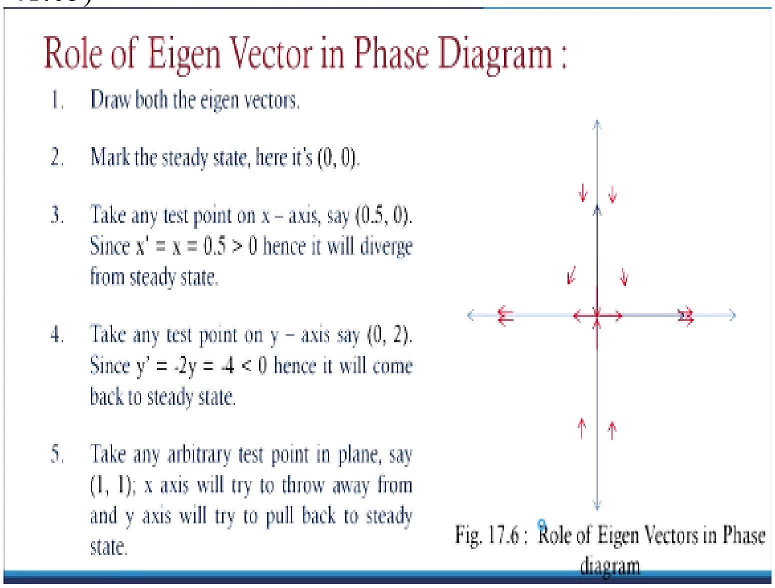

Fig. 17.6 : Role of Eigen Vectors in Phase diagram

So if you will just draw this Eigen vector here so this steady state it is just achieved at 00 and both this x is that is x axis and y axis it represents this vectors here and it will just consider any test point on x axis suppose I am just considering as the point suppose 0.50 suppose.

And at this point if you will just see here so x des equals to x so that is nothing but 0.5 so this slope is positive here so it will just give an unstable steady state there and which will just divorce there and if you will just take any test point in the y axis suppose 0, 2 suppose you can just consider your equation is y des equals to – 2y and then 2 in to -2 this will just give you -4 which is less than 0.

Hence this slope is negative, so it will again back to the steady state so if you will just see this combined form of this vectors here, so first if you will just see here so x axis and y axis we are just considering as the vectors here and at the origin if you see, since the Eigen vector corresponding to y axis it is negative so y axis means it will just come towards the origin here but if you will just see the Eigen vector for, sorry Eigen value for x here then it is just giving a positive Eigen value so it will just move far away from the your steady state point.

Then if combinely if you will just take then forcefully but for this x axis this will just move outwards towards there itself, so that is why you can just find that this arrows are coming towards the origin along the y axis but it is just moving outwards in the x axis there. So that is why it is just written as if you will just consider any arbitrary points suppose 11 here, so you will try to move away from y axis will try to put back to the steady states.

So that is why it is just coming in this fashion here, this means that it is just forced towards the point origin along the y axis and just take away along the axis there.

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Example on Phase Diagram :

Question: Consider a system with matrix $A = \begin{bmatrix} 0.50 & 0 \\ 0.75 & -1 \end{bmatrix}$

Solution: Clearly the eigen values are 0.50 and -1 because it's lower triangular matrix. The eigen vector corresponding to 0.50 is $[1 \;\; 0.5]^T$ and the same corresponding to -1 is $[0 \;\; 1]^T$ (Exercise!). The solution of each dependent variables will be:

$$x(t) = c_1 e^{0.5t} + c_2 e^{-t};\; y(t) = c_3 e^{0.5t} + c_4 e^{-t}.$$

Now find the relation among all c's (constants) by substituting these equations back in system. They will satisfy $x'(t) = 0.5x(t)$ and $y'(t) = 0.75x(t) - y(t)$ simultaneously. After solving for the same you will get,

$$c_1 : c_3 = 2 = 1/0.5 \text{ (observe the eigen vector corresponding to 0.5)},$$

$$c_2 : c_4 = 0 = 0/1 \text{ (observe the eigen vector corresponding to -1)} \textbf{ Interesting!!}.$$

(here you can also observe that if an eigen vector is multiplied by any constant value, it will remain eigen vector corresponding to the eigen value. This means an eigen value has more than one eigen vectors obtained by multiplying primary eigen vector by any constant value.)

So if you will just consider for like martrix where this diagonal entries are existing also all other entries are present there, so like if you will just consider matrix in the form of a equals to 0.50, 0 0.75 and -1. Clearly the Eigen values are if you will just see this is an upper triangular mat5reix hence this diagonal entries are this Eigen values only. So the Eigen values are 0.50 and -1 here and corresponding to this Eigen value 0.50 so the vector is coming as 1 0.5.

And if you will just find this Eigen vector corresponding to -1 it is just coming as 01 here, obviously you can just find it in direct form, m-λ I into v this equals to 0 so since λ is a known to you and then directly you can just find the vector and the solution of each of this dependent variables you can just write it as x of t equals to since first Eigen values it is just giving you as -0.5t +c2 in to the 4 –t here.

So this would be like 0.5 here and second Eigen values since corresponding to this one it will be -1 here, so that is why it is like c2 in to the power –t and y t it can be written as c3 in to the power, sorry 0.5t +c4 in to the power –t if you will just find the relation among this c is by substituting this back in to the system. Then obviously you will have like x des t if you will just find here so this will just written as 0.5xt and y des t it can be written as 0.75x t – y of t.

So after solving this we can just find it as like c1 by c3 this is nothing but 2 you can just find that this is nothing but the ratio of two Eigen vector there but since vectors are not divisible we cannot write in that form and similarly if you just see here c2 by c4 this is 0 this is nothing but 0 by 1 itself and you can just find that this is a Eigen vector corresponding to -1 is a this one 0 and 1 Eigen vector corresponding to 0.5 it is 1 and 0.5.

So this Eigen vectors ads I have told in the previous slide that they are not unique and it can be multiplied by a constant value that it can be transformed and this means that Eigen value has more than 1 Eigen vectors often by multiplying primary Eigen vector by any constant value.

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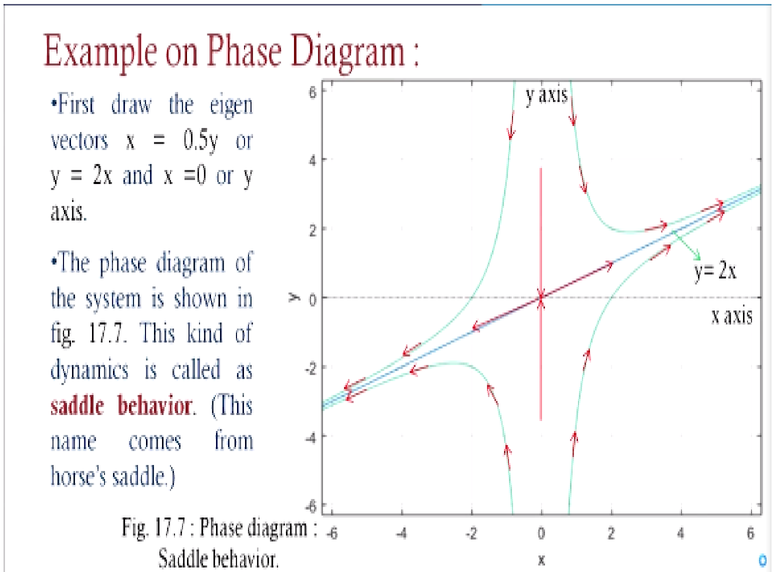

So if you will just draw the phase diagram here that is first draw the Eigen vector for x equals to 0.5y here this means that 2x equals to y we can just say, sorry yes! 2x equals to y or y equal to 2x we can just write in x equals to 0 and y axis all are like the lines presented here so if you will just see here 00 point if you will just noted down here then along the y axis since the Eigen value is negative so it is just approaching towards the point 00 and the Eigen values if you will just see for this x axis here there is a both are like 0.50 it is just present here.

So that is why this values are taken out from this point along the x axis and if you will just analyze here the phase diagram of the system then this kind of behavior it is called like saddle behavior. Since at each of this Connors so this comes from the horse saddle if you will just see this diagram here and especially f you will just see that y axis along the y axis this points are forcing towards the point 00 but it has been taken away since the x is representing the positive Eigen values there.

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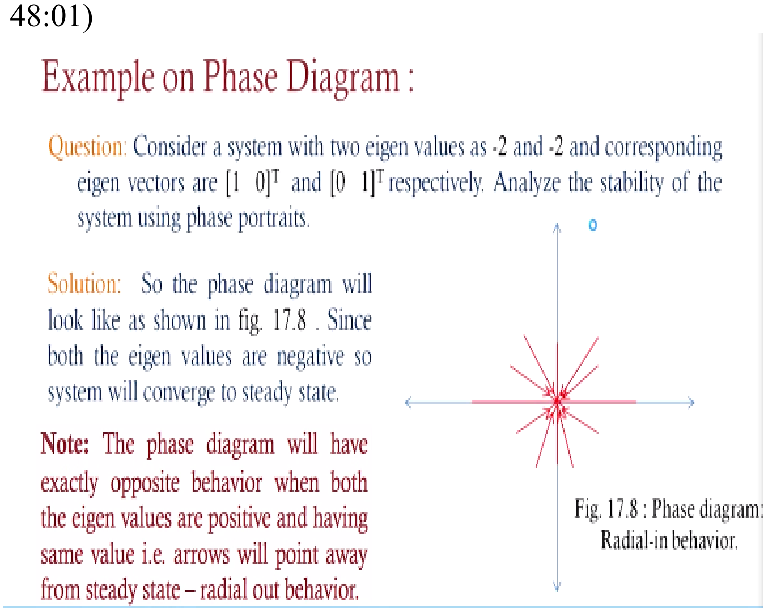

And if you will just go for like further analysis so consider a system with two Eigen values as negative values suppose here so if the two Eigen values are negative here corresponding to this Eigen values you will have the Eigen vectors suppose 10 and 01 suppose. Then x axis a y axis are the like Eigen vectors it is just represented so then if you just see here that obviously -2 are the corresponding Eigen values here.

Then along this x axis you can just find that these values are approaching towards origin here and any arbitrary point if you will just choose like in the closure never hood of 00 then obviously that will also point towards this origin here, so the phase diagram will look like this one here, since both this Eigen values are negative since along the x axis it will be approached towards the 00 point and along the y axis it will also approach towards the 00 point here.

So the phase diagram will have exactly opposite behavior if you just consider both this Eigen values are positive here also, depends that if both this value are positive so this values will go outwards from that and you can just find that if any arbitrary point if you will just chose this point will move outwards also from there.

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Example on Phase Diagram :

Question: Consider a system with two eigen values as -1 ± i2 and corresponding eigen vectors are $[2 \;\; 2]^T \pm i[-1 \;\; 1]^T$ respectively. Analyze the stability of the system using phase portraits.

Solution:

Note: If the eigen values are a ± ib and corresponding eigen vectors are **p** ± i**q**, where **p** and **q** are vectors, then general solution of system is given as

$$\mathbf{x}(t) = e^{at}(c_1\mathbf{p}\sin(bt) + c_2\mathbf{q}\cos(bt)).$$

where c_1, c_2 are constants. Here the value of $a = -1 < 0$ so nature of phase diagram will be spiral.

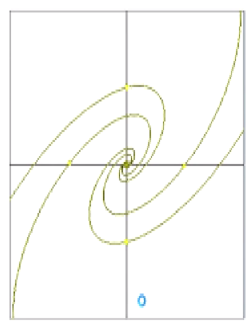

Fig. 17.9 : Spiral behavior.

And if you will just go for analysis of this complex Eigen values so first if you will just consider Eigen values like -1 + or – I2 here, so since we are just considering 2 Eigen values this means that -1 + I of that 2 and -1 -2 her and corresponding Eigen vectors you can just finds it as like suppose 22 + or – I -1 -1 here respectively, if you just analyze this system then we can just represent the Eigen values this is in the form of A + or – I β here.

And the corresponding Eigen vectors especially we can just write it as like suppose P = or – IQ and especially we know that this solution is completely written in the form of X of P equals to E to the power AT c1 p sin BT+ C 2qcos(b t). Already it is explained like differential equations second one as differential equation how we are just done in, this solution and obviously c1 and c2 are constants and if you will just see here the rou7te is coming as in the form of a + or – I β so that is why it is written as b to the power AT c1sinpsin (b t) and c2qcos (b t) here.

And this Eigen vectors which are associated as p and q here itself and if you will just consider like an equals to -1 suppose the nature of the phase diagram will be spiral if you just see, it is just moving like this fashion here. Since this value is negative so always exponentially it will just try this back towards the point 0 here. And if you will just consider the term suppose x 1 t as in the form of psin2t and +qcos2t here.

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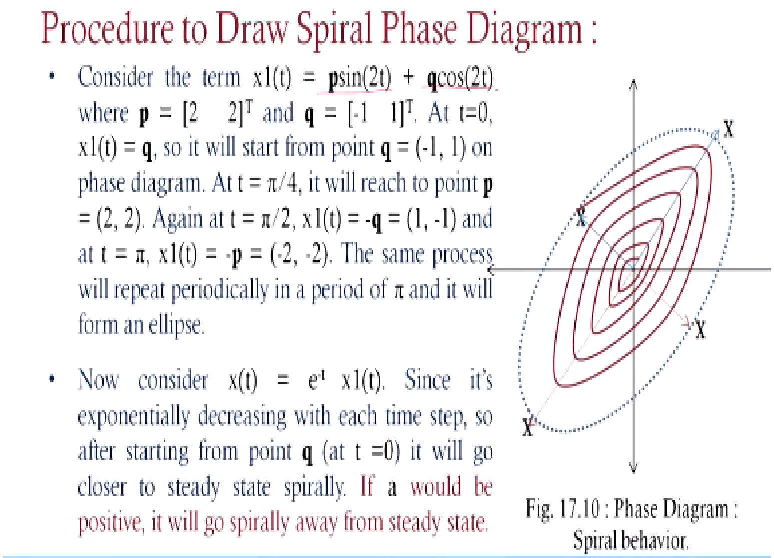

Where p has represented as 22 and q is represented as -11 here and t equals to 0 if you will just see this will just represents in sin0 is in 0 here and cos0 is 1 so that is why x1 equals to q, so we will start from the point like q1 equals to -11 here. So -11 if you will just start at t equals to π by 4 it will reach to the point p that is 22 i8t will just reach. Again at to π by2 we can just find this point x 1t equals to –q that is 1-1 in this reason.

And at t equals to π especially we can again reach to the -2 point. This same process will just repeat and periodically we will just find that it will just go and decreasing and finally it will just form allee there. if you will just consider like x of t equals to e3 bar – t x1t suppose since A equals to -1 if you just consider and it is exponentially it is just going and decreasing after starting from the point that equals to 0 it will go closer to the steady state.

And it will just achieve this steady state spirally if you just see here and if A would be positive then it will just come outwards, so it will just go away from the steady state.

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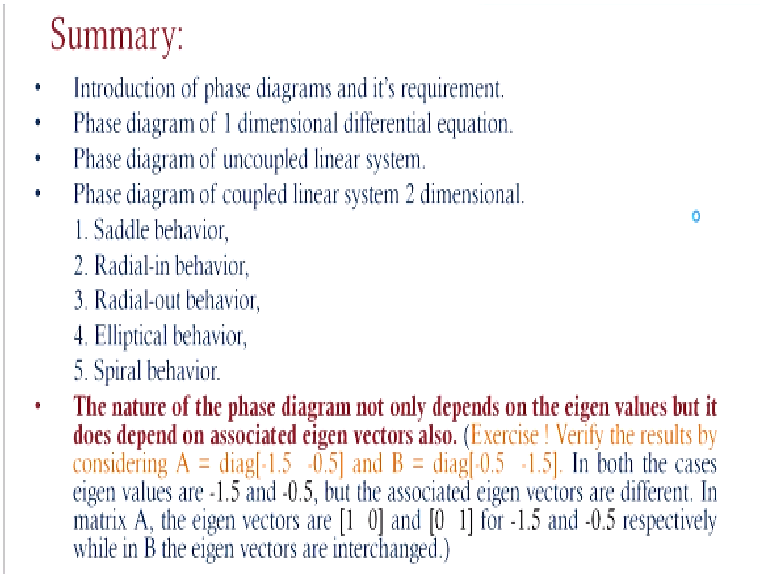

So in this lecture we have introduced this phase diagram as it is required so phase diagram of one dimensional differential equation then phase diagram of un coupled linear system then phase diagram like coupled system then saddle behavior then radial in behavior then radial out behavior elliptical behavior and spiral behavior and the nature of phase diagram not only depends on the Eigen values but it does depend on associated Eigen vectors also sometimes if it is not determined based on this Eigen values.

We are just trying to find the associated Eigen vectors and plotting the Eigen vectors especially we can just determine whether this system is representing a stable solution or unstable solution if you will just consider like case like if the diagonal entries are like -1.5 -0.5 and b is system which represents -0.5 here and -1.5 in both this cases the Eigen values are negative but the associated Eigen vectors are different then we can have like Eigen vectors in cases in matrix A if you will just see here the Eigen vectors are 1 0 and 01 0 1 for -1.5 and – 0.5 while in B the Eigen vectors are interchanged. So in that case also we can just use like this phase diagrams to find this like stability nature of this system. Thank you for listen this lecture.

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