INDIAN INSTITUTE OF TECHNOLOGY ROORKEE
NPTEL
NPTEL ONLINE CERTIFICATION COURSE
Mathematical Modeling:
Analysis and Applications
Lecture- 15
Stability and Linearization of System of ODE's
With
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Welcome to the lecture series on mathematical modeling analysis and applications. In the last lecture we have discussed about this exponential growth of population based on like logistic growth of population and we have discussed also the Malthus model, where there is a non linear growth of population was existing and there itself we have discussed about like stability condition and how this inter gin growth rate is affected by this resources like concentration and all other stops and also we have discussed like chemostat growth there.

So, how this chemostat is acting for this cells culture technique, that is the cell culture technique means either it is a continuous culture medium or like base culture medium. So, based on that we have solved some of this differential equation and in the present lecture we will start about this stability and linearization of system of ordinal differential equations. So, for this lecture first we will just go for

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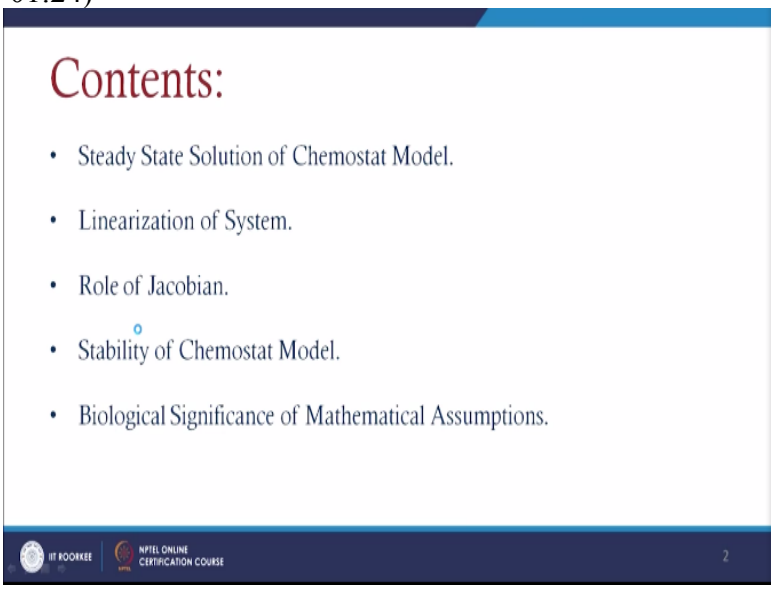

A steady state solution of chemostat model, since in the last lecture we have discussed in detail about this chemostat model and how we can just linearise this system that will also discuss here, then in this system how this the Jacobean comes to the picture and how Jacobean helps out to us to solve the system that will just discuss. Then we will just go for a stability analysis of chemostat model and its biological significance in mathematical assumptions that will also discuss. So, first if we just go for stability,

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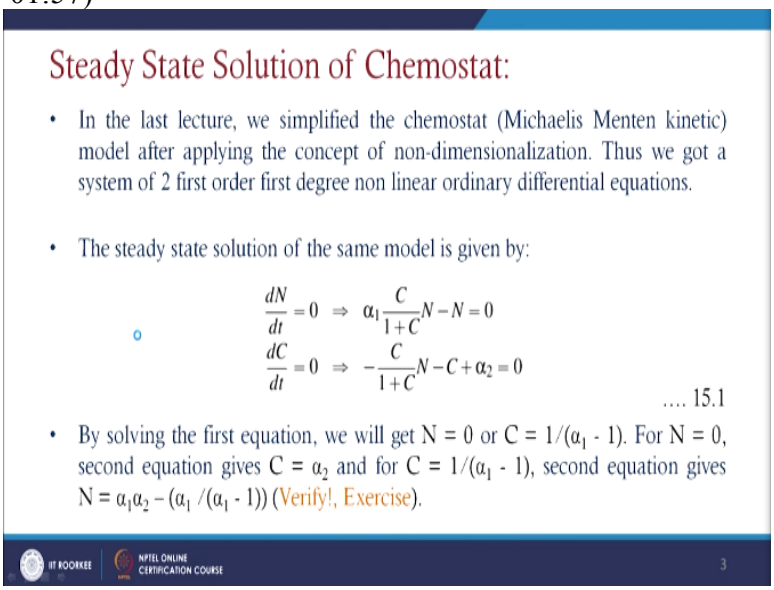

State sorry this steady state solution of chemostat and we have just discussed that, these chemostat models are like michaelis menten kinetic model. It is a concept of non-dimensionalization we have just used to simplify that model and we have obtained like 2 systems of first order on first degree equation in non linear form there. So, that is why we have just written here as we got a system of two first order first degree linear ordinary differential equations.

So these two systems especially, if we just go for a steady state solution of these two equations. So, that can be written as like Dn/dt=0 and Dc/dt=0 and obviously after non-dimensionalization that equation has been reduced as like dN/dt =α_1, dN/dt especially should just see that can be written as like α_1 C/1+C N-N there and also Dc/dt it was written as like –C/1+C N-C+α_2. And if we just find the steady state, this means that this number of population is not get changed to with respective time, then we can just put it as 0 and this comes under level that is also like our resources that is not also changed with respective time that we can just assumed as 0 there. So, that is why this equations are written as α_1 C/1+C N-N =0 and –C/1+C N-C+α_2=0. So, if you just solve this first equation then we can just rewrite this equation in the form of, like N (α_1 C/1+C-1) =0.

Obviously we can say that this is N=0 or α_1C/1+C this =1 then. So, this implies that either N=0 or C= if you just solve these equation here, α_1 C/1+C which can be written as like 1+C/α_1C this =1 here. So, obviously I can just write this on as, this implies, I can just write 1+C/C as α_1 and this implies that 1/C+1 gets α here and 1/C this =α-1 and C=1/α-1. So, that is why it is just written as the C=1/α_1-1 here.

Sorry I have just written masticate α1 and for any =0, if you just put in the second equation here, then we can just find C= α_2 and if you just put like C=1/α_1-1 then the second equation will be reduced as, N=$\alpha_1\alpha_2$-α_1/α_1-1. It is just a simplification we have just made it. So, if you just put directly this values obviously this N value directly you can just get from this equation. So,

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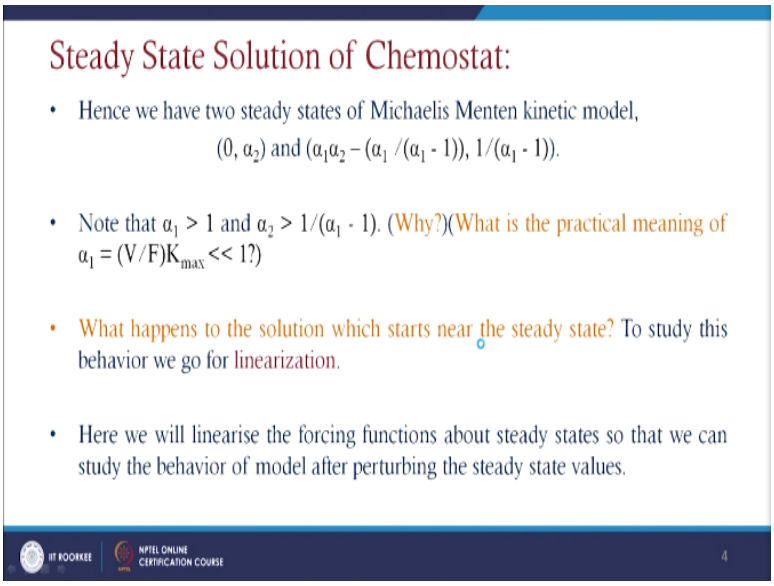

If we just proceed further that we have now two steady states for this equation that is a michaelis menten kinetic model, we will have first value N=0, if you just see here for N=0 we are just getting C=α_2 and for like N=, sorry C=$1/\alpha_1$-1 we are just offending N=$\alpha_1\alpha_2$-α_1/α_1-1. So, both this values have just written here. And if, we will just input some conditions that for practical application oriented or how practical it will just behave.

So, we have to consider that α_1 should be >1and α_2 should be >$1/\alpha_1$-1. Why we are just choosing that one in the next slides, we can just prove that one or verify that one why we are assuming in that form and what is its practical meaning? And in the previous lecture also we have just considered like α_1 is considered as a (V/F) K_{max} which should be considered as max<<1 here also. Even if you can also generalize this one that if we are just putting here like $1/\alpha_1$-1, so if α_1 is <1 here, then we will have a negative quantity.

So, this population level resource should be present there to get a positive quantity there. And that is why we have to assume that α_1 should be >1. And second case if you just see that α_2 we are just assuming here. So, α_2 if you just see from this relationship here, since $1/\alpha_1$-1 it is already we are just assuming as like α_1 >1, definitely α_2 must be >$1/\alpha_1$-1. Since always we will have like positive N value and positive C value there to find this growth of this population neither we cannot have like valid population dynamics there.

Since always we will have certain resources and certain population to find a growth rate. So, that cannot be that like population level or like resources cannot be negative or 0 at best I can say. So, the next question is that what happens to the solution which starts near the steady state? Steady state means we are just finding a state or the critical level so that should be maintained if we just go for like detents to infinity.

To study this behavior we go for linearization, since all this factors whatever we are just getting here that is in non linear form here. And if you just linearise the forcing functions about steady states, so that we can study the behavior of model after perturbing the steady state values. So definitely, if we just perturb these values, it will have influence or after certain time we will have a fixed value that can be considered as the steady state values or the critical values. So,

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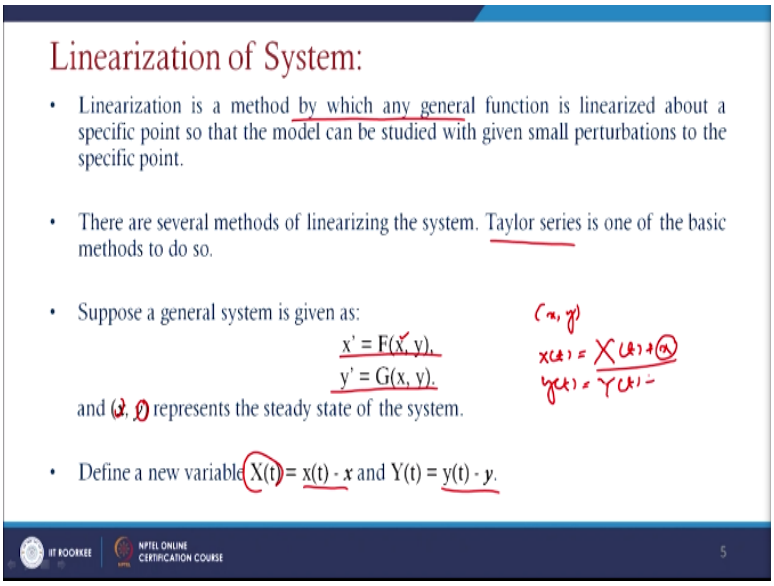

First we will discuss about this linearization of the system: linearization is a method by which any general function is linearized about a specific point, so that the model can be studied with given small perturbations to the specific point. It means if you just introduce a small error close to that point that does not affect this solution process. So, we will just consider for the linearization or linearization of any like non linear terms in that method.

So, there are several methods for linearizing this system, like Newton's linearization technique, Taylor series. So, especially we will use here Taylor series in the linearization techniques, since Taylor series is one of the basic methods to do this linearization. Suppose if you just consider like general system and then we can just put our equations in generalized system to get this linearization of our problem.

So, that is why we have just considered general system that is expressed in the form of x'=F(x, y) and y' it is expressed as G(x, y) here and just to get this values at a point like steady state will achieved at a point, so arbitrarily we are just assigning two values x and y, where we can just get a steady state solution or a steady state of this system can be achieved. So, if you just define suppose two new variables that is capital X (t) is small x- our, this steady state solution system in coordinate.

That is x (t)-x and Y (t) is defined as y (t)-y. If we just follow this basic definition here that is just written as like linearization in the method by which any general function is linearized above the feasible point. Since our feasible point we are just considering as x, y here. And the model can be studied with given small perturbation to the specific point. If you just see here, so x (t) which can be written as capital X (t) +x.

A small perturbation it is just added with this point. And similarly y (t) if you just see it is written as like capital Y (t) +y here. That is the small perturbations it is just added with this solution process to get a linearised form. This means that this factor should not influence the complete solution in that system. Now linearized this function F =F(x,y)

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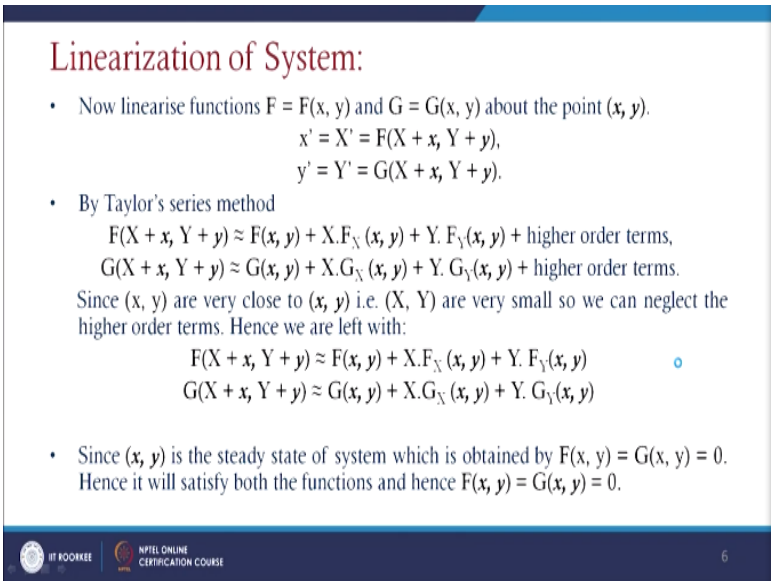

And G=G(x, y) about this point (x, y) which is considered as the steady state point. And if you just write this differentiation of small x' this as capital X', which can represented S like since we have just define in the previous slide here. X' as a capital F(x,y) here. And y' is written as G(x,y) here. So, definitely we are just replacing this X and this x by capital X + *x* and y is replaced by capital Y+ *y*.

So, there exist one difference so I have just put it in this form here so then you can just visualize all this things clearly. So, if you just take a Taylor's series expansion of this function, since a Taylor series function can be applicable when there is an ever ruled point where this function is continues and we can just expand it Taylor series form. So, we are just considering this small ever ruled point as like here as a capital X and capital Y here and small x and small y are the like steady state point where we can just find this functional value solution.

So, that is why can just use Taylor series function or Taylor series expansion at that point. Especially the function should be differentiable at that point neither we cannot use also Taylor series expansion so, that is why we have just written as an F(X+x, Y+y) is like F(x, y) + X is the small ported down there. So, X.Fx(x,y) and Y.Fy(x,y) and + higher order terms like the second order differences in terms or third order differences term that is just assumed as a higher ordered terms here.

Similarly, if you just write like a G(X+x, Y+y) as a G(x,y)+X.Gx(x,y)+ Y.Gy(x,y) whereas Gx and Gy both are like differential function at the point X,Y with this higher order terms which can include like second order and third order or higher order terms. Since small x, small y are very close to this steady state points that is the ported terms like capital X and capital Y are very small so this especially we are just assuming the ever ruled points.

Ever ruled point's means sometimes in mathematical sense we are just using as epsilon there. This can be written as like x+ epsilon and y+ like epsilon there. So, if you just neglect this epsilon terms here, that is capital X and capital Y here, then we are just living with the terms as F(X+x, Y+y) as F(x, y) + X.Fx(x, y) + Y.Fy(x, y) + order of higher terms, it can be neglected

since this capital X all of this power turns it is just neglected and all of this power terms for this Y of G also neglected here.

The same thing it has operand also for this exposure. Since, this *X* and *Y* is the steady state of the system which is obtained by if you just take this the first order differences as =0 then we will have like steady state. So, that is why we are just assuming these points as F(x, y) =0 and G(x, y) =0 for steady state and if you just put both of them =0, this will also satisfy this *x* and *y* yet F and G functions which will take the 0 values. Since these are the ever ruled points so we can just assume that at this point also this concerns or taking also this 0 values. Since at the difference level if you just take that can be just linearise the factors. So, the linearise the equations

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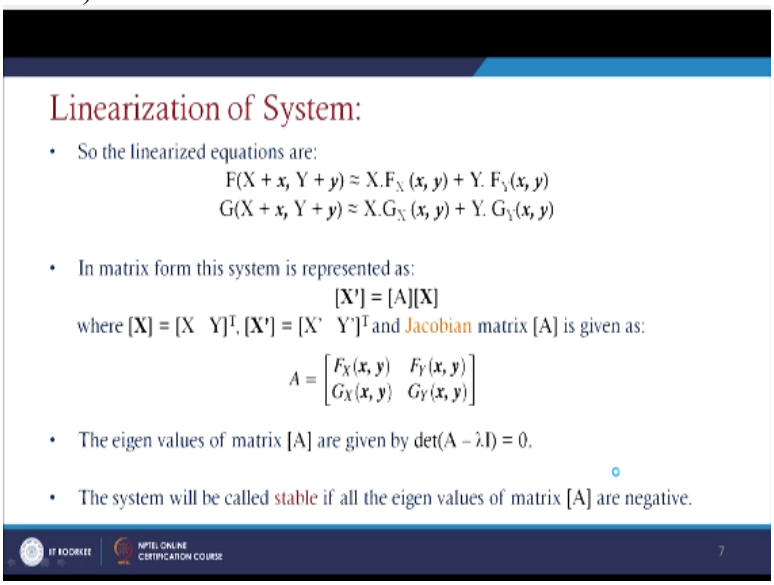

If you just write here, we have already that these terms are =0, these terms are =0 we have left with like F(X+x, Y+y), G(X+x, Y+y) this =X.Fx + Y.Fy, X.Gx + Y.Gy. That we just represented here and in a matrix form if you just write here. So, this is nothing but we have just written there, that has like X' and Y' so, that is why this X' and Y' especially we can just write here as a Xprime here so that is why the right hand side it can be expressed as a matrix A and Matrix X, where Matrix X can be represented as since X and Y.

X and Y it is just involved for this equation there. So, that is why we can just write vector X as like XY and X prime as like X' and Y' prime and the Jacobean of this matrix say, can be written as like A= first Fx then Fy then Gx and Gy if completely, it will be multiplied with this factors like X and Y so reversely this will just give you this spectra here and to find this, like Eigen values of matrix to determine this whether this system is like stable solution or unstable solution, we have to find the eigen values already we have discussed in the previous lectures.

That this Eigen values of matrix A are given by like determinant of A- λ=0 and the system will be called stable if all of this Eigen values of this matrix A are negative. Since we will have like A to the power – λ X which will converts towards the 0 value so that is why it can have a right stable value and if it will be exponentially growth this means λ is positive there then we cannot have a right stable solution. So, if we will just

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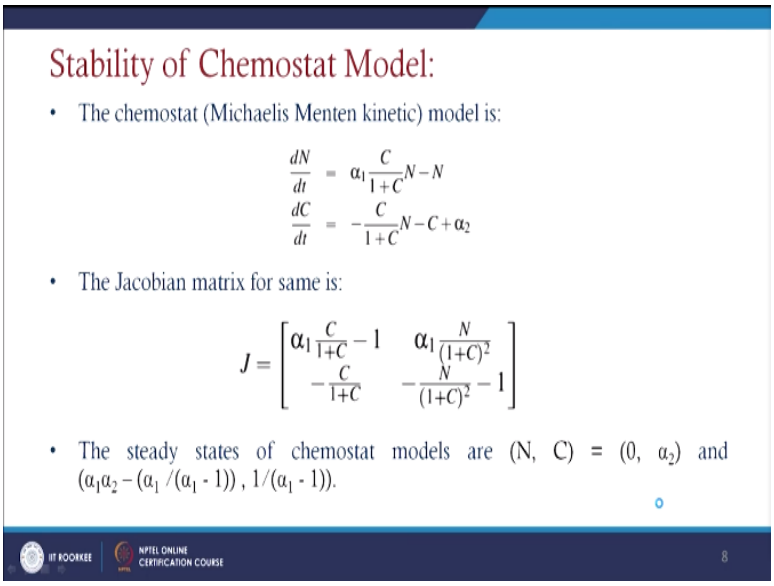

Go for our generalized model for this chemostat model to check the stability the chemostat model already we have defined in our earlier lecture that dN/dt it is just written as α_1 C/1+C N-N and dC /dt which is written as –C/1+C N-C+α_2 here and to get this terms for this Jacobean we can just first difference it since this is nothing but we are just writing here N' this is nothing but C' here.

And this can be expressed as like F of so already we have defined that one here that is F(X+x) and G(X) if you just see here that is F(X+x) and (Y+y) and G(X+x) and (Y+y), this one only F(X,Y) and G(x,y). This is the F(x, y) this is G(x, y). So, to get this first term here we will just differentiate ∂F/ ∂N here and to find this term here we will just differentiate this one ∂f/ ∂C here. Since if you just see here that is just written as FX and FY here. So, that is why I have just written ∂F/ ∂N and ∂F/ ∂C.

So, first differentiation with respect to N it will just provide you α_1 C/1+C -1 here and with respect to C if you just differentiate this one that will just provide you, so this respective will just goes to 0 and you can just find α_1 and this is N/1+C2 and if you just differentiate again this G term, with respective first N then you can just find this one as –C/1+C here and if you just differentiate G term with respective C here then you can just find this one is first term as –N/$(1+C)^2$ and –C this will just give you -1. And the steady state of chemostat model we have just considered as like when N=0 we will have like C goes to α_2 and now when we have just assumed C=1/α_1-1 it is just provide the value of $\alpha_1\alpha_2$-α_1/α_1-1 and

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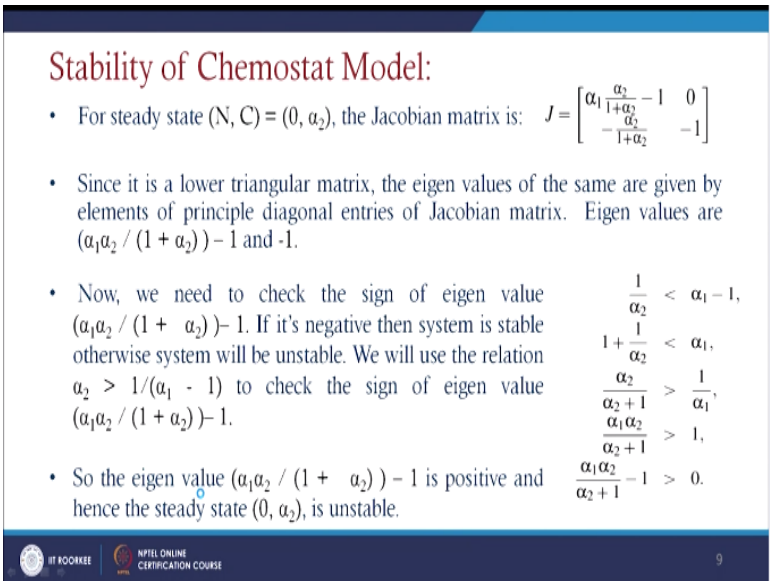

If you just put this values like for steady state N, C =0, α_2 then this Jacobean matrix whatever we have just to get it. So, directly if you just put here like N=0 we will have like α_1 C/1+C-1, this will just completely give you 0 values and if you just put like N goes to 0 here and α_2 for C, so this will just give you α_2/1+α_2- will be here so, this will just completely 0 and this will be -1 here. So, the same thing we have just written here that is $\alpha_1\alpha_2$/1+α_2-0 and –α_2/1+α_2-1.

And if you just see this is nothing but a lower triangular matrix. And the Eigen values of the lower triangular matrix is nothing but the elements present in the like principle diagonal entries of his Jacobean matrix and though Eigen values are especially we can just write it as directly as like $\alpha_1\alpha_2$/1+α_2-1 and-1. Since these are the principle diagonal entries here they are the Eigen values here. Now we will just check whether this Eigen values are negative or positive or it is just representing some like marginal values.

So, if you will just go for the checking of this sign of Eigen values here. The first factor it is just written as like $\alpha_1\alpha_2$/1+α_2-1 here and if you will just see here, if it is negative then the system is stable, already I have explained that if we can just write this expression in the form of a to the power – λ t there. So, whenever we are just approaching towards like T times infinity then it is just approaching towards 0 there.

So, that is why it has like stable solution. Otherwise it will be on stable, so if we will just use this relationship like whatever we have just discussed in the earlier slides here that α_2 should be >1/α_1-1 already we have discussed that one is here α_2 is >1/α_1 -1 here. Then we can just find that whether this Eigen value is representing a positive value or negative value here. So, if we just use this relationship that is in the form of α_2 is >1/α_1-1 to check the sign of this Eigen value of $\alpha_1\alpha_2$/1-1+α_2-1.

We have just do a little manipulation here that is in the form of since α_2 is >1/α_1-1 obviously we can just write 1/α_2 < α1-1 and we can just write this one as like α1 should be kept in the right hand side and 1 can be take to the left hand side here and we can just write α_2+1/α_2, this would be less than α_1 and again we should just take like this fractions here, we can just write α_2/α_2+1 this

would be >1/α_1 and again we will just multiply this α_1 both the sides then we can just write $\alpha_1\alpha_2/\alpha_2$+1 is greater than1.
So, finally we can just, if you just take all the factors to the left hand side $\alpha_1\alpha_2/\alpha_2$+1-1 this is >0. And obviously the factor we have just obtained here is $\alpha_1\alpha_2$/1+α_2-1 which are just representing a positive value and hence the point will give you an unstable solution here. Unstable solution means obviously you can just find that since we are just considering N values as a 0 here, this means total number of population that is like sales or any other structure if it is 0, definitely we cannot find a model there and it will just does not represent anything. So, further if you just go for like for N, C values as like
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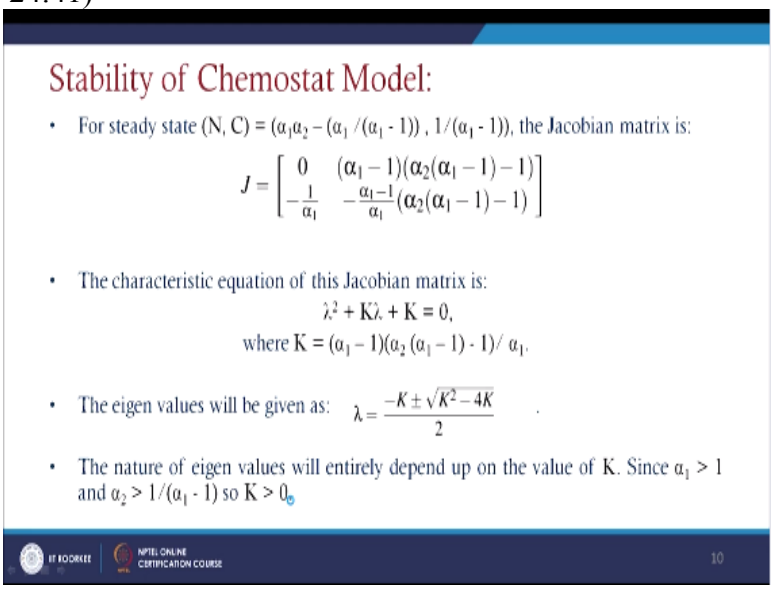

A1α2-α1/α1-1 and 1/α1-1 so, representation of this Jacobean matrix whatever we have just represent here. That is α1 C/1+C-1 and α1 N/1 $(1+C)^2$ and –C/1+C-N/ $(1+C)^2$ -1. So all these factor C and N it will be replaced by these factors here. Then we will have like 0 at the first entry and α1-1(α2 (α1-1-1)) and this factor will be like-1/α1 and this will be –α1 -1/α1 and α2 (α1-1)-1) minus 1 here.
And if we want to find the Eigen value of this Jacobean matrix here we have to subtract λ from this factor- λ this factor and the determinant should =0. And if you just take this determinant=0 then the reduced form of this Jacobean matrix this can be written as like λ2+K λ+K=0, where K is the factor which is represented in the form of like (α1-1) (α2 (α1-1)-1)/α1 here. This is just a simple multiplication you have to do, that means that (– λ) into you can just write (-α1-1/α1 (α2 (α1-1)-1) - λ).
So, -this will we just give a +value 1/α1 (α1-1(α2-1)-1) =0 and from this you can just get this equation. And if you just find this Eigen values, this is the quadratic equation so it will just provide two routes here that is –K+ or – square root of K2-4K/2 here and the nature of Eigen values will entirely depend on the value of K here. Since we have already assumed that α1 is >1 and α2 should be >1/α1-1.

So, if you put all these values here, α1 is greater than1, so this will just give you a positive value here and second one we have just assumed as α2 is >1/α1-1. So it should also be >like 1/α1-1 it will be just cancel it out here and this will be -1 here by α1 so definitely K is greater than0.
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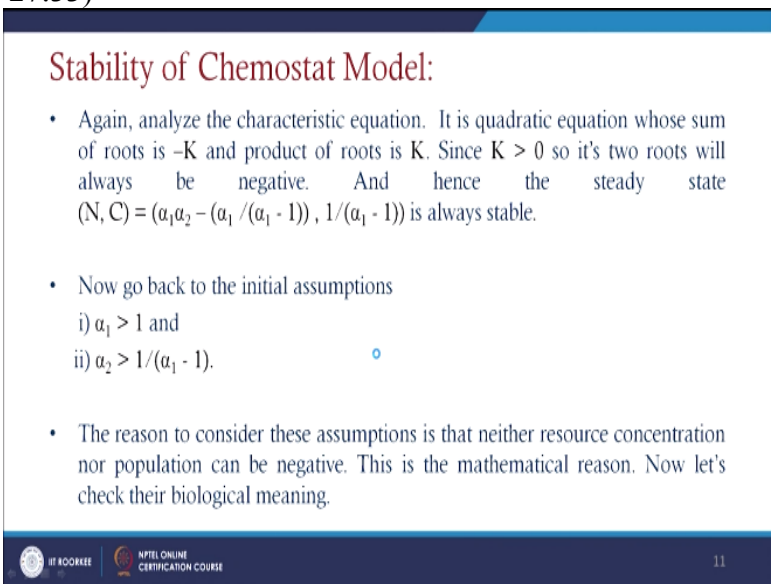

And if we just go for like further analysis of this characteristic equation, it is a quadratic equation whose sum of the roots is like –K and product of the roots is K here. Since for the quadratic equation especially the roots it can be like α and β if are the roots then it can just presented as α+ β =-bya and α* β = cya which is product form. So, if you use that formula definitely you can just get that some of these roots is –K here and product of these roots is K here.
Since we have obtained K as >0 with this substitution of α1 >1 and α2 >1/ α1-1, so its two roots will always be negative and hence the steady state is achieved at N and C values for N=α1-α2-α1/α1 and C is 1/α1-1, which shows the stability at all these steps. Now if we just go back to the initial assumptions so, we have just considered that is α1 >1 and α2>/ α1-1. The reason is that already I have explained that this assumption is that neither resource concentration nor population level, it cannot be negative.
So, if it will be negative then we do not have like population model there. The population will be there then that will be growing up. If there is no population then there will be no grow up and if there is a food or resource then the population will exist there, if there is no resource there, then the population will not exist also. This is the mathematical reason. Now let us check their biological meaning how it just exists.
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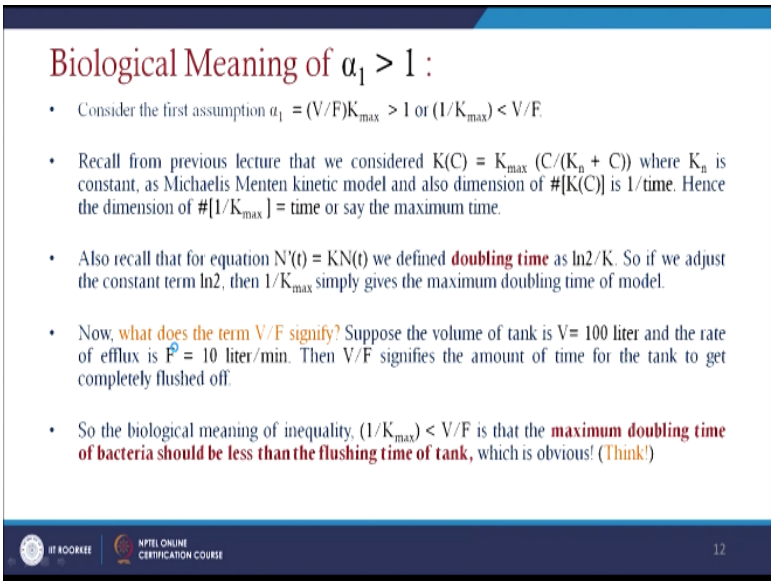

So, if you just go for this biological meaning of α1 greater than1 here. So, consider first the assumptions that α1= it is assumed in a like last lecture we have just considered that as (V/F) K_{max} here which is >1. Obviously if you just consider here $1/K_{max}$ which should be less than volume/flexion or flow. So, from the previous lecture we have considered that is a K has defector of K_{max}(C/Kn+C).

So, that Kn is just constant parameter which is just taking the values like K will assigned as K_{max} when Kn = 0 and this is just taking like half of this K_{max} factor when C= like, sorry Kn = C there. So, this constant Kn is constant as michaelis menten kinetic model and also the dimension of K(C) is 1/time. Time inverses especially we can say. Hence the dimension of $1/K_{max}$, if you just see that is nothing but time or say the minimum time that is sorry the maximum time also if you just see this equation for like population balance modeling for the cell growth, so there itself we have just started this model and N'(t) has written as like KN(t) where K is the rate of like growth rate or decay rate whatever it may be but this change of population size that depends on K and after that also we have just considered like K is a factor of C where C is the resources and depending on that we have just developed that model.

And sometimes we have just find that this doubling time, doubling time means the initial population size it will be just to get doubled from the original size and we have just to find that the time level has achieved as a ln2/K. so, if we adjust this constant term ln2 then $1/K_{max}$ simply gives the maximum doubling time of the model. This means that we are just assuming ln2 as a constant and it has a fixed value you know and 1/K we are just getting so this will just achieved the maximum level if we just double this time period.

Now what does mean by V/F here? So in a dimensional analysis way if you just see that, V/F is nothing but that will just give you else a time level there, that is nothing but $1/K_{max}$. So, if we will just go for the visualization of V/F in a physical sense here. Suppose we will just consider a tank which is containing suppose 100litre of this concentrations. And the rate of efflux is supposing F as like 10 liter per minute then V/F that is the total volume and the flushed out, it is the stock ring in a ratio form.

This signifies the amount of time for the tank to get completely flushed off. This means that if you just consider the total amount of a time it is required to flush out everything then there is a culture should be grow up there, this cannot be possible. So, the biological meaning of inequality that is $1/K_{max}$ which represents the time which should be less than the time that is V/F is the maximum doubling time of bacteria should be less than the flushing time of tank. So, if there is no flush of like this concentration, how these population levels will growths it up? So, it is always required that the population level growing, it is required that there should be a flush of time it is more.

So, if this like the flushing time of the tank it is more then we can just find that the growth rate will be increased. So, neither we cannot find that this maximum doubling time in the factory as would do always will less in that apposite of time, which is obvious. So, obviously we can just think that if there is a flush out then you can just find that there is a like well mixer medium and you can just find the growth rate and if there is no flush is out, how this population level will be just it of there. So, if we just go for this

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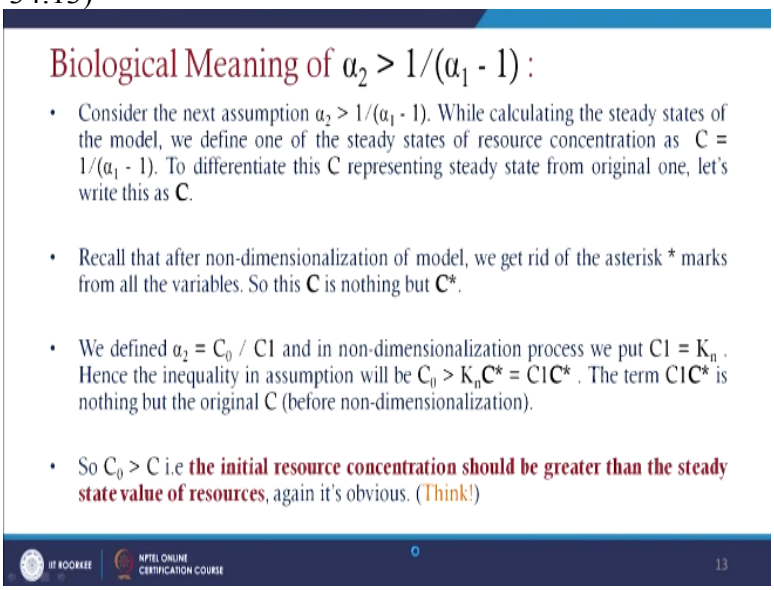

Biological meaning of α2>/α1-1 here so, consider this assumptions like α2>1/ (α1-1). While calculating this steady state of this model we define on of this steady states of the resource concentration as C= suppose 1/α1-1 here. To differentiate this C representing this steady state from original 1, let us write this is as suppose C and if you just see that after non-dimensionalization of the model we have just to get rid of the asterisk* marks from all the variables.

So, this C is nothing but C star here C* and if we just define here, α2which is defined as like C0/C1 here and in non-dimensionalization process we are just putting C1=Kn on the last lecture slides we have just kept C1=Kn there and hence this inequality for this assumptions if you just put here like C0>KnC* here since α2, we are just writing C0/C1 and α2 is > 1/α1-1 and C is representing the steady state for that one, so that is why we are just writing here that is α2>1/α1-

1 here and α2 is obviously written as C0/ C1 here and this is just written as like C is written and C is nothing but we are just putting as C*.
So, that is why we can just write C0>C* into C1 here and where we are just putting C1=Kn so that is why this =C* into Kn and obviously if you just see here, C1C* is nothing but the original C before non-dimensionalization. So, C0 must be > C that is the initial resource concentration should be >the steady state value of resources. Obviously if these population starts from the beginning they will use this concentrations are like in their food to grow of and when they will reach this steady state especially this concentration level will be less compared to the initial concentration level. So, that we want to just show it of why we have just considered α2>1/α1-1.
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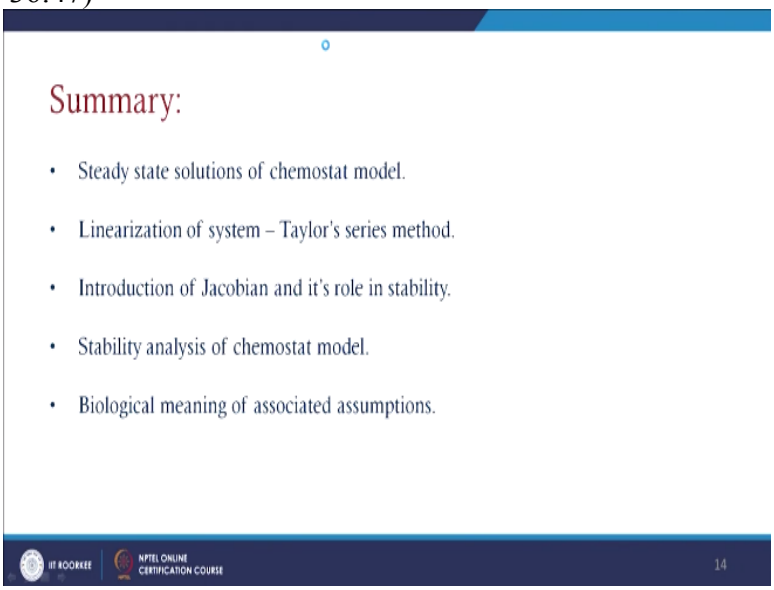

So, in this lecture we have discussed about the steady state solutions of chemostat model, then we have just linearized this model using Taylor's series method and for this linearization we have introduced also this Jacobean to find the stability of this system and its role on stability that we have discussed and then we have just discussed about the stability analysis of chemostat model and its biological meaning where we have just use this assumptions. And after now we have just discussed like different mathematical
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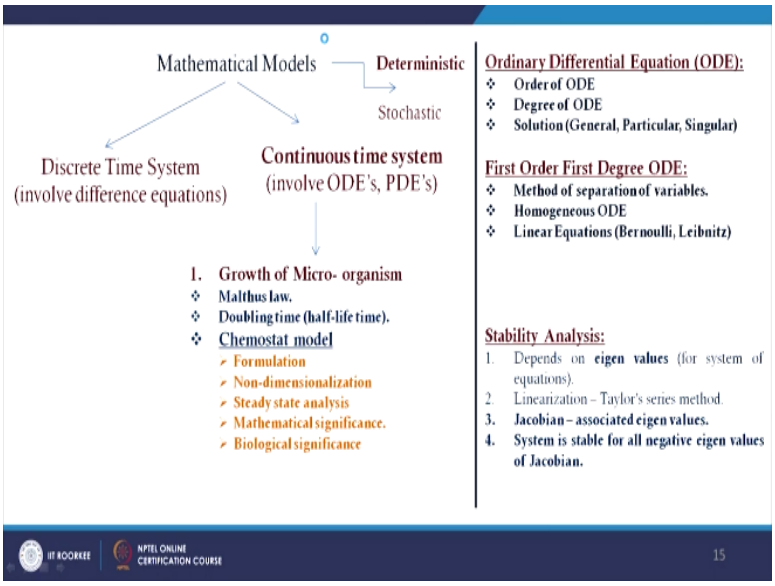

Models like deterministic stochastic and at the beginning of the level we have discussed about discrete time system which involves like difference equations involve one parameter form, two parameter form we have discussed. Then in the continuous time system we have discussed here ordinal differential equations we have not started about PDE's and in the ordinal differential equations since first we have just discussed about growth of micro organism that is based on Malthus model, how this doubling half time is we can calculate for this like the double of this total population size and half of the population size.

Then we have discussed about chemostat model and how we can just formulate a chemostat model and its non-dimensionalization of the equations, then the steady state analysis of this chemostat model equations and then its mathematical significance with its biological relevance we have discussed. And in the ordinal differential equations since we have discussed like order of the ordinal differential equations, degree of the ordinal differential equation, that is general solutions, particular solutions or singular solutions and we have discussed like different methods for the solution of ordinal differential equations that is the method of separation of variables, homogeneous ordinal differential equations linear equations like bernoullis lemniseate form then we have discussed the stability analysis depending on the Eigen value flow lens for the system of equations and for the linearise we have used like Taylor series method and this stability is checked using this Jacobean matrix method and finally we have just tested how the system is stable for all negative Eigen values of Jacobean. Thank you for listen this lecture.

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