INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
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NPTEL ONLINE CERTIFICATION COURSE
Mathematical Modeling:
Analysis and Applications
Lecture- 16
Continuous Time Single Species Models
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Welcome to the lecture serege on mathematical modeling, analysis and application. In the last lectures we have discussed about this population balanced modeling and its solution process and in that level we have discussed like, how this population level will grows like in exponential way or it can just achieved a steady state or how a non linearation technique can be used to linearize the factors that we have discussed. And in the final phase we have concluded as like, how this physical behavior is messing with this mathematical representation of this modeling and in this lecture we will discuss about continuous times single space models.

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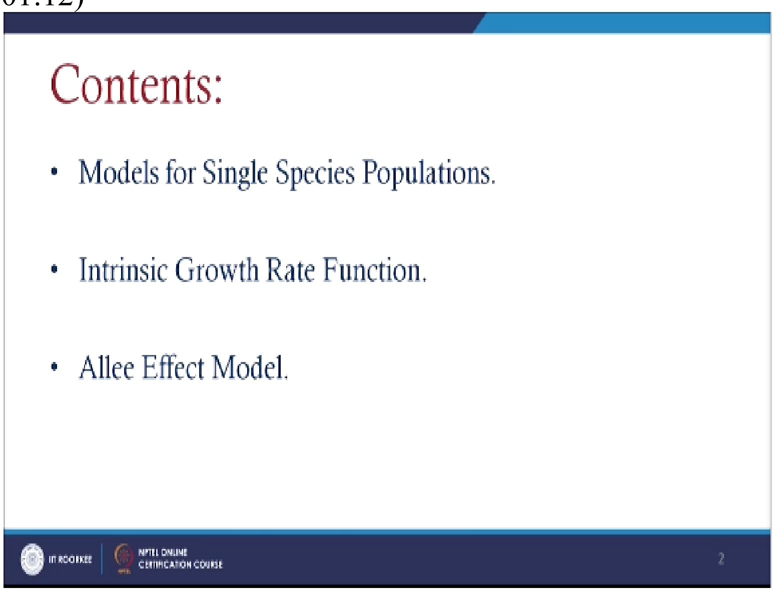

And these models will include for single species populations and then we will just develop this model and based on that we will just test like, the growth rate of function. How it is affecting the population rate and final form we will just discuss about alee effect model. So, if you just see these models for single species model.

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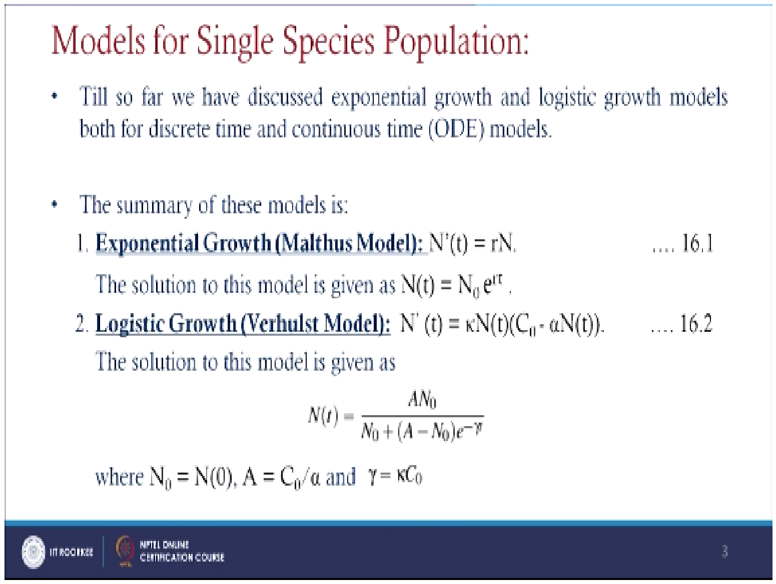

Till now we have discussed like, exponential growth, logistic growth models both for discrete time and continuous levels. So, the models are described either in the exponential growth model as I have told that is basically called Malthus model. And this model especially in mathematical representation it was written as like.

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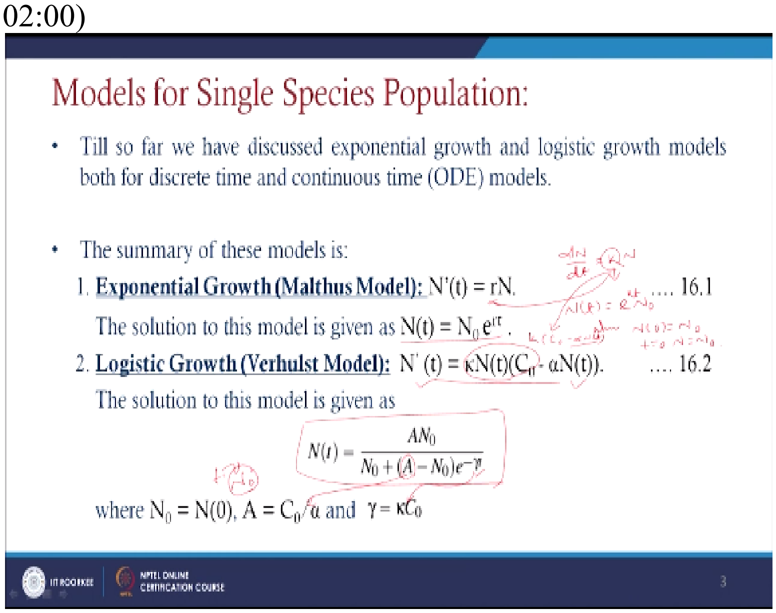

Dn by dt that is nothing but k n we have just written. That is whenever there is a change of this population growth rate or dk rate that depends on this rate of population change especially k and that we have just represented in the form of r here and the solution of this model that is just given you the exponential model if you just see we have just written there as a N (t) as a e^{kt}NO where NO is nothing but the initial population size, we have to just consider, that is t = 0 we will have like N=NO. And that is a specially called Verhulst model.

But the Verhulst model was not relayed always since there are many restrictions available or it can just exist that, the population will not always grow in exponential way or it will not decay in exponential way. Since, whenever we will just consider this like, growth rate we have to consider birth rate, death rate and all other physical aspects or the physical behavior of the system at that time. Maybe you can just visualize that, if there is a population of like, tigers or dears are existing in an atmosphere.

Where tigers are hunting dears and dears are just fed by the grass. So, in those levels sometimes you can just find that, if there is some natural devastation or some cyclones are coming or some other aspects, they are coming. Then both the dears can die or these tigers can die. So, in that level this model cannot be valid. Since, we are just assuming that only these tigers will eat these dears and dears will eat the grass and the population will grow up, but always that is not possible. Maybe, some like disease will eradicate that this can just, this growth rate can be broken down. So, up to certain level we can just assume that this Malthus model can be valid but after that we cannot assume.

That is why this logistic growth model comes to the picture and this logistic model especially it is called Verhulst model where we have just assumed that this K, which is just assumed as factor of like, k (Co- αN^t). This "αN^t" if you just see that is nothing but the restriction factors, that is just objecting the growth rate of this population size and especially, many factors can come to the picture here and if you just see here. First one is a linear model here and second one is non-linear model. Since, if you just multiply these factors so first term it will just give you a linear term here but the second term it will just give you a .

And especially, we have to just find this solution for this differential equation N (t). And, this solution is represented as N (t) = AN0 by N0 + (A – N0) $e^{-\Gamma t}$ Where the parameters like A is defined as a C0/α and Γ is defined as kCo. Where this population at 0th level is T+0 the total population size is adjunct to be N0.

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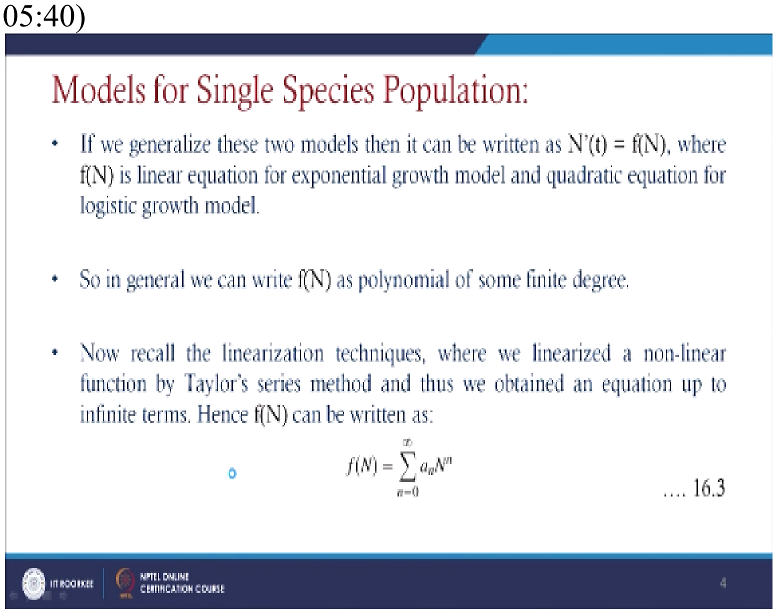

And if you just go for the like, models for single species population that is extended to our earlier case. If, we generalize these two models then it can be written as like N' (t) = f (N) so, always we will have a function f (N) which maybe linear function, which maybe a non-linear function.

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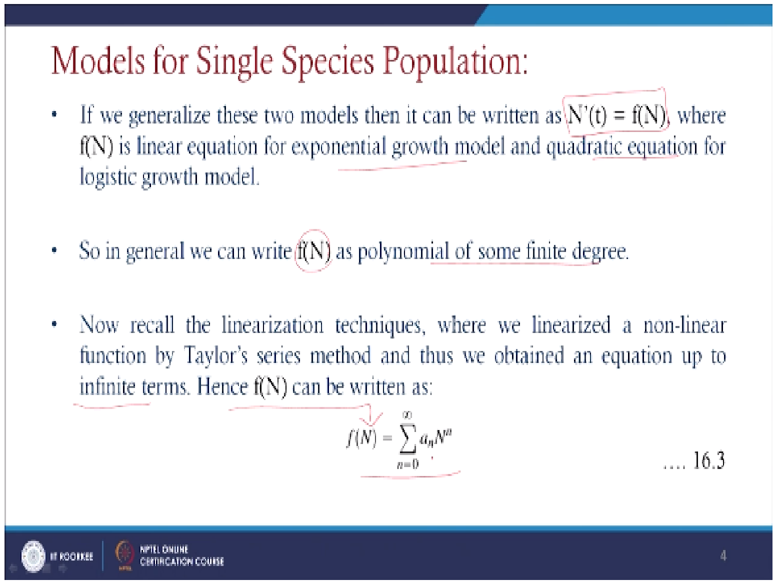

And especially, if you just see f (N) is linear function for exponential growth model and quadratic equation for logistic growth model. So, in general we can write f (N) as polynomial of some finite degree. So, how we can just linearize these functions that we will just consider here. So, if you just recall our earlier lecture, then you can just find that we linearize a non-linear function by Taylor series method using like, certain functions there is a like perturbation of the values we have just assumed like x then we just have to find this jacobian and in that sense we have just find in a matrix form and we have to just linearize this system.

Thus, we obtain an equation of two infinite terms so, if we just assume like finite number of terms then we will have polynomial and if we have like, infinite terms then we will have a series solution. Hence, f (N) we are just representing as sum is N = 0 to infinity $a_n N^n$. So, either way you can just represent that one so if it is has like, finite number of terms you can just say it is polynomial, if it is infinite number of terms then this is like series expansion, that is Taylor's series expansion you can just find in the Taylor series also we are just, finding that this is expanded up to certain number of terms and after that we are just assuming the term is the reminder term or the other term.

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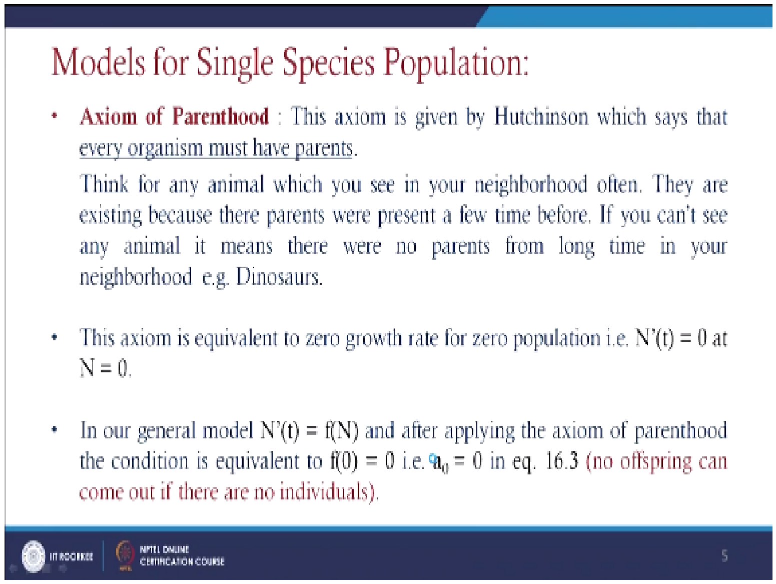

So, if you just go for like, models for single species population here then, one axiom it is called axiom of parenthood it is assumed. This axiom is given by Hutchinson which says that every organism must have parents. If parents are existing then you can just find the growth of population or like their children will exist there. If there are no parents then there is no population. So, that is the basic idea of this axiom of parenthood. Think for any animal which you can see in neighborhood often.

If they have the parents they are existing and if there are more parents then there is no existence of these animals. So, this is the case like, Dinosaurs we can just say. We are just saying that dinosaurs also are existing in the world but now we are not finding any dinosaurs since, they do not have any parents or they do not have any like parent.

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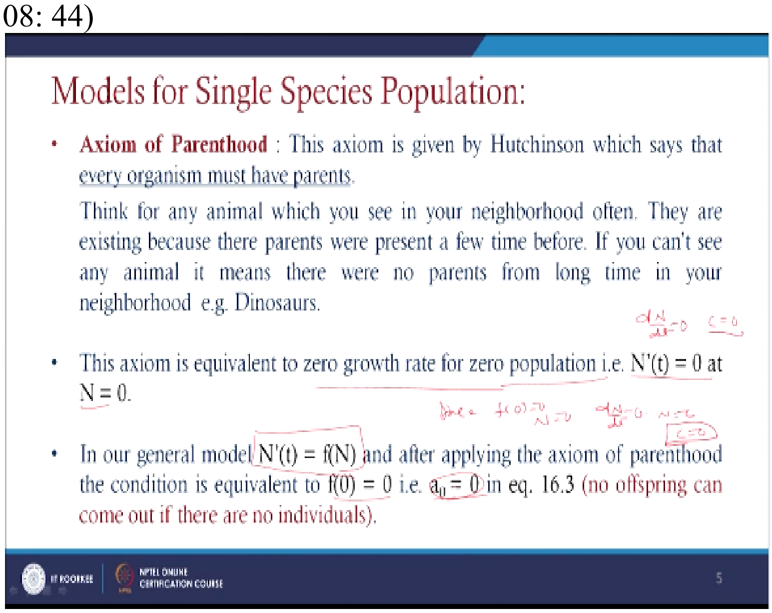

So, these axiom is equivalent to zero growth rate for zero population, that is N' (t) = 0 at N=0. Obviously, if you just assume N=0 then you can just find like dn by dt this is 0 but, otherwise we can just, if we will assume, if we will find the solution whatever the constant we will just get from this N(t) equals to constant this constant is nothing but C=0 there. So, in general model if you just assume, suppose N' (t) = f (N) here and after applying axiom of parenthood. The condition is equivalent to f (0) = 0 here. Since, if you just assume like f (0) = 0 means the population size is 0 suppose. Then, dn by dt this is also 0 and obviously, in reverse wise if you just consider this integration N equals to constant. So obviously, we can just say C=0 if there is no population.

And, in that sense we can just say that if you just take the expansion of the this series here so, first one is a 0 into 0 sorry, this N^n means N^0 here 1 there so, that is why we can just assume as 0=0 here and from this equation 16.3 obviously, we can just say that f (0) especially this is written as a 0 into 0. So, that is why 0 is assumed as 0 for this equation so, no offspring can come out if there are no individuals. If, the population will exist then we can just start our growth of the population it can just grow. But, if there is no population then it cannot be grown. So, that is why this axiom of parenthood can be applicable in this model.

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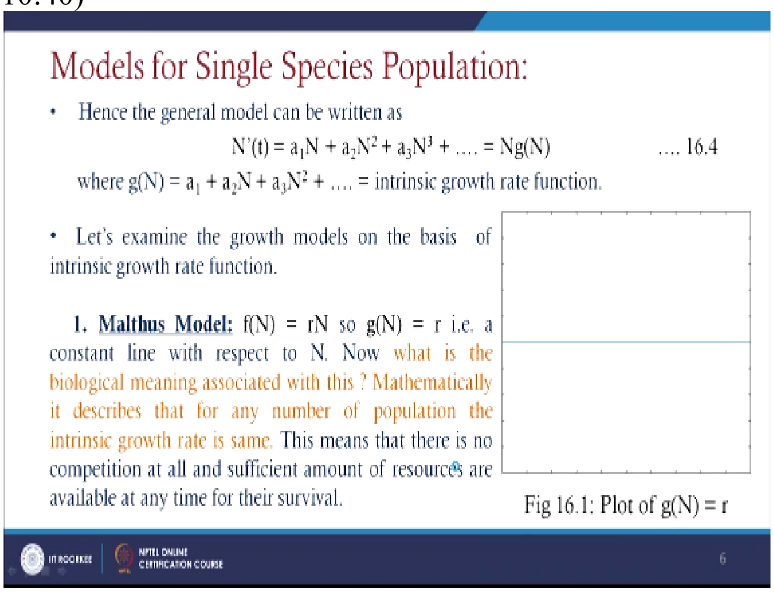

And in genalitions if you just write this function that is N' (t) = f (N) and f (N) is written in the form of some s and N = o to infinity, and $a_n N^n$. Then, we can just write this one as like N' (t) = f (N).

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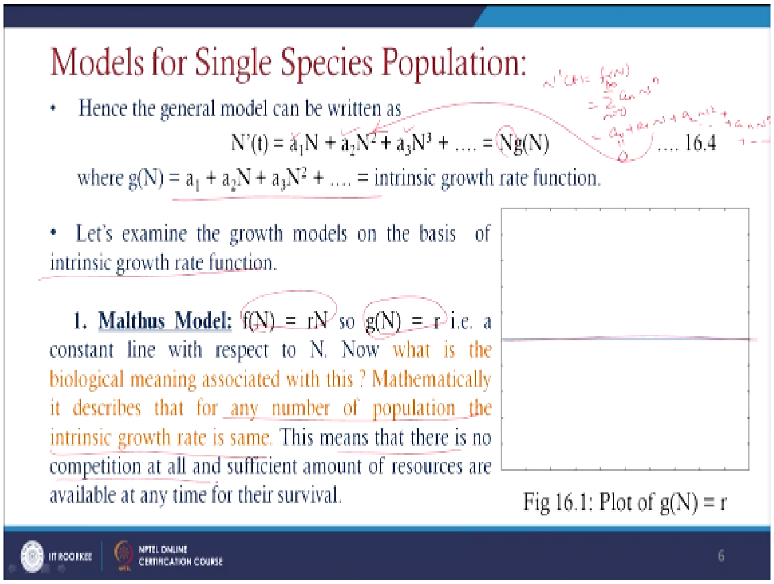

Which can be expressed as N=0 to infinity, an N^n there so, then we can find here the expansion is a0 + a1N a2 N^2 +2 the nth term it will be like aN^n + all other factors. So, if you just here a0 =0 so that is why they say equation is written in this form here a1 N a2N^2 or a3 N^3 so this can be written and if you take common of this N factor here so this can be written as like Ng (N) where g (N) can be expressed like a1 + a2N + a3N^2 likewise we can just write. This is an increasing like intrinsic growth rate function. So, if you just examine these growth models on the basis of intrinsic growth rate function then we can just find that.

The Malthus model will give you like, f(N) = rN here so obviously, we to write that one as the exponential in growth population level so, in this case we can just write g(N) = r there. That is a constant line with respect to N. So, if you just put in a graphical sense this is the constant line, we are just putting g (N) = r. Now, we can just say that this population level will grow or the population level will grow exponentially but this increment rate it is fixed. So, that is why we can just say that, what is the biological meaning associated with this? Mathematically it describes that, for any number of populations the intrinsic growth rate is same.

This means that, there is no competition at all and sufficient amount of resources are available at time for their survival. This means that maybe five, if it is five cells are there sufficient food is available but the increasing rate is like five suppose and even if the population size is suppose hundred the intrinsic growth rate is like five then thousand the intrinsic growth rate it is also five there. So then, it does not hamper that whether this like, resource is deficient to them or due to scarcity of resources the population growth rate is hampered. So, always we can just find that this intrinsic growth rate it will be developed without this dependency of resources there. So, if you just consider this Verhulst model or this non-linear growth model.

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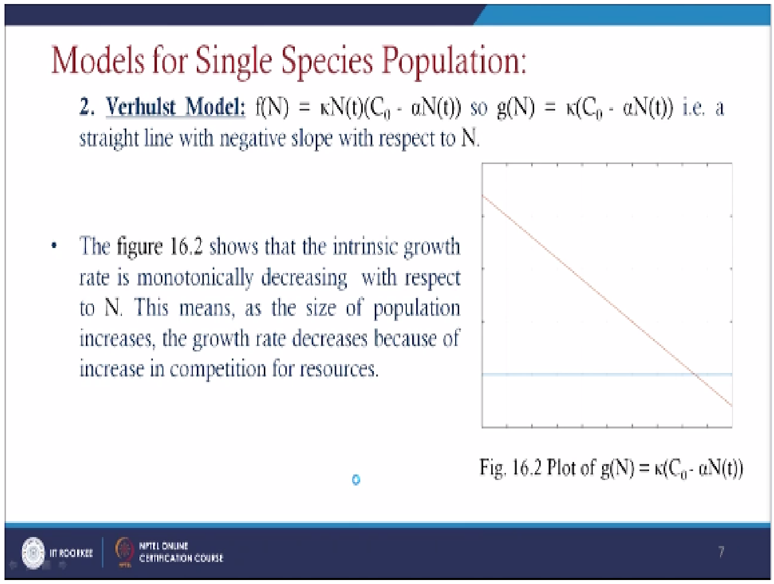

Then, we can just find that this function f(N) is expressed as kN(t) ($C_0 - \alpha N(t)$) s, g(N) which can be written here as, since N is present here so k into $C_0 - \alpha N(t)$ that is a straight line with negative slope with respect to N here.

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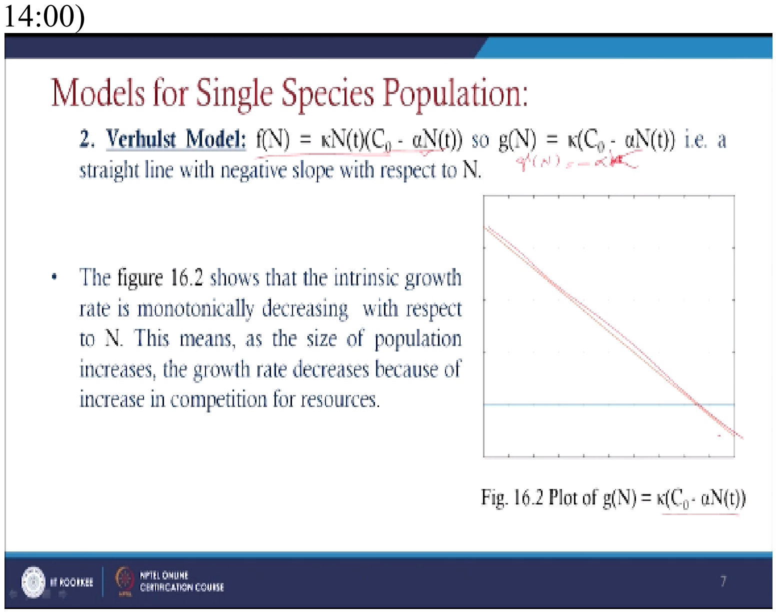

If you just differentiate this one then we can just find that, this line that is if you just differentiate here g' (N) that will just give you –αk here –αk. So, then we can just have like negative slope with respect to this number of population size and this figure this shows that the intrinsic growth rate is monotonically decreasing with respect to N here.

And, this means that the size of the population increases the growth rate decreases because of increase in the competition of resources. So, if the population level will be higher then, they will like consume maximum amount of resources available there and then this population will compete each other to get the food and the population level will decrease. So, that is the basic idea for this non-linear model here. So, even if this intrinsic growth rate is monotonically decreasing to N but the size of this population it will just get increased but the resources it will just get like decreases.

So, this means that the size of the population increases the growth rate decreases because of the increase in the competition of resources. Since, the same food it will be provided to all this like cells or this like animosity just present. So, then they can just compete each other like increase their growth rate but definitely you can just find that this growth rate will decrease. So, that is why if you are just plotting her g (N) that is the total population growth rate. So, it is just representing the declaration of this one with respect to the total number of population.

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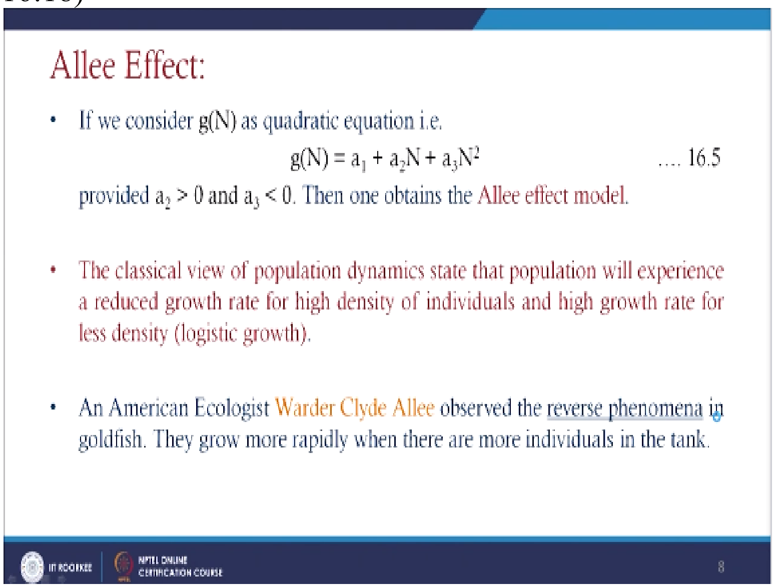

So, next we will just go for a Allee effect here so, this means that if you just consider g(N) as quadratic equation suppose where we are just assuming a2 is greater than 0 a3 is less than 0. Then one can obtains the Allee effect model. So this is the restrictions it should be provided for this quadratic equation then we can just say that it is Allee effect. And the classic view of this population dynamics states that population will experience a reduced growth rate for high density of individuals and high growth rate for less density. So, this means that the population will experience reduced growth rate for high density of individuals.

If lots of gathering is present at a position definitely they will just compete for all these resources there. And high growth rate less density means logistic growth we can just find. But, the American Ecologist this Allee he observed that this reserve phenomena is observed in case of goldfish. If like, the growth is found to be very fast when there are more individuals are present in a tank. So, that is why it is called Allee effect and this is a special case which has been developed by this Warder Clyde Allee so that is why it is called Allee effect. So, if you just consider like different cases in different forms then we will have like different classical models in population dynamics.

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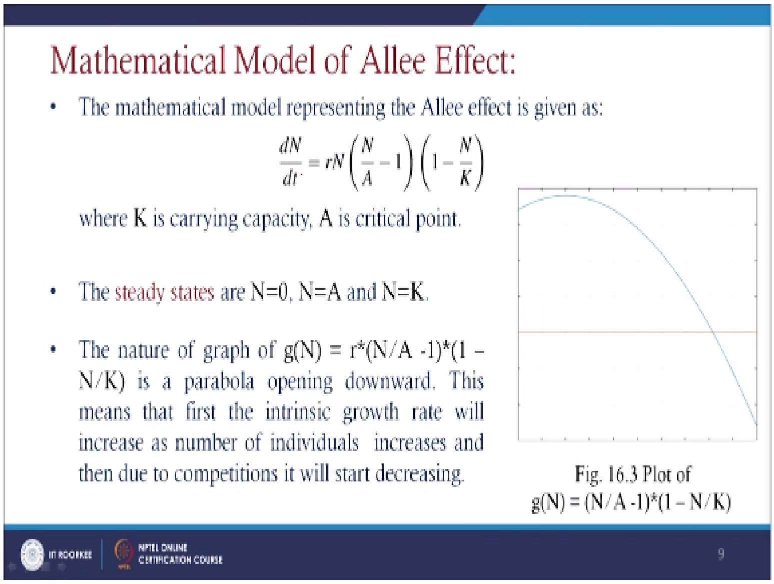

So, if you just go for this solution of Allee effect equation that equation is represented in form the form of like, change of this population is directly proportional to the total number of a population size into this growth rate factor into N / A – 1*1-A/ k.

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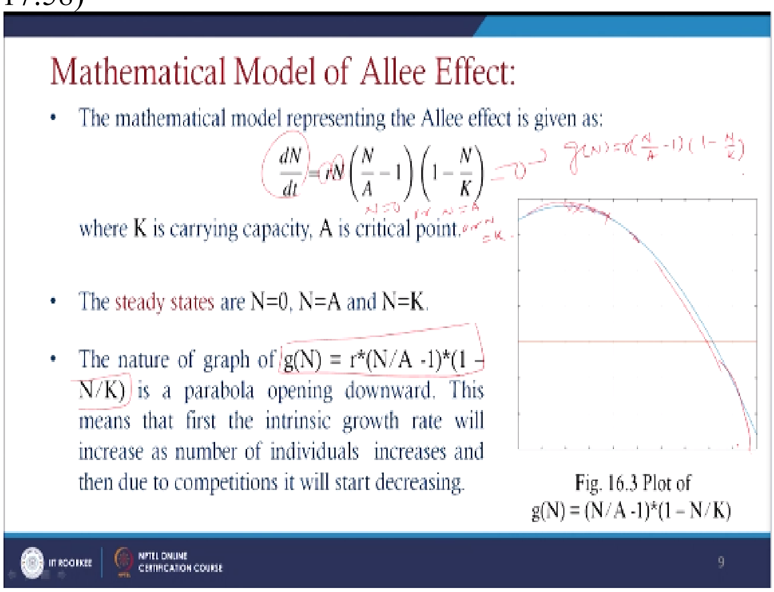

Especially here, we are just assuming that K is the maximum carrying capacity or it is just set to be the critical value here and A is the critical point we have just assumed. And, the steady states are like if you just see here dN/dt = 0. Obviously, we can just write N = 0 or N = A or N = K here. And, the nature of this graph if you just see here, so g(N) is nothing but here, which can be written as like r(N/A – 1) (1-n/k) so, if you just plot this graph of g(N) here that is in the form of r*(N/A-1)*(1-N/K) here.

This will just represent a parabola which is just opening downward here and this means that the first intrinsic growth rate will increase here this is the increasing parabola we are just getting. That is just the number of individual's increases and this rate will decline due to the competitions started here from here onwards there is a decline of this measure. So, this is the general idea about the Allee effect.

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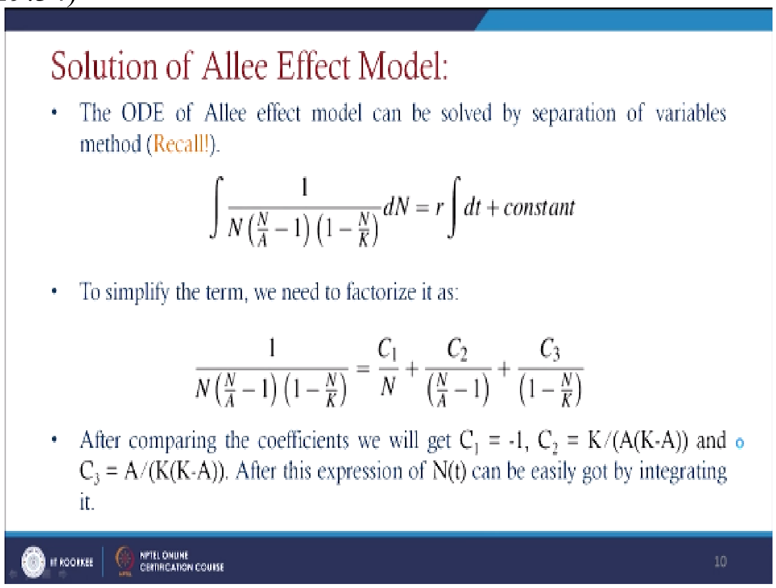

And, if you just go for this solution of this Allee effect model then, we will use this variation of a separation or a variable separation method to get the solution. So, if you just see this equation here dN/ dt and if we want to free N from this right hand side and if we want to keep only this constants and dt terms in the right hand side, we will just keep this equation in the form of like 1/ N (N/A – 1) (1- N/K) dN = r dt +constant.

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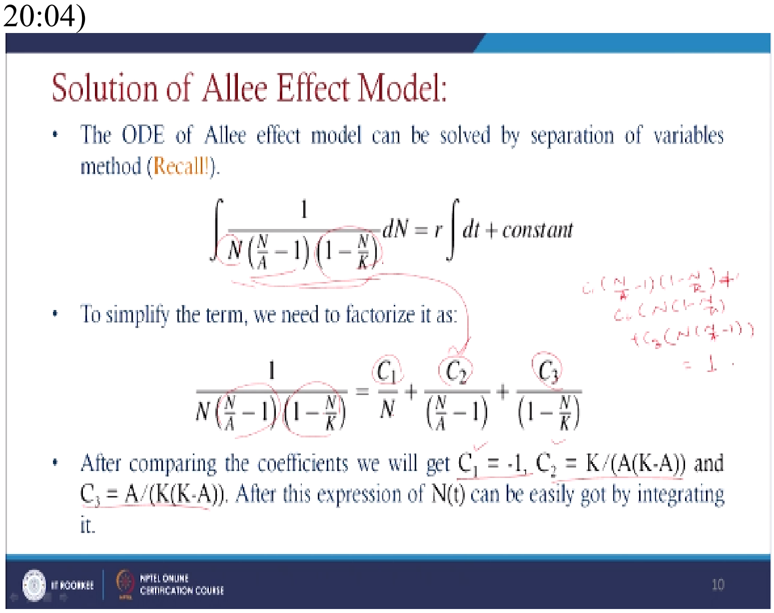

And since, this are all non-linear terms so, we can just factorize this one in the form of like, 1/N(N/A – 1) (1 – N/K) which can be written as since, three n factors are there so, that is why we can just write C1 / N here + C2 / (N/A) + C3 / (1 – N/K) here. And if you just compare both the sides then we can just find this, C1, C2, C3 so, if you just compare here this means that this denominator part it will remain same.

So, first C1 (N/A – 1\) (1-N/K) this is just considered as + C2 can be considered as C2 (N (1-N/K) + C3 we can just write as (N (N/A – 1)) this = 1. And, if you just compare all the powers of N then we can just find C1 as -1 C2 as K / (A (K – A)) and C3 as K/, A/ (K (K-A)). After this expression so, once you are just getting this values of C1, C2 and C3 then we can just integrate and we can obtain the values of N (t).
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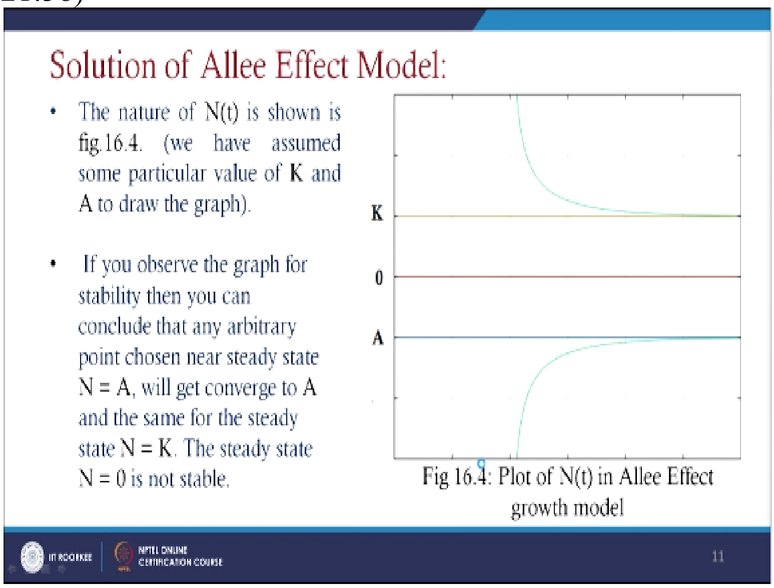

And, if you just go for this solution of Allee effect in a graphical sense, the nature of N (t) is shown as that for some particular value of K that is a maximum level of population it can be maintained when tens to infinity. This can be assumed to a constant level we are just finding here and if you observe the graph for stability then you can just conclude that, any arbitrary point chosen near the steady state N = A, will converge towards the point A here. And, the same for study state N = K also. And, the study state N = 0 is not stable since if N = 0 then there is no growth rate or no decay rate since population level should be present there.
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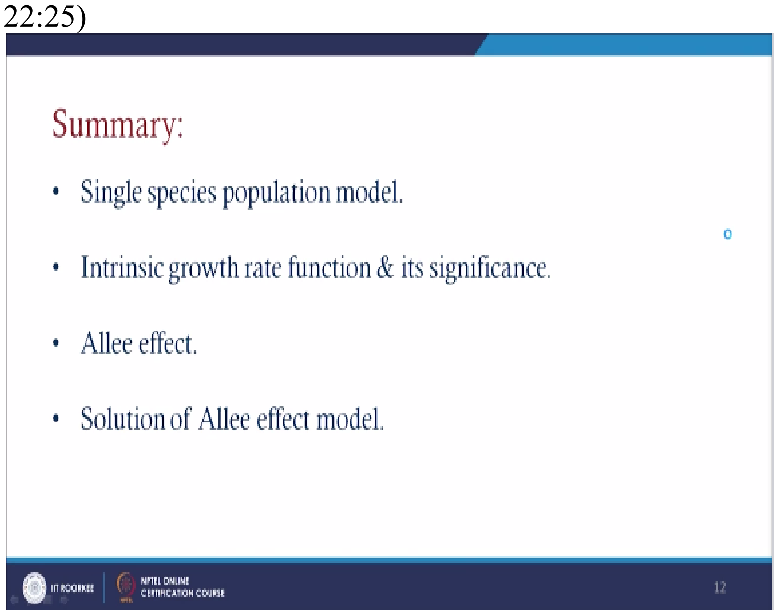

So, in this lecture we have discussed about single species population model. Then intrinsic growth rate growth function and its significance and in the exponential growth rate we have just find that, this intrinsic growth rate function that is independent of the resources available there. This means that sufficient resources it will be available there and this intrinsic growth rate it will not get sense even if this population size is five or hundred or like thousand it does not affect it.

But, if we are just assuming like non-linear model in that case we have just observed that this population size there just competing each other and this population level is just getting declined. And, in the last phase we have just discussed about this Allee effect which is effective in case of certain population levels that is goldfish as it just found that if the certain individual numbers maximum amount of numbers are gathered in a place then we are just finding this population level is just getting increased.

And, then we have just found the solution of Allee effect model. So, thank you for listen this lecture.

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